INEQUALITIES FOR SYMMETRIC MEANS

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ABSTRACT. We study Muirhead-type generalizations of families of inequalities due to Newton, Maclaurin and others. Each family is defined in terms of a commonly used basis of the ring of symmetric functions in n variables. Inequalities corresponding to elementary symmetric functions and power sum symmetric functions are characterized by the same simple poset which generalizes the majorization order. Some analogous results are also obtained for the Schur, homogeneous, and monomial cases.

1. INTRODUCTION

Commonly used bases for the vector space Λ_n^r of homogeneous of degree r symmetric functions in n variables $\mathbf{x} = (x_1, \ldots, x_n)$ are the *monomial* symmetric functions $\{m_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, *elementary* symmetric functions $\{e_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, (complete) *homogeneous* symmetric functions $\{h_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, *power sum* symmetric functions $\{p_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, and *Schur* functions $\{s_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$. (See [12, Ch. 7] for definitions.)

To each element $g_\lambda(\mathbf{x})$ of these bases, we will associate a *term-normalized* symmetric function $G_\lambda(\mathbf{x})$ and a *mean* $\mathfrak{G}_\lambda(\mathbf{x})$ by

$$(1.1) \qquad G_\lambda(\mathbf{x}) = \frac{g_\lambda(\mathbf{x})}{g_\lambda(1, \ldots, 1)}, \qquad \mathfrak{G}_\lambda(\mathbf{x}) = \sqrt[r]{G_\lambda(\mathbf{x})}.$$

Thus, for example $E_\lambda(\mathbf{x})$ and $\mathfrak{E}_\lambda(\mathbf{x})$ are associated with the elementary symmetric function $e_\lambda(\mathbf{x})$. Note that $\{G_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$ forms a basis of Λ_n^r, and that the functions $\{\mathfrak{G}_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, while symmetric, are not polynomials in $\mathbf{x}$ and therefore do not belong to the *ring of symmetric functions* Λ_n. In the definition of $\mathfrak{G}_\lambda(\mathbf{x})$, we assume $r > 0$.

The functions $\mathfrak{G}_\lambda(\mathbf{x})$ are examples of *symmetric means* (see, e.g., [2, p. 62]). By definition, these are symmetric functions in $x_1, \ldots, x_n$ satisfying

(1) $\min(\mathbf{a}) \leq \mathfrak{G}(\mathbf{a}) \leq \max(\mathbf{a})$,
(2) $\mathbf{a} \leq \mathbf{b}$ (componentwise) implies $\mathfrak{G}(\mathbf{a}) \leq \mathfrak{G}(\mathbf{b})$,
(3) $\lim_{\mathbf{b} \to \mathbf{0}} \mathfrak{G}(\mathbf{a} + \mathbf{b}) = \mathfrak{G}(\mathbf{a})$,

(4) $\mathfrak{G}(c\mathbf{a}) = c\mathfrak{G}(\mathbf{a})$,

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{\geq 0}^n$ and $c \in \mathbb{R}_{\geq 0}$.

This paper will explore inequalities between symmetric means. For fixed n and two means $\mathfrak{F}$, $\mathfrak{G}$, we will write $\mathfrak{F}(\mathbf{x}) \leq \mathfrak{G}(\mathbf{x})$ or $\mathfrak{G}(\mathbf{x}) - \mathfrak{F}(\mathbf{x}) \geq 0$ if we have $\mathfrak{F}(\mathbf{a}) \leq \mathfrak{G}(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{R}_{\geq 0}^n$. We define the inequality $F(\mathbf{x}) \leq G(\mathbf{x})$ analogously. Note that if the degrees of $F(\mathbf{x})$ and $G(\mathbf{x})$ are equal, then we have $F(\mathbf{x}) \leq G(\mathbf{x})$ if and only if $\mathfrak{F}(\mathbf{x}) \leq \mathfrak{G}(\mathbf{x})$.

The study of inequalities of symmetric means has a long history. (See, e.g., [2], [5].) Perhaps the best known such inequality is that of the *arithmetic* and *geometric* means,

$$\mathfrak{E}_1(\mathbf{x}) \geq \mathfrak{E}_n(\mathbf{x}).$$

See [2] for many proofs of this result. Another example is *Muirhead's inequality* [8]: if λ and μ are partitions of r, then

$$M_\lambda(\mathbf{x}) \leq M_\mu(\mathbf{x}) \qquad \text{if and only if } \mu \text{ majorizes } \lambda; \text{ equivalently,}$$
$$\mathfrak{M}_\lambda(\mathbf{x}) \leq \mathfrak{M}_\mu(\mathbf{x}) \qquad \text{if and only if } \mu \text{ majorizes } \lambda.$$

See Section 2 for a definition and further discussion of the majorization order (also known as dominance order) on partitions. Muirhead's inequality will serve as a prototype for many of the results in this paper.

Other classical inequalities are due to

(1) Maclaurin [6]: For $1 \leq i \leq j \leq n$,
$$\mathfrak{E}_i(\mathbf{x}) \geq \mathfrak{E}_j(\mathbf{x}),$$

(2) Newton [9, p. 173]: For $1 \leq k \leq n-1$,
$$E_{k,k}(\mathbf{x}) \geq E_{k+1,k-1}(\mathbf{x}); \text{ equivalently,}$$
$$\mathfrak{E}_{k,k}(\mathbf{x}) \geq \mathfrak{E}_{k+1,k-1}(\mathbf{x}),$$

(3) Schlömilch [11]: For $1 \leq i \leq j$,
$$\mathfrak{P}_i(\mathbf{x}) \leq \mathfrak{P}_j(\mathbf{x}),$$

(4) Gantmacher [4, p. 203]: For $k \geq 1$,
$$p_{k,k}(\mathbf{x}) \leq p_{k+1,k-1}(\mathbf{x}); \text{ equivalently,}$$
$$P_{k,k}(\mathbf{x}) \leq P_{k+1,k-1}(\mathbf{x}); \text{ equivalently,}$$
$$\mathfrak{P}_{k,k}(\mathbf{x}) \leq \mathfrak{P}_{k+1,k-1}(\mathbf{x}),$$

(5) Popoviciu [10]: For $1 \leq i \leq j$,
$$\mathfrak{H}_i(\mathbf{x}) \leq \mathfrak{H}_j(\mathbf{x}),$$

(6) Schur [5, p. 164]: For $k \geq 1$,

$$H_{k,k}(\mathbf{x}) \leq H_{k+1,k-1}(\mathbf{x}); \text{ equivalently,}$$
$$\mathfrak{H}_{k,k}(\mathbf{x}) \leq \mathfrak{H}_{k+1,k-1}(\mathbf{x}).$$

Note that term-normalized symmetric functions and means are defined only for a finite number n of variables. Nevertheless, we may essentially eliminate dependence upon n from the inequalities enumerated above by considering them to be inequalities in sequences of functions,

$$G = (G(x_1), G(x_1, x_2), G(x_1, x_2, x_3), \dots),$$
$$\mathfrak{G} = (\mathfrak{G}(x_1), \mathfrak{G}(x_1, x_2), \mathfrak{G}(x_1, x_2, x_3), \dots).$$

We will define partial orders on such sequences by declaring $F \leq G$ if we have $F(\mathbf{x}) \leq G(\mathbf{x})$ for all $n > 0$, and $\mathfrak{F} \leq \mathfrak{G}$ if we have $\mathfrak{F}(\mathbf{x}) \leq \mathfrak{G}(\mathbf{x})$ for all $n > 0$.

Our strategy will be to classify the above partial orders on the infinite sets $\{\mathfrak{G}_\lambda \mid \lambda \vdash 1, 2, \dots\}$ corresponding to the common bases of the ring of symmetric functions. Our principal results and conjectures (Theorem 3.2, Theorem 4.2, Conjecture 5.1, Theorem 7.3, Conjecture 7.4) can be viewed as either analogs or extensions of Muirhead's inequality. Stating them in full generality requires the introduction of two new partial orders on partitions: the *normalized majorization order*, and the *double (normalized) majorization order*. These will be defined in Section 2 .

Strangely, these partial orders seem to have escaped study within the extensive and venerable literature on symmetric means. The classical inequalities listed above solve many special cases of the problem we have posed above, but fall short of a complete classification. For example, Muirhead's inequalities apply only to pairs of polynomials having the same degree. Maclaurin's inequalities allow different degrees, but only deal with partitions having a single part.

