RESEARCH PLAN

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Many current problems in algebraic combinatorics concern inequalities satisfied by matrix minors or by components of integer sequences. My work for the past five years has been to explain such inequalities arising in the areas of total nonnegativity, symmetric functions, and polynomials with real zeros. To continue this work I hope to collaborate with faculty in algebra and discrete mathematics and to supervise undergraduate research.

1. Total nonnegativity

Let $x = (x_{ij})$ be a generic square matrix and define $\Delta_{I,I'}(x)$ to be the (I, I') minor of x, i.e., the determinant of the submatrix of x corresponding to rows I and columns I'. A real matrix is called *totally nonnegative* (TNN) if each of its minors is nonnegative. A polynomial $p(x_{11}, \dots, x_{nn})$ in n^2 variables is called totally nonnegative if it satisfies $p(A) \geq 0$ for every $n \times n$ TNN matrix A.

Some recent interest in total nonnegativity concerns a set of polynomials known in quantum Lie theory as the *dual canonical basis* of type A_{n-1} [2], and the search for a concise description of this set. Berenstein, Gelfand and Zelevinsky [3, 11] have developed machinery to enumerate the basis elements for small n, Lusztig [15] has proved that they are TNN polynomials, and some evidence suggests that they are expressable as subtraction-free Laurent polynomials in matrix minors [9]. Further progress on the problem of describing the dual canonical basis seems to be hindered by the difficulty of constructing families of TNN polynomials, even those not having the subtraction-free Laurent property.

A few authors have constructed such families by considering polynomials known as *immanants.* Natural generalizations of the determinant, these are defined in terms of functions $f : S_n \to \mathbb{R}$ by

$$\mathrm{Imm}_f(x) = \sum_{\sigma \in S_n} f(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

Providing one family, Fallat, Gekhtman and Johnson [7, Thm. 4.6] characterized all TNN immanants of the form $\Delta_{J,J}(x)\Delta_{\overline{J},\overline{J}}(x) - \Delta_{I,I}(x)\Delta_{\overline{I},\overline{I}}(x)$. I then generalized this result [18], characterizing all TNN immanants of the form

$$\sum_{(I,I')} c_{I,I'} \Delta_{I,I'}(x) \Delta_{\overline{I},\overline{I'}}(x). \tag{1.1}$$

These immanants are related to functions $f_\tau : S_n \to \mathbb{R}$ which B. Rhoades and I defined using elements τ of the Temperley-Lieb algebra. Specifically, we have the following.

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Theorem 1.1. *An immanant of the form (1.1) is TNN if and only if it is equal to a nonnegative linear combination of Temperley-Lieb immanants*

$$\mathrm{Imm}_{f_\tau}(x) = \sum_{\sigma \in S_n} f_\tau(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}. \tag{1.2}$$

Providing another family of TNN polynomials, Stembridge [26] proved the total nonnegativity of immanants defined in terms of irreducible characters χ^λ of S_n. Continuing in this direction, I generalized a result of Lindström [14] by obtaining a simple formula for $\mathrm{Imm}_{\chi^\lambda}(A)$ when λ is a hook partition [20]. I also proved that when λ has at most two columns, $\mathrm{Imm}_{\chi^\lambda}(x)$ is a nonnegative linear combination of Temperley-Lieb immanants.

The above results suggest that Temperley-Lieb immanants play an important role in the characterization of TNN immanants. It is therefore interesting that these immanants have never failed to have the subtraction-free Laurent property in my preliminary experimentation. I am investigating this matter further with undergraduates at Dartmouth.

2. Schur nonnegativity

Let H be the infinite upper triangular matrix whose (i, j) entry (for $i \leq j$) is the complete homogeneous symmetric function h_{j-i}. (See [24, Ch. 7] for definitions.) Square submatrices of H, called *Jacobi-Trudi* matrices, have properties analogous to those of TNN matrices [12]. The determinant of each Jacobi-Trudi matrix is a symmetric function called a (skew) *Schur function*, and is denoted by $s_{\lambda/\beta}$ for certain integer sequences λ, β. Each Schur function is nonnegative in the sense that it is equal to a nonnegative linear combination of monomial symmetric functions.

