BITABLEAUX AND ZERO SETS OF DUAL CANONICAL BASIS ELEMENTS

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Abstract. We state new results concerning the zero sets of polynomials belonging to the dual canonical basis of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$. As an application, we show that this basis is related by a unitriangular transition matrix to the simpler bitableau basis popularized by Désarménien-Kung-Rota. It follows that spaces spanned by certain subsets of the dual canonical basis can be characterized in terms of products of matrix minors, or in terms of their common zero sets.

1. Introduction

Let $x = (x_{i,j})$ be an $n \times n$ matrix of variables and consider the polynomial ring $\mathbb{C}[x] = \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ as an infinite dimensional complex vector space. It is often convenient to construct a basis of $\mathbb{C}[x]$ by modifying and combining bases $(\mathcal{B}_r)_{r \geq 0}$ of spaces

$$\mathrm{span}_{\mathbb{C}}\{y_{1,w_1} \cdots y_{r,w_r} \,|\, w \in S_r\} \subset \mathbb{C}[y_{1,1}, \ldots, y_{r,r}], \tag{1.1}$$

where $y = (y_{i,j})$ is an $r \times r$ matrix of variables, for r arbitrarily large. Following [56], we will call these spaces *immanant spaces*. For example, it is possible to show that the natural basis

$$\{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \,|\, a_{1,1}, \ldots, a_{n,n} \in \mathbb{N}\}$$

of $\mathbb{C}[x]$ may be constructed as above. A second basis which may be constructed as above was made popular by Désarménien-Kung-Rota [12] and has a rather simple description in terms of Young tableaux. A third basis which is of great interest in the representation theory of quantum groups was shown in [53] also to have a construction as above. Known as the *dual canonical basis*, it arose naturally from Kashiwara's [28] and Lusztig's [35] work on canonical bases, and currently has no elementary description.

Du [17] expressed the dual canonical basis elements in terms of Kazhdan-Lusztig polynomials, which led to the term *Kazhdan-Lusztig immanants* used in [50] for those elements belonging to the immanant space of $\mathbb{C}[x]$. Since then, nonnegativity properties of Kazhdan-Lusztig immanants studied in [50] have played an important role

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in establishing inequalities satisfied by Littlewood-Richardson coefficients [31] and in creating a representation theoretic model for the combinatorial action of jeu-de-taquin promotion [48], thus proving conjectures of Fomin-Fulton-Li-Poon [20], Reiner-Stanton-White [47] and others.

While the problem of providing an explicit and elementary description of dual canonical basis elements is widely believed to be difficult, one can obtain interesting partial results by describing spaces spanned by nested subsets of these polynomials, by studying the corresponding algebraic varieties, and by relating the dual canonical basis to any simpler basis. Indeed, we will continue such work begun in [49], [50], [51] by considering a family of nested subsets defined in terms of a partial order on S_n which we call *iterated dominance*. This partial order helps to describe spaces spanned by dual canonical basis elements, their zero sets, and a unitriangular transition matrix relating the bitableau and dual canonical bases.

In Sections 2-3, we discuss a multigrading of $\mathbb{C}[x]$ and the natural, bitableau, and dual canonical bases. We also include standard facts about the symmetric group S_n, several partial orders and Kazhdan-Lusztig preorders. In Section 4, we give new sufficient conditions for a matrix to belong to the zero sets of certain dual canonical basis elements. We state these conditions in terms of repetition patterns among matrix rows and columns. In Section 5, we define the iterated dominance order on S_n. This partial order then allows us to combine results in Section 4 with those of Greene and others to prove the unitriangularity of transition matrices relating the bitableau and dual canonical bases. We finish in Section 6 by showing that natural filtrations of the immanant space (1.1) have similar descriptions in terms of the two bases.