This program is only partially complete. We obtain complete results for the elementary and power sum cases: the posets $\{\mathfrak{E}_\lambda\}$ and $\{\mathfrak{P}_\lambda\}$ are classified in Sections 3 and 4, using an analog of Muirhead's inequality based on the normalized majorization order. For the monomial poset $\{\mathfrak{M}_\lambda\}$, we conjecture a characterization that extends Muirhead's inequality to pairs of functions with different degree, using the double-majorization order. In Section 5 we establish the necessity of this condition, and prove its sufficiency in many cases. For homogeneous symmetric functions, it is easy to prove a sufficient condition for the partial order $\{\mathfrak{H}_\lambda\}$, but its necessity is open. For Schur functions, the situation is reversed: necessity of the corresponding condition is easy but sufficiency is open. We discuss the status of these and other questions in Section 7.

2. Majorization and its extensions

The following definition is classical and has a vast literature (see, e.g. [7]): if λ and μ are partitions of n, then we write $\lambda \preceq \mu$ and say that λ is *majorized* by μ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \qquad \text{for all } i \geq 1.$$

In this definition, we tacitly regard λ and μ as sequences of the same length, adding zeros if necessary. Let $\mathcal{P}_n$ denote the poset of all partitions of n, under the majorization order. It is well-known that $\mathcal{P}_n$ is a lattice, and that it is self-dual. More precisely,

(2.1) $$\lambda \preceq \mu \text{ if and only if } \lambda^\top \succeq \mu^\top,$$

where $\lambda^\top$ denotes the transpose (or conjugate) of λ defined by $\lambda_j^\top = \max\{i \,|\, \lambda_i \geq j\}$. See [1] for more discussion of the lattice structure of $\mathcal{P}_n$.

The notion of majorization extends readily to rational sequences: if $\mathcal{Q}_*$ denotes the set of weakly decreasing sequences of nonnegative rationals, and $\alpha, \beta \in \mathcal{Q}_*$, we say that $\alpha \preceq \beta$ if $\alpha_1 + \cdots + \alpha_i \leq \beta_1 + \cdots + \beta_i$ for all i. The set $\mathcal{Q}_*$ is a lattice under this ordering, with meets defined as in $(\mathcal{P}_n, \preceq)$ using partial sums, i.e., if $S_i(\lambda) = \lambda_1 + \cdots + \lambda_i$, then

$$S_i(\lambda \wedge \mu) = \min(S_i(\lambda), S_i(\mu)) \qquad \text{for all } i \geq 1.$$

However, there is no analog of (2.1), and $\mathcal{Q}_*$ is not self-dual. The subset $\mathcal{Q}_1 \subseteq \mathcal{Q}_*$ consisting of sequences whose entries sum to 1 is a sublattice of $\mathcal{Q}_*$. We will refer to the elements of $\mathcal{Q}_1$ as *rational partitions of* 1.

For each integer n, $\mathcal{P}_n$ embeds naturally in $\mathcal{Q}_1$ under the map

$$\lambda \hookrightarrow \bar{\lambda} = \frac{\lambda}{|\lambda|}.$$

This allows us to define a new relation, called *normalized majorization*, on the set $\mathcal{P}_*$ of all integer partitions. If λ and μ are partitions, possibly of different integers, we write

$$\lambda \sqsubseteq \mu \text{ if } \bar{\lambda} \preceq \bar{\mu}, \text{ i.e., } \frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}.$$

It is important to note that $(\mathcal{P}_*, \sqsubseteq)$ is preorder, not a partial order: for example, if λ is any partition, then $\lambda \sqsubseteq k\lambda$ and $k\lambda \sqsubseteq \lambda$ for any positive integer k. Let $\overline{\mathcal{P}}_* = (\overline{\mathcal{P}}_*, \sqsubseteq)$ denote the quotient of $\mathcal{P}_*$ with respect to the relation $\alpha \sim \beta$ iff $\alpha \sqsubseteq \beta$ and $\beta \sqsubseteq \alpha$. If n is a positive integer, let $\overline{\mathcal{P}}_{\leq n}$ denote the subposet of $\overline{\mathcal{P}}_*$ consisting of elements corresponding to partitions of integers less than or equal to n. Several similarities and differences between the normalized majorization order and ordinary majorization orders are easy to see.

Observation 2.1. *Normalized majorization satisfies the following properties:*

(1) For all n, $(\mathcal{P}_n, \preceq)$ embeds isomorphically in $\overline{\mathcal{P}}_{\leq n}$ (and hence in $\overline{\mathcal{P}}_*$) as a sub-poset.

(2) For all n, $\overline{\mathcal{P}}_{\leq n}$ is a finite poset; for $n \geq 5$ it is not a lattice and is not self-dual; for $n \geq 6$ it is not ranked.

(3) $\overline{\mathcal{P}}_*$ is a lattice, isomorphic to the infinite sublattice of $\mathcal{Q}_1$ consisting of sequences with finite support; it is not locally finite (in fact every interval has infinite length), and it is not self-dual.

Figure 2.1 shows the poset $\overline{\mathcal{P}}_{\leq 6}$, with each element represented by the corresponding integer partition in "lowest terms". Thus, for example, $\{3, 3\}$ and $\{2, 2\}$ are both represented by $\{1, 1\}$. Partitions of integers dividing 6 have been emphasized to show the embedding of $(\overline{\mathcal{P}}_6, \preceq)$ in $\overline{\mathcal{P}}_{\leq 6}$.
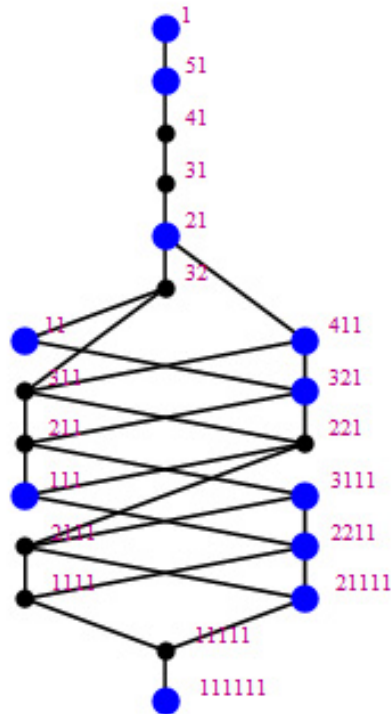

FIGURE 2.1. The poset $\overline{\mathcal{P}}_{\leq 6}$, with an embedding of $(\overline{\mathcal{P}}_6, \preceq)$.

Parts (1) and (2) of Observation 2.1 are straightforward, as are most of the claims made in (3). To verify that $\overline{\mathcal{P}}_*$ is not locally finite, suppose that $\alpha, \beta \in \mathcal{Q}_1$ are distinct partitions with finite support, such that $\alpha \prec \beta$. Then the partition

$$\gamma = \frac{\alpha + \beta}{2} = (\frac{\alpha_1 + \beta_1}{2}, \frac{\alpha_2 + \beta_2}{2}, \dots)$$

lies strictly between α and β, and also has finite support.

We will also introduce another partial order on $\mathcal{P}_*$, called the *double (normalized) majorization order*. If λ and μ are partitions, possibly of different numbers, define

$$\lambda \trianglelefteq \mu \text{ iff } \lambda \sqsubseteq \mu \text{ and } \lambda^\top \sqsupseteq \mu^\top,$$

or in other words,

$$\lambda \trianglelefteq \mu \text{ iff } \frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|} \text{ and } \frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}.$$

Let $\mathcal{DP}_* = (\mathcal{P}_*, \trianglelefteq)$. It is worth noting that the conditions $\lambda \sqsubseteq \mu$ and $\lambda^\top \sqsupseteq \mu^\top$ are not equivalent in general; for example, if $\lambda = \{2,2\}$ and $\mu = \{2,1\}$, then $\lambda \sqsubseteq \mu$ but $\lambda^\top \not\sqsupseteq \mu^\top$. However, when $|\lambda| = |\mu|$, $\lambda \trianglelefteq \mu$ if and only if $\lambda \preceq \mu$, and thus double majorization is equivalent to ordinary majorization in this case. We note some basic properties of $\mathcal{DP}_*$:

Observation 2.2. *Let λ and μ be integer partitions.*

(1) *If $\lambda \trianglelefteq \mu$ and $\mu \trianglelefteq \lambda$, then $\lambda = \mu$; hence $\mathcal{DP}_*$ is a partial order.*
(2) *For all n, $(\mathcal{P}_n, \preceq)$ embeds isomorphically in $\mathcal{DP}_*$ as a subposet.*
(3) *$\lambda \trianglelefteq \mu$ if and only if $\lambda^\top \trianglerighteq \mu^\top$; hence $\mathcal{DP}_*$ is self-dual.*
(4) *$\mathcal{DP}_*$ is an infinite poset without universal bounds; it is locally finite, but is not locally ranked.*
(5) *$\mathcal{DP}_*$ is not a lattice.*

Figure 2.2 shows the restriction of this poset to integer partitions of of $1, \ldots, 5$. Embeddings of $(\mathcal{P}_n, \preceq)$ in $\mathcal{DP}_*$ appear as vertical columns in the diagram, for $n = 1, \ldots, 5$.