Some recent interest in Schur functions concerns their connection to the cohomology ring of the Grassmannian variety. In particular, the desire to describe phenomena in this ring has led Fomin, Fulton, Li and Poon [10, Conj. 2.8, Conj. 5.1], and Bergeron, Biagioli and Rosas [4, Conj. 1.8] to conjecture that certain symmetric functions of the form

$$s_{\mu/\gamma} s_{\nu/\delta} - s_{\kappa/\alpha} s_{\lambda/\beta} \tag{2.1}$$

are equal to nonnegative linear combinations of Schur functions.

Such conjectures suggest the study of *Schur nonnegative* (SNN) polynomials, those polynomials $p(x_{11}, \dots, x_{nn})$ in n^2 variables for which $p(A)$ is equal to a nonnegative linear combination of Schur functions whenever A is an $n \times n$ Jacobi-Trudi matrix. *Monomial nonnegative* (MNN) polynomials are defined analogously. While nontrivial classes of polynomials which are known to be SNN or MNN are scarce, examples are provided somewhat mysteriously by TNN immanants [13, 25, 27]. For instance, all TNN immanants of the form (1.1) are SNN. This is a corollary of the following result [17] which I proved using the Kazhdan-Lusztig basis for $\mathbb{C}[S_n]$.

Theorem 2.1. *Each Temperley-Lieb immanant (1.2) is SNN.*

I expect that the Schur nonnegativity of the polynomials (1.1) will imply special cases of the conjectures concerning the symmetric functions (2.1).

While Schur nonnegativity implies monomial nonnegativity, the apparently strong connection between these properties and total nonnegativity is not well understood. Providing more evidence of this connection, B. Drake, S. Gerrish and I have obtained the following characterizations of the Bruhat order [5, 6].

Theorem 2.2. *Let π and σ be permutations in S_n. The following are equaivalent.*

(1) $\pi \leq \sigma$ in the Bruhat order.
(2) The polynomial $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is TNN.
(3) The polynomial $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is SNN.
(4) The polynomial $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is MNN.

3. Simplicial complexes and polynomials having real zeros

Given a $(d-1)$-dimensional simplicial complex Σ, we define the *f-polynomial* of Σ to be $1 + f_0 z + f_1 z^2 + \cdots + f_{d-1} z^d$, where f_i is the number of i-dimensional faces of Σ. A similar definition applies to posets. (See [22, p. 99], [23, Ch. 3] for definitions.) Several interesting open problems concern f-polynomials having only real zeros. One of these, related to a conjecture of Hopf, has recently been settled by Stembridge [28]. (See also [16], [22, p. 100].)

To aid in settling more problems, it would be desirable to realize many polynomials

$$a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z] \tag{3.1}$$

having only real zeros as f-polynomials of simplicial complexes or posets. Two sources of f-polynomials with real zeros are the posets called *unit interval orders*, and the more general $(\mathbf{3}+\mathbf{1})$*-free posets*. (See [19].) While the second of these classes lacks some nice combinatorial properties of the first, I proved that the corresponding sets of f-polynomials are equal [21, Cor. 4.5]. J. Bell and I have also found a class of combinatorial objects whose f-polynomials include *all* polynomials of the form (3.1) having only real zeros [1, Thm. 3.6].

Theorem 3.1. *Each polynomial (3.1) having only real zeros is the f-polynomial of a multicomplex.*

(See [22, p. 55] for definitions.) As a corollary we have that for each such polynomial $a(z)$ and for any nonnegative integer c, there exists a Cohen-Macaulay ring having Hilbert series $a(z)/(1-z)^c$. Furthermore, Bell, S. Gerrish, J. McClain and I have gathered evidence in favor of the following.

Conjecture 3.2. *Each polynomial (3.1) having only real zeros is the f-polynomial of a simplicial complex.*

Other methods of proving the reality of the zeros of a polynomial (3.1) employ TNN immanants, Schur functions, and permutation statistics. A conjecture of mine concerning this third class of tools has recently been proved by Foata and Han [8, Thm. 2.4].