2. A multigrading of $\mathbb{C}[x]$ and two bases

$\mathbb{C}[x]$ has a natural grading by degree,

$$\mathbb{C}[x] = \bigoplus_{r \geq 0} \mathcal{A}_r,$$

where $\mathcal{A}_r = \mathcal{A}_r(x)$ is the span of all monomials of total degree r. It is easy to see that the natural basis $\{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \,|\, a_{1,1}, \ldots, a_{n,n} \in \mathbb{N}\}$ of $\mathbb{C}[x]$ is a disjoint union

$$\bigcup_{r \geq 0} \{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \,|\, a_{1,1} + \cdots + a_{n,n} = r\}$$

of bases of the homogeneous components $\{\mathcal{A}_r \,|\, r \geq 0\}$. One may further decompose each homogeneous component $\mathcal{A}_r$ by considering pairs (L, M) of r-element multisets of $[n] \underset{\text{def}}{=} \{1, \ldots, n\}$, written as weakly increasing sequences

$$L = (\ell(1), \ldots, \ell(r)), \qquad M = (m(1), \ldots, m(r)).$$

In particular, define the multigrading

$$\mathcal{A}_r = \bigoplus_{\substack{L,M \\ |L|=|M|=r}} \mathcal{A}_{L,M}, \tag{2.1}$$

where $\mathcal{A}_{L,M}$ is the linear span of monomials whose row indices and column indices (with multiplicity) are given by the multisets L and M, respectively. The component $\mathcal{A}_{[n],[n]}$ is the immanant space (1.1) corresponding to $r = n$, and we will call any element of $\mathcal{A}_{[n],[n]}$ an $n \times n$ *immanant*.

If the numbers $1, \ldots, n$ appear in the multiset L with multiplicities $\alpha = (\alpha_1, \ldots, \alpha_n)$, we write $L = 1^{\alpha_1} \cdots n^{\alpha_n}$. Just as the $\mathbb{Z}$-graded components $\mathcal{A}_r$ and $\mathcal{A}_s$ satisfy $\mathcal{A}_r\mathcal{A}_s \subset \mathcal{A}_{r+s}$, the multigraded components $\mathcal{A}_{L,M}$ and $\mathcal{A}_{L',M'}$ satisfy

$$\mathcal{A}_{L,M}\mathcal{A}_{L',M'} \subset \mathcal{A}_{L \Cup L', M \Cup M'},$$

where $\Cup$ denotes the *multiset union* of two multisets,

$$1^{\alpha_1} \cdots n^{\alpha_n} \Cup 1^{\alpha'_1} \cdots n^{\alpha'_n} \underset{\mathrm{def}}{=} 1^{\alpha_1+\alpha'_1} \cdots n^{\alpha_n+\alpha'_n}. \tag{2.2}$$

The convenience of the multigrading (2.1) manifests itself in the situation that one can do the following.

(1) Construct for all $r \geq 0$ a basis $\mathcal{B}_r$ of the $r \times r$ immanant space (1.1).
(2) For each pair (L, M) of r-element multisets of $[n]$, evaluate the immanants in $\mathcal{B}_r$ at the generalized submatrix

$$x_{L,M} \underset{\mathrm{def}}{=} \begin{bmatrix} x_{\ell(1),m(1)} & \cdots & x_{\ell(1),m(r)} \\ x_{\ell(2),m(1)} & \cdots & x_{\ell(2),m(r)} \\ \vdots & & \vdots \\ x_{\ell(r),m(1)} & \cdots & x_{\ell(r),m(r)} \end{bmatrix}, \tag{2.3}$$

i.e., substitute $y = x_{L,M}$.
(3) Select from the resulting polynomials a basis of the component $\mathcal{A}_{L,M}$ of $\mathbb{C}[x]$.

The constructed basis of $\mathbb{C}[x]$ is then a union of bases of $\mathcal{A}_{L,M}$ over all $r \geq 0$ and all pairs (L, M) of r-element multisets of $[n]$. Thus this basis of $\mathbb{C}[x]$ requires knowledge only of bases of immanant spaces (1.1).

Three such bases of $\mathbb{C}[x]$ may be described in terms of integer partitions, the symmetric group, and Young tableaux. We define these and related partial orders as follows. (For more information see, e.g., [21], [52], [54], [55].)