Claims (1)–(3) in Observation 2.2 are immediate or straightforward. To verify the claim in (4) that $\mathcal{DP}_*$ is locally finite, note that if $\lambda \trianglelefteq \theta \trianglelefteq \mu$, then it follows from the definition of double majorization that

$$\theta_1^\top \leq \lambda_1^\top \text{ and } \theta_1 \leq \mu_1.$$

Hence the Ferrers diagram of θ fits inside a box of size $\lambda_1^\top \times \mu_1$. Since there are only a finite number of such θ, the interval $[\lambda, \mu]$ is finite.

The statement in (5) that $\mathcal{DP}_*$ is not a lattice may be verified by inspection of Figure 2.2. Note for example, that the partitions $\{2\}$ and $\{3,1\}$ do not have a greatest lower bound; any such partition would have to lie in the interval $[\{2,1\}, \{2\}]$, and that entire interval is displayed in the diagram.

We continue by listing some technical observations that will be useful in later sections. Some of these will involve the operations of *dilation* and *replication*, defined
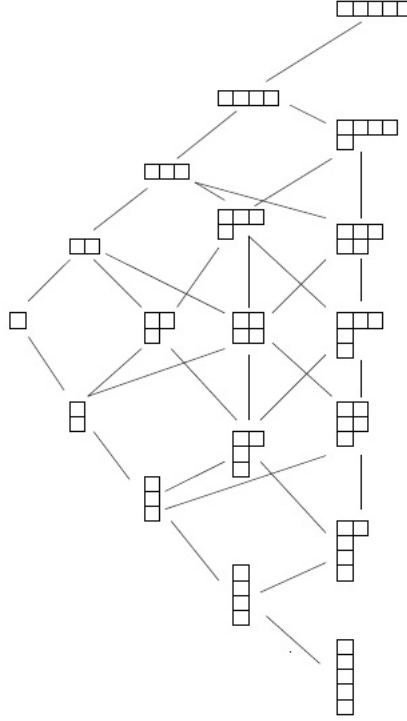
FIGURE 2.2. Double majorization of partitions of $1, \ldots, 5$.

for partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ as follows: if c is a positive integer, then

$$
\begin{aligned}
c\lambda &= (c\lambda_1, \ldots, c\lambda_\ell), \\
\lambda^c &= (\underbrace{\lambda_1, \ldots, \lambda_1}_{c}, \ldots, \underbrace{\lambda_\ell, \ldots, \lambda_\ell}_{c}).
\end{aligned}
$$

The operations of dilation and replication extend to rational values of c, provided that the resulting part sizes and multiplicities are integral. The following facts are easy consequences of the above definitions.

Observation 2.3. *Let λ, μ be integer partitions with $|\lambda| = |\mu|$, let c and d be positive rational numbers. Then we have*

(1) $(\lambda^d)^\top = d\lambda^\top$,
(2) $(d\lambda)^\top = (\lambda^\top)^d$,
(3) $\lambda \preceq \mu \Longleftrightarrow c\lambda \preceq c\mu \Longleftrightarrow \lambda^d \preceq \mu^d \Longleftrightarrow \lambda^\top \succeq \mu^\top$,

assuming all of these sequences are defined.

The following results are routine but require a little more effort, and we leave their verification to the reader.

Observation 2.4. *Suppose that λ and μ are arbitrary integer partitions.*

(1) *If $|\lambda| \leq |\mu|$, then $\lambda \trianglelefteq \mu$ if and only if $\lambda \sqsubseteq \mu$, i.e., $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$.*

(2) *If $|\lambda| \geq |\mu|$, then $\lambda \trianglelefteq \mu$ if and only if $\lambda^\top \sqsupseteq \mu^\top$, i.e., $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$.*

(3) *If c is a positive rational such that $c\lambda$ is defined, then $\lambda \trianglelefteq c\lambda$ if and only if $c \geq 1$.*

(4) *If c is a positive rational such that λ^c is defined, then $\lambda \trianglerighteq \lambda^c$ if and only if $c \geq 1$.*

We conclude this section with an alternate characterization of the majorization order that will be used in Sections 3 and 4. If λ is a partition, define the function $\psi_\lambda : \mathbb{N} \to \mathbb{N}$ by

$$(2.2) \qquad \psi_\lambda(j) = \max_{1 \leq k \leq \ell}\{\lambda_1 + \cdots + \lambda_k - kj\}.$$

Lemma 2.5. *Two integer partitions λ, μ of r satisfy $\lambda \preceq \mu$ if and only if we have $\psi_\lambda(j) \leq \psi_\mu(j)$ for $j = 1, \ldots, r$. Furthermore, for any fixed j we have $\psi_\lambda(j) > \psi_\mu(j)$ if and only if $\lambda_1^\top + \cdots + \lambda_j^\top < \mu_1^\top + \cdots + \mu_j^\top$.*

Proof. Note that $\psi_\lambda(j)$ is equal to the number of boxes in columns $j+1, \ldots, r$ of the Young diagram of λ,

$$\psi_\lambda(j) = r - (\lambda_1^\top + \cdots + \lambda_j^\top).$$

Thus the condition $\psi_\lambda(j) \leq \psi_\mu(j)$ for $j = 1, \ldots, r$ is equivalent to the condition $\lambda^\top \succeq \mu^\top$, which in turn is equivalent to $\lambda \preceq \mu$. $\qquad \square$

This result can be generalized easily to rational partitions of 1.

Corollary 2.6. *Two integer partitions λ, μ satisfy $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$ if and only if we have $\frac{1}{|\lambda|}\psi_\lambda(j) \leq \frac{1}{|\mu|}\psi_\mu(j)$ for $j = 1, \ldots, r$. Furthermore, for any fixed index j we have $\frac{1}{|\lambda|}\psi_\lambda(j) > \frac{1}{|\mu|}\psi_\mu(j)$ if and only if $\frac{\lambda_1^\top}{|\lambda|} + \cdots + \frac{\lambda_j^\top}{|\lambda|} < \frac{\mu_1^\top}{|\mu|} + \cdots + \frac{\mu_j^\top}{|\mu|}$.*

3. ELEMENTARY MEANS

Maclaurin's and Newton's inequalities state that

$$\mathfrak{E}_1 \geq \cdots \geq \mathfrak{E}_n,$$
$$E_{k,k} \geq E_{k+1,k-1}, \qquad \text{for } k = 1, \ldots, n-1.$$

In this section we generalize these to inequalities of the form $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$, where λ, μ are integer partitions and $\mathfrak{E}_\lambda$, $\mathfrak{E}_\mu$ are means corresponding to elementary symmetric functions.

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash r$, the term-normalized elementary symmetric function $E_\lambda(\mathbf{x})$ is given by the formula

$$E_\lambda(\mathbf{x}) = \frac{e_\lambda(\mathbf{x})}{\binom{n}{\lambda_1} \cdots \binom{n}{\lambda_\ell}},$$

and the corresponding elementary mean is the rth root of this, $\mathfrak{E}_\lambda(\mathbf{x}) = \sqrt[r]{E_\lambda(\mathbf{x})}$. Since $e_{\lambda^d}(\mathbf{x}) = e_\lambda(\mathbf{x})^d$ and λ^d is a partition of dr, we have the following stability property of elementary means under the replication operation.

Observation 3.1. *For any partition λ and integer $d \geq 1$ we have $\mathfrak{E}_\lambda = \mathfrak{E}_{\lambda^d}$.*

We can now state and prove the main result of this section.

Theorem 3.2. *If λ and μ are integer partitions with $|\lambda| = |\mu|$, then*

$$E_\lambda \leq E_\mu \text{ iff } \lambda \succeq \mu \text{ iff } \lambda^\top \preceq \mu^\top.$$

If λ and μ are arbitrary integer partitions, then

$$\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu \text{ iff } \lambda^\top \sqsubseteq \mu^\top \text{ i.e., } \frac{\lambda^\top}{|\lambda|} \preceq \frac{\mu^\top}{|\mu|}.$$

The partial order on $\{E_\lambda | \lambda \vdash n\}$ is isomorphic to $(\mathcal{P}_n, \preceq)$. The partial order on $\{\mathfrak{E}_\lambda\}$ is isomorphic to $(\overline{\mathcal{P}}_*, \sqsubseteq)$.