References

[1] J. Bell and M. Skandera. Multicomplexes and polynomials with real zeros, 2003. In progress.

[2] A. Berenstein and A. Zelevinsky. String bases for quantum groups of type A_r. *Adv. Sov. Math.*, **16** (1993) pp. 51–89.

[3] A. Berenstein and A. Zelevinsky. Canonical bases for the quantum groups of type A_r and piecewise linear combinatorics. *Duke Math. J.*, **82** (1996) pp. 473–502.

[4] F. Bergeron, R. Biagioli, and M. Rosas. On a conjecture concerning Littlewood-Richardson coefficients, 2003. In progress.

[5] B. Drake, S. Gerrish, and M. Skandera. Nonnegativity properties and the Bruhat order, 2004. In progress.

[6] B. Drake, S. Gerrish, and M. Skandera. Two new criteria for comparison in the Bruhat order. *Electron. J. Combin.*, **11** (2004). Note 6, 4 pp. (electronic).

[7] S. M. Fallat, M. I. Gekhtman, and C. R. Johnson. Multiplicative principal-minor inequalities for totally nonnegative matrices. *Adv. Appl. Math.*, **30** (2003) pp. 442–470.

[8] D. Foata and G. Han. Une nouvelle transformation pour les statistiques Euler-Mahoniennes ensemblistes. *Mosc. Math. J.*, **4** (2004) pp. 131–152.

[9] S. Fomin, 2002. Personal communication.

[10] S. Fomin, W. Fulton, C. Li, and Y. Poon. Eigenvalues, singular values, and Littlewood-Richardson coefficients, 2003. Preprint math.AG/03013078 on ArXiv.

[11] I. Gelfand and A. Zelevinsky. Canonical basis in irreducible representations of gl_3 and its applications. In *Group Theoretical Methods in Physics, Vol. II (Jurmala, 1985)*. VNU Sci. Press, Utrecht, 1986 pp. 127–146.

[12] I. Gessel and G. Viennot. Determinants and plane partitions, 1989. Preprint.

[13] M. Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6** (1993) pp. 569–595.

[14] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, **5** (1973) pp. 85–90.

[15] G. Lusztig. Total positivity in reductive groups. In *Lie Theory and Geometry: in Honor of Bertram Kostant*, vol. 123 of *Progress in Mathematics*. Birkhäuser, Boston, 1994 pp. 531–568.

[16] V. Reiner and V. Welker. On the Charney-Davis and Neggers-Stanley conjectures, 2003. Preprint available at http://www.math.umn.edu/∼reiner/Papers/papers.html.

[17] B. Rhoades and M. Skandera. Kazhdan-Lusztig immanants, 2004. In progress.

[18] B. Rhoades and M. Skandera. Temperley-Lieb immanants, 2004. Submitted.

[19] M. Skandera. A characterization of $(\mathbf{3}+\mathbf{1})$-free posets. *J. Combin. Theory Ser. A*, **93** (2001) pp. 231–241.

[20] M. Skandera. Path tableaux and immanants related to representations of S_n, 2004. In progress.

[21] M. Skandera and B. Reed. Total nonnegativity and $(\mathbf{3}+\mathbf{1})$-free posets. *J. Combin. Theory Ser. A*, **103** (2003) pp. 237–256.

[22] R. Stanley. *Combinatorics and Commutative Algebra*. Birkhäuser, Boston, MA, 1996.

[23] R. Stanley. *Enumerative Combinatorics*, vol. 1. Cambridge University Press, Cambridge, 1997.

[24] R. Stanley. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge, 1999.

[25] R. Stanley and J. R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted positions. *J. Combin. Theory Ser. A*, **62** (1993) pp. 261–279.

[26] J. Stembridge. Immanants of totally positive matrices are nonnegative. *Bull. London Math. Soc.*, **23** (1991) pp. 422–428.

[27] J. Stembridge. Some conjectures for immanants. *Can. J. Math.*, **44** (1992) pp. 1079–1099.

[28] J. Stembridge, 2004. Personal communication.