Call a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers which sum to r an *integer partition of* r and write $\lambda \vdash r$ or $|\lambda| = r$. The components of λ are called *parts*. A left-justified array of boxes with λ_i boxes in row i $(1 \leq i \leq \ell)$ is called a *Young diagram of shape* λ. Transposing this diagram as one would transpose a matrix, we obtain a diagram whose shape is another integer partition of r which we denote by $\lambda^\top$. (This is often called the *conjugate* of λ.)

We define the *dominance order* on partitions of r by declaring $\lambda \preceq \mu$ if we have

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i,$$

for $i = 1, \ldots, r$ (with λ_i and μ_j defined to be zero for i, j larger than the number of parts of these partitions). It is well known that we have $\lambda \preceq \mu$ if and only if $\lambda^\top \succeq \mu^\top$. Filling a Young diagram of shape $\lambda \vdash r$ with positive integers, we obtain a *Young tableau* T of shape λ. T is called *injective* if no number appears more than once in T, *column-(semi)strict* if entries (weakly) increase downward in columns, *row-(semi)strict* if entries (weakly) increase to the right in rows, *semistandard* if it is column-strict and row-semistrict, and *standard* if it is injective, semistandard, and has entries $1, \ldots, r$.

To define bases of the immanant space, we will find it convenient to associate an n-letter word, two tableaux, and a partition of n to each element of the symmetric group S_n as follows.

Let $s_1, \ldots, s_{n-1}$ be the standard generators of S_n, satisfying the relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j, && \text{if } |i-j| = 1, \\ s_i s_j &= s_j s_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

Let S_n act on rearrangements of the letters $[n]$ by

$$s_i \circ v_1 \cdots v_n \underset{\text{def}}{=} v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n.$$

For each permutation $w = s_{i_1} \cdots s_{i_\ell} \in S_n$ we define the *one-line notation* of w to be the word

$$w_1 \cdots w_n \underset{\text{def}}{=} s_{i_1} \circ (\cdots (s_{i_\ell} \circ (1 \cdots n)) \cdots). \tag{2.4}$$

For example, we define the one-line notation of $s_1 s_2$ in S_3 to be 312.

We associate two tableaux $P(w)$, $Q(w)$ to w by applying the Robinson-Schensted *column* insertion map to $w_1 \cdots w_n$,

$$w_1 \cdots w_n \mapsto (P(w), Q(w)). \tag{2.5}$$

These two tableaux necessarily have the same shape, which we declare also to be the *shape of* w,

$$\mathrm{sh}(w) \underset{\text{def}}{=} \mathrm{sh}(P(w)) = \mathrm{sh}(Q(w)).$$

It is well known that w has a decreasing subsequence of length k if and only if the first part of $\mathrm{sh}(w)$ is at least k. Similarly, w has an increasing subsequence of length k if and only if $\mathrm{sh}(w)$ has at least k parts.

In this paper, the notation $P(w)$ and $Q(w)$ will always refer to the tableaux in Equation (2.5) defined in terms of the one-line notation given in Equation (2.4). In the literature, the one-line notation of $w = s_{i_1} \cdots s_{i_\ell}$ is often defined to be the word we associate in (2.4) to $w^{-1} = s_{i_\ell} \cdots s_{i_1}$. (See, e.g., [21, p. 83].) Furthermore, the notation $(P(w), Q(w))$ is more often associated with the Robinson-Schensted *row insertion* of w. (See, e.g., [21, Sec. 4].) Since changing from column to row insertion transposes a tableau, and since we have $P(w^{-1}) = Q(w)$ by either insertion method, our notation $(P(w), Q(w))$ corresponds to a pair of tableaux which might more traditionally be denoted $(Q(w)^\top, P(w)^\top)$.

Given a permutation $w \in S_r$ expressed in terms of generators as $w = s_{i_1} \cdots s_{i_\ell}$, call w *reduced* if it cannot be expressed as a shorter product of generators. Call $\ell = \ell(w)$ the *length* of w. We define the Bruhat order on S_r by $v \leq w$ if some reduced expression for w contains a reduced expression for v as a subexpression. It is easy to see that we have $v \leq w$ if and only if $v^{-1} \leq w^{-1}$. (See [6] for more information.)