Proof. First we consider the case that $|\lambda| = |\mu|$, which was first proved in [3, Thm. 5.7].

Suppose that $\lambda^\top \not\preceq \mu^\top$. Then for some index i we have

$$\lambda_1^\top + \cdots + \lambda_i^\top > \mu_1^\top + \cdots + \mu_i^\top.$$

Choosing a number $n \geq \max(\lambda_1, \mu_1)$ and specializing the symmetric functions $E_\lambda(\mathbf{x})$, $E_\mu(\mathbf{x})$ at

$$x_1 = \cdots = x_i = t,$$
$$x_{i+1} = \cdots = x_n = 1,$$

we obtain polynomials in $\mathbb{N}[t]$ of degrees $\lambda_1^\top + \cdots + \lambda_i^\top$ and $\mu_1^\top + \cdots + \mu_i^\top$, respectively. Thus we have

$$\lim_{t \to \infty} [E_\lambda(\underbrace{t, \ldots, t}_{i}, \underbrace{1, \ldots, 1}_{n-i}) - E_\mu(\underbrace{t, \ldots, t}_{i}, \underbrace{1, \ldots, 1}_{n-i})] = \infty,$$

which implies that $E_\lambda \not\leq E_\mu$.

Conversely, suppose that $\lambda^\top \preceq \mu^\top$ and write $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. If λ covers μ in the majorization order, then there exist indices $1 \leq j < k \leq \ell$ for which we have

$$\mu = (\lambda_1, \ldots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \ldots, \lambda_{k-1}, \lambda_k + 1, \lambda_{k+1}, \ldots, \lambda_\ell).$$

For arbitrary n, we therefore have that $E_\mu(\mathbf{x}) - E_\lambda(\mathbf{x})$ is equal to

$$\frac{E_\lambda(\mathbf{x})}{E_{\lambda_j}(\mathbf{x})E_{\lambda_k}(\mathbf{x})}(E_{\lambda_j-1}(\mathbf{x})E_{\lambda_k+1}(\mathbf{x}) - E_{\lambda_j}(\mathbf{x})E_{\lambda_k}(\mathbf{x})).$$

Rewriting Newton's inequalities as

$$\frac{E_1(\mathbf{x})}{E_0(\mathbf{x})} \geq \frac{E_2(\mathbf{x})}{E_1(\mathbf{x})} \geq \frac{E_3(\mathbf{x})}{E_2(\mathbf{x})} \geq \cdots ,$$

we see that $E_{\lambda_j}(\mathbf{x})E_{\lambda_k}(\mathbf{x}) \leq E_{\lambda_j-1}(\mathbf{x})E_{\lambda_k+1}(\mathbf{x})$. Thus, $E_\lambda(\mathbf{x}) \leq E_\mu(\mathbf{x})$. If λ does not cover μ in the majorization order, then there exists a sequence of partitions

$$\mu = \nu^{(0)} \leq \nu^{(1)} \leq \cdots \leq \nu^{(m)} = \lambda,$$

in which each comparison of consecutive partitions is a covering relation. Thus we have

$$E_\mu(\mathbf{x}) - E_\lambda(\mathbf{x}) = \sum_{i=0}^{m-1}(E_{\nu^{(i)}}(\mathbf{x}) - E_{\nu^{(i+1)}}(\mathbf{x})) \geq 0,$$

and again $E_\lambda(\mathbf{x}) \leq E_\mu(\mathbf{x})$. Since this argument is independent of n, we have $E_\lambda \leq E_\mu$.

Now consider the case that $|\lambda|$ and $|\mu|$ are not equal. By Observation 3.1, we have

$$\mathfrak{E}_\lambda(\mathbf{x}) = \mathfrak{E}_{\lambda^{|\mu|}}(\mathbf{x}), \qquad \mathfrak{E}_\mu(\mathbf{x}) = \mathfrak{E}_{\mu^{|\lambda|}}(\mathbf{x}).$$

Since $\lambda^{|\mu|}$ and $\mu^{|\lambda|}$ are both partitions of $|\lambda| \cdot |\mu|$, we have $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$ if and only if $(\lambda^{|\mu|})^\top \preceq (\mu^{|\lambda|})^\top$. By Observation 2.3 this is equivalent to the condition $\frac{\lambda^\top}{|\lambda|} \preceq \frac{\mu^\top}{|\mu|}$. $\square$

The isomorphism between $\{\mathfrak{E}_\lambda\}$ and $(\overline{\mathcal{P}}_*, \sqsubseteq)$ is given by the map $\mathfrak{E}_\lambda \mapsto \lambda^\top$. Consequently, the labeling of elements in Figure 2.1 does not represent the partial order on the $\mathfrak{E}_\lambda$. Figure 3.1 shows another copy of the diagram in which the labels have been corrected to show the ordering of elementary means.

The proof of Theorem 3.2 shows that Newton's inequalities imply those of Maclaurin and those of the form $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$ in the following strong algebraic sense. Define the *Newton semiring* to be the set of all nonnegative linear combinations of products of symmetric functions of the forms

$$\{E_{j,i}(\mathbf{x}) - E_{j+1,i-1}(\mathbf{x}) \mid 1 \leq i \leq j \leq n-1\} \cup \{E_i(\mathbf{x}) \mid 1 \leq i \leq n\}.$$

Corollary 3.3. *Each difference $E_\mu(\mathbf{x}) - E_\lambda(\mathbf{x})$ with $|\lambda| = |\mu|$ and $\mu \preceq \lambda$ belongs to the Newton semiring.*

Note that Corollary 3.3 includes differences of the form $E_{\mu^{|\lambda|}}(\mathbf{x}) - E_{\lambda^{|\mu|}}(\mathbf{x})$ with $\frac{\lambda^\top}{|\lambda|} \preceq \frac{\mu^\top}{|\mu|}$, even if $|\lambda| \neq |\mu|$. This shows that Newton's inequalities imply Maclaurin's inequality and relatives of the form $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$.
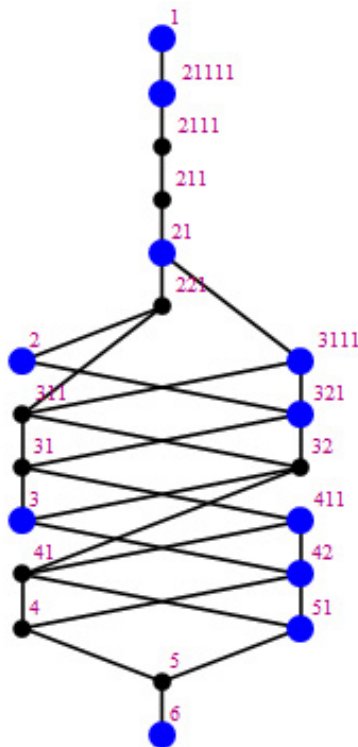
FIGURE 3.1. Equivalence classes of partitions of $1, \ldots, 6$ ordered by inequalities of elementary means.

4. Power sum means

The inequalities of Schlömilch and Gantmacher state that

$$\mathfrak{P}_1 \leq \mathfrak{P}_2 \leq \cdots,$$

$$P_{k,k} \leq P_{k+1,k-1}, \qquad \text{for } k = 1, 2, \ldots.$$

We will generalize these to inequalities of the form $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$, where λ, μ are integer partitions and $\mathfrak{P}_\lambda$, $\mathfrak{P}_\mu$ are means corresponding to power sum symmetric functions.

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash r$, the term-normalized power sum symmetric function $P_\lambda(\mathbf{x})$ is given by

$$P_\lambda(\mathbf{x}) = \frac{p_\lambda(\mathbf{x})}{n^\ell},$$

and the corresponding power sum mean is the rth root of this, $\mathfrak{P}_\lambda(\mathbf{x}) = \sqrt[r]{P_\lambda(\mathbf{x})}$. Like the elementary basis $\{e_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$ of Λ_n^r, the power sum basis $\{p_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$ is multiplicative. We therefore have the following equalities.