A subgroup of S_r generated by some subset J of $\{s_1, \ldots, s_{r-1}\}$ is called *parabolic* and is denoted W_J. In particular, given a multiset $M = (m(1), \ldots, m(r))$, we may use the generators $\iota(M) = \{s_i \,|\, m(i) = m(i+1)\}$ to define a parabolic subgroup $W_{\iota(M)}$ of S_r. If $M = 1^{\alpha_1} \cdots r^{\alpha_r}$, then we have $W_{\iota(M)} \cong S_{\alpha_1} \times \cdots \times S_{\alpha_r}$. For each partition $\lambda \vdash r$ and the corresponding multiset $M = 1^{\lambda_1} \cdots r^{\lambda_r}$, we write $S_\lambda \underset{\text{def}}{=} W_{\iota(M)}$.

We may use the symmetric group to express the natural basis of $\mathbb{C}[x]$ in terms of immanants as follows. Given an $r \times r$ matrix $y = (y_{i,j})$, define the $r \times r$ immanant $y^w = y_{1,w_1} \cdots y_{r,w_r}$. Then the set

$$\{(x_{L,M})^w \,|\, w \in S_r\} \tag{2.6}$$

of at most $r!$ distinct monomials forms a basis of $\mathcal{A}_{L,M}$. The union of these sets over all pairs (L, M) of r-element multisets of $[n]$ forms a basis of $\mathcal{A}_r$. Using the fact that each double coset $W_{\iota(L)} w W_{\iota(M)} \subset S_r$ has a unique Bruhat-maximal element, and that for each element $v \in W_{\iota(L)} w W_{\iota(M)}$, we have $(x_{L,M})^v = (x_{L,M})^w$, one may express the basis (2.6) more precisely as $\{(x_{L,M})^w \,|\, w \text{ Bruhat-maximal in } W_{\iota(L)} w W_{\iota(M)} \subset S_r\}$.

A second basis of $\mathbb{C}[x]$, consisting of polynomials parametrized by pairs of Young tableaux, is called the *bitableau basis*. Appearing in the work of Mead [41] and others (see [9, pp. 488-489]), it was later popularized by Désarménien-Kung-Rota [12], who substantially improved our understanding of it and used it to advance combinatorial methods in invariant theory. Outside of invariant theory, the bitableau basis appears often in papers treating problems in representation theory and quantum groups. (See, e.g., [1], [32].)

Let T, U be column-strict tableaux of the same shape and having k columns, and recall the submatrix notation (2.3). We define the *bitableau* $(T{:}U)(x)$ to be the product of k minors

$$\det(x_{I_1,J_1}) \cdots \det(x_{I_k,J_k}),$$

where $I_1, \dots, I_k$ are the sets of entries in columns $1, \dots, k$ of T and $J_1, \dots, J_k$ are the sets of entries in columns $1, \dots, k$ of U. For example, we have

$$\begin{aligned}\left(\begin{smallmatrix} 1 & 1 & 2 \\ 3 & 3 & \end{smallmatrix} : \begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & \end{smallmatrix}\right)(x) &= \det(x_{13,12}) \det(x_{13,23}) \det(x_{2,3}) \\ &= (x_{1,1}x_{3,2} - x_{1,2}x_{3,1})(x_{1,2}x_{3,3} - x_{1,3}x_{3,2})x_{2,3}.\end{aligned}$$

We call a bitableau $(T{:}U)(x)$ *column-strict*, *semistandard*, etc., if both T and U have these properties.

Define the *content* of a bitableau $(T{:}U)(x)$ to be the pair of multisets of entries of T and U. Mead [41] showed that for all pairs (L, M) of multisets of $[n]$, one obtains a basis of $\mathcal{A}_{L,M}$ by taking the set of all semistandard bitableaux $(T{:}U)(x)$ having content (L, M). (See also [12, Thm. 2.2], [15, Thm. 3] and references cited in [9, p. 486].)

Theorem 2.1. *The set of semistandard bitableaux*

$$\{(T{:}U)(x) \,|\, \mathrm{content}(T, U) = (L, M)\}$$

forms a $\mathbb{Z}$-basis of $\mathcal{A}_{L,M}$.

In particular, the set of all standard bitableaux forms a $\mathbb{Z}$-basis of the immanant space $\mathcal{A}_{[n],[n]}$. Using the conventions (2.4), (2.5), we will index standard bitableaux by permutations as

$$R_w(x) \underset{\mathrm{def}}{=} (Q(w){:}P(w))(x).$$

Denoting the transpose of the matrix x by $x^\top$, we have the following elementary identity.

Proposition 2.2. *For all $v \in S_n$ we have $R_v(x) = R_{v^{-1}}(x^\top)$.*

Proof. It is clear that $(T{:}U)(x) = (U{:}T)(x^\top)$ for any tableaux T, U. Since the Robinson-Schensted column insertion map satisfies $(P(v^{-1}), Q(v^{-1})) = (Q(v), P(v))$, we have

$$R_v(x) = (Q(v){:}P(v))(x) = (P(v){:}Q(v))(x^\top) = (Q(v^{-1}){:}P(v^{-1}))(x^\top) = R_{v^{-1}}(x^\top).$$

$\square$

It is easy to see that each semistandard bitableau can be expressed as the evaluation of a standard bitableau at a generalized submatrix of x. Conversely, the evaluation of a standard bitableau at a generalized submatrix of x is either zero or is equal to a semistandard bitableau. Thus for $|L| = |M| = r$, a basis of $\mathcal{A}_{L,M}$ is given by the nonzero elements of the set

$$\{R_w(x_{L,M}) \,|\, w \in S_r\}. \tag{2.7}$$

We will show in Corollary 5.11 that this basis may be expressed more precisely as

$$\{R_w(x_{L,M}) \,|\, w \text{ Bruhat maximal in } W_{\iota(L)} w W_{\iota(M)}\}.$$

(The relationship between the standard and semistandard bitableaux bases above can be described in terms of the Robinson-Schensted-Knuth correspondence and a variation of the standardization procedure in [55, Sec. 7.11].)

3. The Hecke algebra and dual canonical basis of $\mathbb{C}[x]$

The *Hecke algebra (of type A)*, denoted $H_n(q)$, is the noncommutative $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra generated by the set $\{T_{s_i} \,|\, 1 \leq i \leq n-1\}$, subject to the relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + q, && \text{for } i = 1, \dots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, && \text{if } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned}$$

If $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for $w \in S_n$ we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$. (This element does not depend upon the chosen reduced expression for w. See, e.g., [27].) We also define $T_e = 1$. We call the elements $\{T_w \,|\, w \in S_n\}$ the *natural basis* of $H_n(q)$ as a $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module. Specializing $H_n(q)$ at $q^{\frac{1}{2}} = 1$, we obtain the classical group algebra $\mathbb{C}[S_n]$ of the symmetric group, with each natural basis element T_w specializing to the permutation w.

In [29], Kazhdan and Lusztig defined another basis $\{C'_w(q) \,|\, w \in S_n\}$,

$$C'_w(q) = q^{-\ell(w)/2} \sum_{v \leq w} P_{v,w}(q) T_v,$$

where $\{P_{v,w}(q) \,|\, v, w \in S_n\}$ are the unique polynomials in $\mathbb{N}[q]$ satisfying a certain recursive formula. (See, e.g., [29].) The basis and polynomials are known as the *Kazhdan-Lusztig basis* and *Kazhdan-Lusztig polynomials*, respectively. Neither has an entirely elementary description. Even the integers $C'_w(1)$ and $P_{v,w}(1)$ have no simple descriptions. (See [5, Ch. 6] for a summary of interpretations of the polynomials, and [4] for recent progress.) The Kazhdan-Lusztig basis has many interesting properties, including that multiplication of the basis elements is described by structure constants belonging to $\mathbb{N}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. (See [26, Appendix] and references there.)