Observation 4.1. *For any partition λ and integer $d \geq 1$ we have $\mathfrak{P}_\lambda = \mathfrak{P}_{\lambda^d}$.*

Theorem 4.2. *If λ and μ are integer partitions with $|\lambda| = |\mu|$, then*

$$P_\lambda \leq P_\mu \text{ iff } \lambda \preceq \mu \text{ iff } \lambda^\top \succeq \mu^\top.$$

If λ and μ are arbitrary integer partitions, then

$$\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu \text{ iff } \lambda^\top \sqsupseteq \mu^\top \text{ i.e., } \frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}.$$

The partial order on $\{P_\lambda | \lambda \vdash n\}$ is isomorphic to $(\mathcal{P}_n, \preceq)$. The partial order on $\{\mathfrak{P}_\lambda\}$ is isomorphic to the dual of $(\overline{\mathcal{P}}_*, \sqsubseteq)$.

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_m)$ be integer partitions. Let us first assume that $|\lambda| = |\mu|$.

Suppose that $\lambda \not\preceq \mu$, and let ψ_λ, ψ_μ be the functions defined in (2.2). Using Lemma 2.5, choose an index j such that $\psi_\lambda(j) > \psi_\mu(j)$, and consider the functions which we temporarily denote by $\phi_\lambda(t)$, $\phi_\mu(t)$ and which we define by

$$\phi_\lambda(t) = P_\lambda(t, \underbrace{1, \ldots, 1}_{t^j}) = \prod_{i=1}^{\ell} \frac{t^{\lambda_i} + t^j}{1 + t^j} = \frac{1}{(t^j + 1)^\ell} \sum_{k=0}^{\ell} \sum_{\{i_1, \ldots, i_k\}} t^{\lambda_{i_1} + \cdots + \lambda_{i_k} + (\ell - k)j},$$

$$\phi_\mu(t) = P_\mu(t, \underbrace{1, \ldots, 1}_{t^j}) = \prod_{i=1}^{m} \frac{t^{\mu_i} + t^j}{1 + t^j} = \frac{1}{(t^j + 1)^m} \sum_{k=0}^{m} \sum_{\{i_1, \ldots, i_k\}} t^{\mu_{i_1} + \cdots + \mu_{i_k} + (m - k)j}.$$

These are rational functions in t which asymptotically approach polynomials in t of degrees $\psi_\lambda(j)$ and $\psi_\mu(j)$, respectively. Thus we have

$$\lim_{t \to \infty} [\phi_\lambda(t) - \phi_\mu(t)] = \infty,$$

which implies $P_\lambda \not\leq P_\mu$.

Conversely, suppose that $\lambda \preceq \mu$. Then we proceed as in the proof of Theorem 3.2 with Gantmacher's inequalities replacing those of Newton and conclude that $P_\lambda \leq P_\mu$.

In the case that $|\lambda|$ and $|\mu|$ are not equal, we again proceed as in the proof of Theorem 3.2. Specifically, we apply Observation 4.1 to see that we have

$$\mathfrak{P}_\lambda(\mathbf{x}) = \mathfrak{P}_{\lambda^{|\mu|}}(\mathbf{x}), \qquad \mathfrak{P}_\mu(\mathbf{x}) = \mathfrak{P}_{\mu^{|\lambda|}}(\mathbf{x})$$

for all n. Thus we have $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$ if and only if $\lambda^{|\mu|} \preceq \mu^{|\lambda|}$, or equivalently, if and only if $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$. $\qquad\square$

The similarity of Theorems 3.2 and 4.2 is somewhat curious. In fact, we have $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$ if and only if $\mathfrak{E}_\lambda \geq \mathfrak{E}_\mu$. It would be interesting to find a more direct proof of this fact.

The proof of Theorem 4.2 shows that Gantmacher's inequalities imply those of Schlömilch and those of the form $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$, just as as Corollary 3.3 shows that

Newton's inequalities imply those of Maclaurin. To be more precise, we define the *Gantmacher semiring* to be the set of all nonnegative linear combinations of products of symmetric functions of the forms

$$\{P_{j+1,i-1}(\mathbf{x}) - P_{j,i}(\mathbf{x}) \,|\, 1 \leq i \leq j\} \cup \{P_i(\mathbf{x}) \,|\, i \geq 1\}.$$

Corollary 4.3. *Each difference $P_\mu(\mathbf{x}) - P_\lambda(\mathbf{x})$ with $|\lambda| = |\mu|$ and $\lambda \preceq \mu$ belongs to the Gantmacher semiring.*

As was the case with Corollary 3.3, the last result also covers cases where $|\lambda| \neq |\mu|$, since it implies that $P_{\mu|\lambda|}(\mathbf{x}) - P_{\lambda|\mu|}(\mathbf{x})$ when $\frac{\lambda^\top}{|\lambda|} \preceq \frac{\mu^\top}{|\mu|}$.

Finally, we note that the inequalities of Schlömilch and Gantmacher hold in greater generality than we have considered here. For example, Schlömilch's inequalities hold for power sum means indexed by any two real numbers [11]. (See also [5, p. 26].) It also is easy to see that Gantmacher's inequalities $\mathfrak{P}_{k,k} \leq \mathfrak{P}_{k+1,k-1}$ hold for k real. Furthermore, these are just a small part of a much larger family of inequalities derived from minors of matrices. See [4, p. 203].

5. Monomial means

We next turn to the case of monomial means, and look for inequalities of the form $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ where λ, μ are arbitrary integer partitions and $\mathfrak{M}_\lambda, \mathfrak{M}_\mu$ are means corresponding to monomial symmetric functions. The prototype is Muirhead's inequality, which states that

$$\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu \qquad \text{if and only if} \qquad \lambda \preceq \mu,$$

provided that $|\lambda| = |\mu|$.

Note that a formula for $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash r$, the term-normalized monomial symmetric function is given by

$$M_\lambda(\mathbf{x}) = \frac{m_\lambda(\mathbf{x})}{\binom{n}{\alpha_1, \ldots, \alpha_r, n-\ell}},$$

where α_j is equal to the number of parts of λ which are equal to j. The corresponding monomial mean is the rth root of this, $\mathfrak{M}_\lambda(\mathbf{x}) = \sqrt[r]{M_\lambda(\mathbf{x})}$.

Unlike the elementary and power sum bases of Λ_n^r, the monomial basis is not multiplicative. Nonetheless, evidence suggests that a characterization analogous to Theorem 3.2 and Theorem 4.2 exists for the poset $\{\mathfrak{M}_\lambda\}$ as well.

Conjecture 5.1. *Given integer partitions λ and μ, we have*

$$(5.1) \qquad \mathfrak{M}_\lambda \leq \mathfrak{M}_\mu \text{ if and only if } \lambda \trianglelefteq \mu, \text{ i.e., } \frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|} \text{ and } \frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}.$$

Equivalently, $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ if and only if $\mathfrak{E}_{\lambda^\top} \leq \mathfrak{E}_{\mu^\top}$ and $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$. The poset $\{\mathfrak{M}_\lambda\}$ is isomorphic to $\mathcal{DP}_*$.

Much of Conjecture 5.1 may be proved using the methods of the proof of Theorem 3.2. In particular, we will show that the conditions on λ and μ in (5.1) are necessary for the inequality $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$, i.e., that $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ implies $\lambda \trianglelefteq \mu$. As in the proof of Theorem 3.2, we consider specializations of M_λ, M_μ at vectors of the form $(t, \ldots, t, 1, \ldots, 1)$. First we prove the necessity in (5.1) of the conditions $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$.

Proposition 5.2. *If $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$, then $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$.*

Proof. Suppose $\frac{\lambda}{|\lambda|} \not\preceq \frac{\mu}{|\mu|}$. Then there exists an index j such that

$$\frac{\lambda_1}{|\lambda|} + \cdots + \frac{\lambda_j}{|\lambda|} > \frac{\mu_1}{|\mu|} + \cdots + \frac{\mu_j}{|\mu|}.$$

Choosing $n > \max(\lambda_1^\top, \mu_1^\top)$ and specializing the symmetric functions $M_\lambda(\mathbf{x})$, $M_\mu(\mathbf{x})$ at

$$x_1 = \cdots = x_j = t,$$
$$x_{j+1} = \cdots = x_n = 1,$$

we obtain polynomials in $\mathbb{N}[t]$ of degrees $\lambda_1 + \cdots + \lambda_j$ and $\mu_1 + \cdots + \mu_j$, respectively. It follows that

$$\lim_{t \to \infty} [\mathfrak{M}_\lambda(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{n-j}) - \mathfrak{M}_\mu(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{n-j})] = \infty,$$

which implies $\mathfrak{M}_\lambda \not\leq \mathfrak{M}_\mu$. $\square$

To prove the necessity in (5.1) of the conditions $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$, we will need to look more closely at specializations of M_λ, M_μ at $x = (t, \ldots, t, 1, \ldots, 1)$.