Kazhdan and Lusztig used the basis $\{C'_w(q) \,|\, w \in S_n\}$ and a preorder $\leq_L$ on S_n to construct irreducible representations of $H_n(q)$. We define the *(Kazhdan-Lusztig) left preorder* $\leq_L$ on S_n to be the transitive closure of the relation $\lessdot_L$ defined by $v \lessdot_L u$ if $C'_v(q)$ appears with nonzero coefficient in the Kazhdan-Lusztig expansion of $T_w C'_u(q)$ for some $w \in S_n$. Following the treatment given in [26, Appendix], one constructs an irreducible $H_n(q)$-module indexed by a partition $\lambda \vdash n$ by first choosing any fixed standard Young tableau T of shape λ, and any permutation w satisfying $P(w) = T$. One then lets $H_n(q)$ act by left multiplication on the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module

$$\mathrm{span}\{C'_v(q) \,|\, P(v) = T\},$$

regarded as the quotient

$$\mathrm{span}\{C'_v(q)\,|\,v \leq_L w\}/\mathrm{span}\{C'_v(q)\,|\,v \leq_L w, w \not\leq_L v\}.$$

Analogous to the left preorder is a *right preorder* $\leq_R$, defined to be the transitive closure of the relation $\lessdot_R$ defined by $v \lessdot_R u$ if $C'_v(q)$ appears with nonzero coefficient in the Kazhdan-Lusztig expansion of $C'_u(q)T_w$ for some $w \in S_n$. It is easy to see that $v \leq_R u$ if and only if $v^{-1} \leq_L u^{-1}$. A third preorder, the *left-right preorder* $\leq_{LR}$, is the transitive closure of the relation $\lessdot_{LR}$ defined by $v \lessdot_{LR} u$ if $v \lessdot_L u$ or $v \lessdot_R u$. The preorders defined above are closely related to the dominance order on partitions. In particular, we have the equivalence of $v \leq_{LR} u$ and $\mathrm{sh}(u) \preceq \mathrm{sh}(v)$, which is often attributed to [2]. Thus,

$$(v \leq_L u \text{ or } v \leq_R u) \Rightarrow v \leq_{LR} u \Leftrightarrow \mathrm{sh}(u) \preceq \mathrm{sh}(v). \tag{3.1}$$

Building upon results in [13] and [25], Du [17, Sec. 2] gave an expression for the dual canonical basis of $\mathbb{C}[x]$, relating it to the natural basis of $\mathbb{C}[x]$ by a transition matrix whose entries are alternating sums [17, Sec. 1] of Kazhdan-Lusztig polynomials. (See also [18].) Not surprisingly, the dual canonical basis has no entirely elementary description. In [53, Thm. 2.1] these basis elements were described in terms of immanants and (single) Kazhdan-Lusztig polynomials,

$$\mathrm{Imm}_v(x) \underset{\mathrm{def}}{=} \sum_{w \geq v} (-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1) x_{1,w_1} \cdots x_{n,w_n},$$

where w_0 is the longest element of S_n, with one-line notation $n \cdots 1$.

The inversion formula [29, Sec. 3] for the matrix of Kazhdan-Lusztig polynomials implies the identity

$$\sum_w f(w) x_{1,w_1} \cdots x_{n,w_n} = \sum_w f(C'_w(1)) \mathrm{Imm}_w(x), \tag{3.2}$$

for immanants defined in terms of a linear function $f : S_n \to \mathbb{C}$. Other properties of Kazhdan-Lusztig polynomials imply that matrix transposition has the same effect on Kazhdan-Lusztig immanants [50, Prop. 15] that it has on elements of the bitableau basis. (Compare to Proposition 2.2.)

Proposition 3.1. *For all* $v \in S_n$ *we have* $\mathrm{Imm}_v(x) = \mathrm{Imm}_{v^{-1}}(x^\top)$.