Proposition 5.3. *Fix an integer partition $\theta = (\theta_1, \ldots, \theta_\ell)$ and a nonnegative integer $j \geq \ell$. Then we have*

$$(5.2) \qquad M_\theta(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{n-j}) = \sum_{k=0}^{\ell} \frac{\binom{j}{k}\binom{n-j}{\ell-k}}{\binom{n}{\ell}} \sum_{\rho} a(\theta, \rho) t^{|\rho|}$$

where the second sum is over subsequences $\rho = (\rho_1, \ldots, \rho_k) = (\theta_{i_1}, \ldots, \theta_{i_k})$ of θ and $a(\theta, \rho)$ is a constant which depends upon θ and ρ.

Proof. We have

$$(5.3) \qquad m_\theta(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{n-j}) = \sum_{k=0}^{\ell} \binom{j}{k}\binom{n-j}{\ell-k} \sum_{(\rho_1, \ldots, \rho_k)} b(\theta, \rho) t^{\rho_1 + \cdots + \rho_k},$$

where $b(\theta, \rho)$ is the number of rearrangements of θ whose first k components are a rearrangement of $(\rho_1, \ldots, \rho_k)$. Similarly,

$$m_\theta(\underbrace{1, \ldots, 1}_{n}) = \binom{n}{\ell} c(\theta),$$

where $c(\theta)$ is the number of rearrangements of θ. The ratio of these two expressions therefore has the desired form. $\qquad \square$

Choosing the number of ones in this specialization to be a function of $t \in \mathbb{R}$, we obtain the following sharper result.

Lemma 5.4. *Fix an integer partition $\theta = (\theta_1, \ldots, \theta_\ell)$ and nonnegative integers $j \geq \ell$ and b. Then the function*

(5.4)
$$M_\theta(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{t^b - j})$$

in $\mathbb{Q}(t)$ behaves asymptotically like a constant times $t^{\psi_\theta(b)}$, where ψ_θ is the function defined in (2.2).

Proof. For $k = 0, \ldots, \ell$, the rational function

$$\frac{\binom{j}{k}\binom{n-j}{\ell-k}}{\binom{n}{\ell}}$$

appearing in (5.3) is a ratio $q_1(n)/q_2(n)$ of polynomials satisfying $\deg q_1 - \deg q_2 = -k$. Substituting $n = t^b$ in (5.3) and observing that each subsequence $(\theta_{i_1}, \ldots, \theta_{i_k})$ of θ satisfies $\theta_{i_1} + \cdots + \theta_{i_k} \leq \theta_1 + \cdots + \theta_k$, we see that the function (5.4) behaves asymptotically like a constant times

$$t^{\max_k\{\theta_1 + \cdots + \theta_k - kb\}} = t^{\psi_\theta(b)}.$$

$\qquad \square$

We can now complete the final step in showing necessity of the conditions in (5.1).

Proposition 5.5. *If $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$, then $\frac{\lambda^\intercal}{|\lambda|} \succeq \frac{\mu^\intercal}{|\mu|}$.*

Proof. Suppose that $\frac{\lambda^\intercal}{|\lambda|} \not\succeq \frac{\mu^\intercal}{|\mu|}$. Then by Corollary 2.6 we may choose an index b such that $\psi_\lambda(b)/|\lambda| > \psi_\mu(b)/|\mu|$. Fix a nonnegative integer $j \geq \max(\lambda_1^\intercal, \mu_1^\intercal)$ and temporarily define the functions $\phi_\lambda(t)$, $\phi_\mu(t)$ of t by

$$\phi_\lambda(t) = \mathfrak{M}_\lambda(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{t^b - j}), \qquad \phi_\mu(t) = \mathfrak{M}_\mu(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{t^b - j}).$$

By Lemma 5.4 we have

$$\lim_{t \to \infty} \frac{\phi_\lambda(t)}{\phi_\mu(t)} = \lim_{t \to \infty} \frac{t^{\psi_\lambda(b)/|\lambda|}}{t^{\psi_\mu(b)/|\mu|}} = \infty,$$

which implies that $\mathfrak{M}_\lambda \not\leq \mathfrak{M}_\mu$. $\square$

Thus we have proved "half" of Conjecture 5.1, and we summarize by stating this result as a corollary.

Corollary 5.6. *If $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$, then $\lambda \trianglelefteq \mu$.*

It remains to prove sufficiency, i.e. that $\lambda \trianglelefteq \mu$ implies $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$. A significant number of cases may be proved easily using the technique of plethystic substitution. In particular, we will show that the implication is true whenever $|\lambda| \leq |\mu|$.

Recall from Observation 2.4 (3) that for $c \in \mathbb{Q}$ and $c\lambda$ an integer partition, we have $\lambda \trianglelefteq c\lambda$ if and only if $c \geq 1$. Inequalities for monomial means satisfy a similar condition; in fact we do not even need to assume that $c\lambda$ is an integer partition.

Proposition 5.7. *For $c \in \mathbb{Q}$, we have $\mathfrak{M}_\lambda \leq \mathfrak{M}_{c\lambda}$ if and only if $c \geq 1$.*

Proof. Fix $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and an integer $n \geq \ell$. Let $D \subset \mathbb{N}^n$ be the set of all rearrangements of $(\lambda_1, \ldots, \lambda_\ell, 0, \ldots, 0)$. It is clear that we have

$$(5.5) \qquad \frac{\sum_{\delta \in D} x_1^{\delta_1} \cdots x_n^{\delta_n}}{|D|} = M_\lambda(x),$$

since the numerator and denominator of the fraction on the left-hand side are equal to $m_\lambda(x)$ and $m_\lambda(1^n)$, respectively. Similarly for any $c \in \mathbb{R}$, we have

$$\frac{\sum_{\delta \in D} x_1^{c\delta_1} \cdots x_n^{c\delta_n}}{|D|} = M_{c\lambda}(x).$$

Now observe that for all $\mathbf{a} \in \mathbb{R}_{\geq 0}^n$ we may define a sequence $\mathbf{b} \in \mathbb{R}_{\geq 0}^{|D|}$ whose components are the numbers $a_1^{\delta_1} \cdots a_n^{\delta_n}$ obtained by letting δ vary over D (in any order). Furthermore, the evaluations of $\mathfrak{P}_1(y_1, \ldots, y_{|D|})$ and $\mathfrak{P}_c(y_1, \ldots, y_{|D|})$ at $\mathbf{b}$ are equal to $\mathfrak{M}_\lambda(\mathbf{a})$ and $\mathfrak{M}_{c\lambda}(\mathbf{a})$, respectively. We may then apply Schlömilch's inequality (or Theorem 4.2) to obtain

$$\mathfrak{M}_\lambda(\mathbf{a}) = \mathfrak{P}_1(\mathbf{b}) \leq \mathfrak{P}_c(\mathbf{b}) = \mathfrak{M}_{c\lambda}(\mathbf{a})$$

if and only if $c \geq 1$. Thus we have $\mathfrak{M}_\lambda \leq \mathfrak{M}_{c\lambda}$ if and only if $c \geq 1$. $\square$

We note that a result equivalent to Proposition 5.7 appears in [2, p. 361], with essentially the same proof. We can now complete the proof of the sufficiency of the conditions on λ and μ in (5.1), as follows.

Proposition 5.8. *Assume $|\lambda| \leq |\mu|$. If $\lambda \trianglelefteq \mu$ then $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$.*

Proof. By Observation 2.4 (1), the condition $\lambda \trianglelefteq \mu$ is equivalent to $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$, when $|\lambda| \leq |\mu|$. Suppose therefore that we have $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$, or equivalently, $\frac{|\mu|}{|\lambda|}\lambda \preceq \mu$. By Muirhead's Theorem, we then have $\mathfrak{M}_{k\lambda} \leq \mathfrak{M}_\mu$, where $k = \frac{|\mu|}{|\lambda|}$. By Proposition 5.7 we also have $\mathfrak{M}_\lambda \leq \mathfrak{M}_{k\lambda}$ and thus $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$. $\square$

Thus the only remaining part of Conjecture 5.1 which needs to be proved is the sufficiency of the condition $\lambda \trianglelefteq \mu$ when $|\lambda| > |\mu|$. We state this as a separate conjecture:

Conjecture 5.9. *Assume $|\lambda| > |\mu|$. If $\lambda \trianglelefteq \mu$ then $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$.*

Observation 2.4 (4) states that $\lambda \trianglerighteq \lambda^d$ for all rationals $d \geq 1$ such that λ^d is an integer partition. From this we obtain the following simple special case of Conjecture 5.9.

Conjecture 5.10. $\mathfrak{M}_\lambda \geq \mathfrak{M}_{\lambda^d}$ *for all rationals $d \geq 1$ such that λ^d is an integer partition. Equivalently, $\mathfrak{M}_{\lambda^a} \geq \mathfrak{M}_{\lambda^b}$ for all pairs of nonnegative integers $a \leq b$.*

We do not have a proof of Conjecture 5.10 in general, but the special case $d \in \mathbb{N}$ (or $a = 1$) can be proved by applying Muirhead's inequalities.

Proposition 5.11. $\mathfrak{M}_\lambda \geq \mathfrak{M}_{\lambda^d}$ *for all integers $d \geq 1$.*

Proof. Observe that $(M_\lambda)^d$ is a convex combination of $\{M_\mu \,|\, \lambda^d \preceq \mu \preceq d\lambda\}$, and therefore by Muirhead's inequalities satisfies $M_{\lambda^d} \leq (M_\lambda)^d \leq M_{d\lambda}$. Thus we have

$$\mathfrak{M}_{\lambda^d} = \sqrt[d|\lambda|]{M_{\lambda^d}} \leq \sqrt[d|\lambda|]{(M_\lambda)^d} = \sqrt[|\lambda|]{M_\lambda} = \mathfrak{M}_\lambda.$$

$\square$

Also, the special "rectangular" case $\lambda = (k)$ of Conjecture 5.10 can be proved by plethysm.

Proposition 5.12. $\mathfrak{M}_{k^a} \geq \mathfrak{M}_{k^b}$ *if $a, b, k \in \mathbb{N}$, $a \leq b$.*

Proof. Define $\mathbf{y} = (x_1^k, \ldots, x_n^k)$. Then we have

$$\mathfrak{M}_{k^a}(\mathbf{x}) = \mathfrak{E}_a(\mathbf{y}) \geq \mathfrak{E}_b(\mathbf{y}) = \mathfrak{M}_{k^b}(\mathbf{x}).$$

$\square$

Another special case of Conjecture 5.9 can be derived from the following property of the double majorization order.

Observation 5.13. *If $\lambda \trianglelefteq \mu$ then $\lambda \cup \mu \trianglelefteq \mu$.*

Proof. If $\lambda \trianglelefteq \mu$ then we have $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$, or equivalently,

$$\frac{\lambda_1^\top + \cdots + \lambda_i^\top}{|\lambda|} \geq \frac{\mu_1^\top + \cdots + \mu_i^\top}{|\mu|}$$

for all i. Then we have

$$\frac{(\lambda \cup \mu)_1^\top + \cdots + (\lambda \cup \mu)_i^\top}{|\lambda \cup \mu|} = \frac{\lambda_1^\top + \cdots + \lambda_i^\top + \mu_1^\top + \cdots + \mu_i^\top}{|\lambda| + |\mu|} \geq \frac{\mu_1^\top + \cdots + \mu_i^\top}{|\mu|}$$

for all i. Thus,

$$\frac{(\lambda \cup \mu)^\top}{|\lambda \cup \mu|} \succeq \frac{\mu^\top}{|\mu|}.$$

By Observation 2.4 (2), this implies $\lambda \cup \mu \trianglelefteq \mu$. $\square$

The pairs $\{(\lambda \cup \mu, \mu) \,|\, \lambda \trianglelefteq \mu\}$ of partitions above provide examples of monomial means for which Conjecture 5.9 is true.

Proposition 5.14. *If* $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$, *then* $\mathfrak{M}_{\lambda \cup \mu} \leq \mathfrak{M}_\mu$.

Proof. We have

$$\mathfrak{M}_{\lambda \cup \mu} = (M_{\lambda \cup \mu})^{\frac{1}{(|\lambda| + |\mu|)}} \leq (M_\lambda M_\mu)^{\frac{1}{(|\lambda| + |\mu|)}} = (M_\lambda^{|\mu|} M_\mu^{|\mu|})^{\frac{1}{|\mu|(|\lambda| + |\mu|)}},$$

since $M_\lambda M_\mu$ is a convex combination of $\{M_\nu \,|\, \mu \cup \lambda \preceq \nu \preceq \mu + \lambda\}$. Since the condition $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ is equivalent to $M_\lambda^{|\mu|} \leq M_\mu^{|\lambda|}$, the last expression above is less than or equal to

$$(M_\mu^{|\lambda|} M_\mu^{|\mu|})^{\frac{1}{|\mu|(|\lambda| + |\mu|)}} = M_\mu^{\frac{1}{|\mu|}} = \mathfrak{M}_\mu.$$

$\square$

For example, if $\lambda = 1$ and $\mu = n$, we have $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ by Schlömilch's inequality. Hence it follows from Proposition 5.14 that $\mathfrak{M}_n \geq \mathfrak{M}_{n,1}$ for all $n \geq 1$.

6. The Muirhead Cone and Muirhead Semiring

In Sections 3 and 4 we defined the Newton and Gantmacher semirings, and showed that these contained the differences $E_\mu(\mathbf{x}) - E_\lambda(\mathbf{x})$ and $P_\mu(\mathbf{x}) - P_\lambda(\mathbf{x})$, respectively, when $|\lambda| = |\mu|$. Our conclusion was that the main results of those sections (Theorems 3.2 and 4.2) could be derived algebraically from the classical inequalities of Newton and Gantmacher. In this section we show that Muirhead's inequality is even stronger algebraically.

Define the *Muirhead cone* to be the set of polynomials in n variables that are non-negative linear combinations of *Muirhead differences*, which by definition are symmetric functions of the form

$$M_\mu(\mathbf{x}) - M_\lambda(\mathbf{x}), \quad \lambda, \mu \vdash d \geq 0 \quad \lambda \preceq \mu.$$

Define the *Muirhead semiring* to be the set of all nonnegative linear combinations of products of functions in the set

$$\bigcup_d \{M_\mu(\mathbf{x}) - M_\lambda(\mathbf{x}) \mid \lambda, \mu \vdash d, \lambda \preceq \mu\} \cup \bigcup_d \{M_\lambda(\mathbf{x}) \mid \lambda \vdash d\}.$$

The main result of this section is the following:

Theorem 6.1. *The Newton and Gantmacher differences lie in the Muirhead cone. More precisely, if $1 \le i \le k$, then*

$$(6.1) \qquad P_{k+1,i-1}(\mathbf{x}) - P_{k,i}(\mathbf{x}) = \frac{n-1}{n}(M_{k+1,i-1}(\mathbf{x}) - M_{k,i}(\mathbf{x})),$$

and

$$(6.2) \qquad E_{k,i}(\mathbf{x}) - E_{k+1,i-1}(\mathbf{x}) = \sum_{j=0}^{k-1} d_j \big(M_{2^i - j 1^{k-i+2j}}(\mathbf{x}) - M_{2^{i-j-1} 1^{k-i+2j+2}}(\mathbf{x}) \big),$$

where

$$(6.3) \qquad d_j = \frac{(k-i+1)(i-j)(n-k-j)}{i(n-k)(k-i+j+1)} \frac{\binom{i}{j}\binom{n-i}{k-i+j}}{\binom{n}{k}}.$$

Proof. Equation (6.1) is an elementary computation. To prove (6.2), note first that the left-hand side is equal to

$$\sum_{j=0}^{i} \left(\frac{\binom{k-i+2j}{j}}{\binom{n}{i}\binom{n}{k}} - \frac{\binom{k-i+2j}{j-1}}{\binom{n}{i-1}\binom{n}{k+1}} \right) m_{2^{i-j}1^{k-i+2j}}(\mathbf{x})$$

$$= \sum_{j=0}^{i} \left(\frac{\binom{k-i+2j}{j}}{\binom{n}{i}\binom{n}{k}} - \frac{\binom{k-i+2j}{j-1}}{\binom{n}{i-1}\binom{n}{k+1}} \right) \binom{n}{k-j, 2j, n-k-j} M_{2^{i-j}1^{k-i+2j}}(\mathbf{x}),$$

which after a bit of manipulation gives

$$(6.4) \qquad \sum_{j=0}^{i} \left(\frac{(k-i+1)(ni-ki-nj-j)}{i(n-k)(k-i+j+1)} \frac{\binom{i}{j}\binom{n-i}{k-i+j}}{\binom{n}{k}} \right) M_{2^{i-j}1^{k-i+2j}}(\mathbf{x}).$$

On the other hand, the right-hand side is equal to

$$(6.5) \qquad \sum_{j=0}^{k} (d_j - d_{j-1}) M_{2^{i-j}1^{k-i+2j}}(\mathbf{x}).$$

Further manipulation using the defintion of d_j in (6.3) eventually shows that (6.4) and (6.5) are equal. $\qquad \square$

Corollary 6.2. *The Newton semiring and Gantmacher semiring are contained in the Muirhead semiring.*

Thus our principal results concerning elementary and power sum means (Theorems 3.2 and 4.2) may be viewed as algebraic consequences of Muirhead's inequality. We do not know whether the differences $M_\mu^{|\lambda|}(\mathbf{x}) - M_\lambda^{|\mu|}(\mathbf{x})$ are contained in the Muirhead semiring, when $\lambda \trianglelefteq \mu$. Proving this result would establish Conjecture 5.1 in a strong form.

It is worth noting that there exist nonnegative symmetric functions that are not contained in the the Muirhead semiring. For example, a classical result known as *Schur's inequality* [5, p.64] states that the function

$$f(x_1, x_2, x_3) \;=\; x_1(x_1 - x_2)(x_1 - x_3) + x_2(x_2 - x_1)(x_2 - x_3) + x_3(x_3 - x_1)(x_3 - x_2)$$
$$=\; m_3(\mathbf{x}) - m_{2,1}(\mathbf{x}) + 3\, m_{1,1,1}(\mathbf{x})$$

is nonnegative for all $\mathbf{x} \geq 0$. It is not difficult to show that f does not lie in the degree 3 component of the Muirhead semiring, which is the nonnegative span of $m_3(\mathbf{x}), m_{2,1}(\mathbf{x}), m_{1,1,1}(\mathbf{x}), m_{2,1}(\mathbf{x}) - 6m_{1,1,1}(\mathbf{x})$, and $2m_3(\mathbf{x}) - m_{2,1}(\mathbf{x})$.

7. OPEN QUESTIONS

In this section we collect various partial results and conjectures reflecting our state of knowledge about the corresponding questions for homogeneous symmetric functions and Schur functions. We consider the homogeneous case first.

A formula for the term-normalized homogeneous symmetric function is given by

$$H_\lambda(\mathbf{x}) = \frac{h_\lambda(\mathbf{x})}{\left(\!\binom{n}{\lambda_1}\!\right) \cdots \left(\!\binom{n}{\lambda_\ell}\!\right)},$$

where $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$. We define $\mathfrak{H}_\lambda(\mathbf{x}) = \sqrt[|\lambda|]{H_\lambda(x)}$.

Like the elementary and power sum bases, the homogeneous basis is multiplicative. We therefore have the following equalities.

Observation 7.1. *For any partition λ and integer $d \geq 1$ we have $\mathfrak{H}_\lambda = \mathfrak{H}_{\lambda^d}$.*

Evidence suggests that a result analogous to Theorems 3.2 and 4.2 is true.

Conjecture 7.2. *Given integer partitions λ, μ,*

$$\mathfrak{H}_\lambda \leq \mathfrak{H}_\mu \text{ if and only if } \lambda \sqsubseteq \mu, \;\; i.e., \;\; \frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}.$$

We can prove this result in one direction:

Theorem 7.3. *Given integer partitions λ and μ, we have*

$$\mathfrak{H}_\lambda \leq \mathfrak{H}_\mu \qquad if \qquad \lambda \sqsubseteq \mu.$$

Proof. In the case that $|\lambda|$ and $|\mu|$ are equal, suppose that $\lambda \preceq \mu$. Then we proceed as in the proof of Theorem 3.2 with Schur's inequalities replacing those of Newton and conclude that $H_\lambda \leq H_\mu$.

In the case that $|\lambda|$ and $|\mu|$ are not equal, we again proceed as in the proof of Theorem 3.2. Specifically, we apply Observation 7.1 to see that we have

$$\mathfrak{H}_\lambda(\mathbf{x}) = \mathfrak{H}_{\lambda^{|\mu|}}(\mathbf{x}), \qquad \mathfrak{H}_\mu(\mathbf{x}) = \mathfrak{H}_{\mu^{|\lambda|}}(\mathbf{x})$$

for all n. Thus we have $\mathfrak{H}_\lambda \leq \mathfrak{H}_\mu$ if $\lambda^{|\mu|} \preceq \mu^{|\lambda|}$, or equivalently, if $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$. $\qquad\square$

We have not established the converse of Theorem 7.3 even when $|\lambda| = |\mu|$, which would mean proving that $H_\lambda \leq H_\mu$ implies $\lambda \preceq \mu$. We have verified this by explicit computation up through $|\lambda| = |\mu| = 7$, but several degree 8 cases remain unresolved. We invite the reader to help complete this argument by showing, for example, that $H_\lambda \not\leq H_\mu$ when $\lambda = \{5, 2, 1\}, \mu = \{4, 4\}$.

Next we turn to the case of Schur functions. A formula for the term-normalized Schur function is given by

$$S_\lambda(\mathbf{x}) = \frac{s_\lambda(\mathbf{x})}{d_\lambda},$$

where d_λ is equal to the number of semistandard Young tableaux of shape λ and having entries $1, \ldots, n$. It would be natural to define $\mathfrak{S}_\lambda(\mathbf{x}) = \sqrt[|\lambda|]{S_\lambda(x)}$, and establish inequalities for these "Schur means" analogous to those obtained for the families $\{\mathfrak{E}_\lambda\}$, $\{\mathfrak{P}_\lambda\}$, $\{\mathfrak{M}_\lambda\}$, and $\{\mathfrak{H}_\lambda\}$. However, we have a conjecture only for the equal-degree case, i.e., when $|\lambda| = |\mu|$. The question of characterizing inequalities among the $\mathfrak{S}_\lambda$ remains open. Computational evidence supports the following:

Conjecture 7.4. *Given integer partitions λ and μ with $|\lambda| = |\mu|$,*

$$S_\lambda \leq S_\mu \text{ if and only if } \lambda \preceq \mu.$$

We can prove that the condition is necessary:

Theorem 7.5. *Given integer partitions λ and μ with $|\lambda| = |\mu|$, we have*

$$S_\lambda \leq S_\mu \text{ only if } \lambda \preceq \mu.$$

Proof. Suppose $\lambda \not\preceq \mu$. Then there exists an index j such that

$$\lambda_1 + \cdots + \lambda_j > \mu_1 + \cdots + \mu_j.$$

Specializing the symmetric functions $S_\lambda(\mathbf{x})$, $S_\mu(\mathbf{x})$ at

$$x_1 = \cdots = x_j = t,$$
$$x_{j+1} = \cdots = x_n = 1,$$

we obtain polynomials in $\mathbb{N}[t]$ of degrees $\lambda_1 + \cdots + \lambda_j$ and $\mu_1 + \cdots + \mu_j$, respectively. It follows that

$$\lim_{t \to \infty} [S_\lambda(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{n-j}) - S_\mu(\underbrace{t, \ldots, t}_{j}, \underbrace{1, \ldots, 1}_{n-j})] = \infty,$$

which implies $S_\lambda \not\le S_\mu$.   □

When $|\lambda| \ne |\mu|$, it would be natural to conjecture that

$$\mathfrak{S}_\lambda \le \mathfrak{S}_\mu \text{ if and only if } \lambda \sqsubseteq \mu,$$

or perhaps

$$\mathfrak{S}_\lambda \le \mathfrak{S}_\mu \text{ if and only if } \lambda \trianglelefteq \mu.$$

However, both of these statements are false. For example, if $\lambda = \{3, 2\}$ and $\mu = \{2, 1\}$, then $\lambda \sqsubseteq \mu$ but $\mathfrak{S}_\lambda$ and $\mathfrak{S}_\mu$ are incomparable. Also, if $\lambda = \{3, 1\}$ and $\mu = \{2\}$, then λ and μ are incomparable in the double-majorization order, but $\mathfrak{S}_\lambda \le \mathfrak{S}_\mu$.

It would be interesting to express the appropriate homogeneous and Schur differences $H_\lambda(\mathbf{x}) - H_\mu(\mathbf{x})$ and $S_\lambda(\mathbf{x}) - S_\mu(\mathbf{x})$ as nonnegative linear combinations of Muirhead differences $M_\lambda(\mathbf{x}) - M_\mu(\mathbf{x})$. The authors have obtained partial results suggesting that this is possible in many cases.

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