The dual canonical basis of $\mathbb{C}[x]$ may be expressed as a union of bases of $\mathcal{A}_{L,M}$, over all r-element multisets of $[n]$, for $r \geq 0$. In particular, the dual canonical basis elements in each component $\mathcal{A}_{L,M}$ are the nonzero polynomials in the set $\{\mathrm{Imm}_v(x_{L,M})\,|\,v \in S_r\}$, or more precisely

$$\{\mathrm{Imm}_v(x_{L,M})\,|\,v \text{ Bruhat maximal in } W_{\iota(L)} v W_{\iota(M)} \subset S_r\}. \tag{3.3}$$

(See [53] for more details.) In particular, each matrix minor $\det(x_{I,J})$ belongs to the dual canonical basis.

Nonnegativity properties [35], [37] of the dual canonical basis imply various inequalties in matrix minors, symmetric functions, and characters. (See, e.g., [7], [14], [16], [31], [50], [51].) Furthermore, just as the structure constants describing multiplication of Kazhdan-Lusztig basis elements in $\mathbb{C}[S_n]$ belong to $\mathbb{N}$, so do those describing multiplication

$$\mathrm{Imm}_v(x_{L,M})\mathrm{Imm}_{v'}(x_{L',M'}) = \sum_w \hat{m}_w^{v,v'} \mathrm{Imm}_w(x_{L \uplus L', M \uplus M'}) \tag{3.4}$$

of dual canonical basis elements [38]. (We have supressed from our notation the dependence of these coefficients upon the multisets L, L', M, M'.) More precisely, in an appropriate quantization of the polynomial ring $\mathbb{C}[x]$, the resulting structure constants belong to $\mathbb{N}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. This nonnegativity result is a known special case of the more general conjecture [36, Conj. 25.4.2]. (See also [36, Sec. 29.5].)

4. Zero sets of basis elements

While the bitableau and dual canonical bases have quite different definitions, the two bases have rather similar vanishing properties. Let us consider several conditions on a permutation w and a matrix A which imply that $\mathrm{Imm}_w(A) = 0$ or $R_w(A) = 0$. Since $\det(x) = R_e(x) = \mathrm{Imm}_e(x)$ is an element of both bases, it is natural to expect that repetition of rows and/or columns in a matrix A causes some basis elements to vanish on A. Indeed we have the following proposition [50, Cor. 17] and observation concerning the equality of two rows or columns of a matrix.

Proposition 4.1. *If rows i and $i+1$ of A are equal and $s_i w > w$, then* $\mathrm{Imm}_w(A) = 0$*; if columns i and $i+1$ of A are equal and $ws_i > w$, then* $\mathrm{Imm}_w(A) = 0$.

Observation 4.2. *If rows i and $i+1$ of A are equal and i, $i+1$ appear in the same column of $Q(w)$, then $R_w(A) = 0$; if columns i and $i+1$ of A are equal and i, $i+1$ appear in the same column of $P(w)$, then $R_w(A) = 0$.*

We also have the following proposition [51, Prop. 2.2] and observation concerning the equality of k rows of a matrix.

Proposition 4.3. *If A has k equal rows or k equal columns and w has no decreasing subsequence of length k, then* $\mathrm{Imm}_w(A) = 0$.

Observation 4.4. *If A has k equal rows or k equal columns and w has no decreasing subsequence of length k, then $R_w(A) = 0$.*

Proof. Suppose w has no decreasing subsequence of length k. Then each of the tableaux $P(w)$ and $Q(w)$ has at most $k-1$ entries in its first row. If A has k equal rows or k equal columns, then two of the corresponding k indices appear together in a single column of $P(w)$ and two appear together in a single column of $Q(w)$. It follows that $R_w(A) = 0$. □

In order to state stronger results about vanishing properties of standard bitableaux and dual canonical basis elements, we will use a partial order on Young tableaux known as the *chain order*, and a family of preorders on S_n introduced by Geck.

The chain order on tableaux, in some sense induced by dominance on partitions, was introduced by Melnikov [42] in her work on orbital varieties of $\mathfrak{sl}(n, \mathbb{C})$, and also by Van Leeuwen [58]. (See [43, Sec. 3.7] and [57].) We define the chain order (actually dual to that defined in [42], [58]) as follows.

Given indices i, j satisfying $1 \leq i \leq j \leq n$, let $[i, j]$ be the interval of integers $\{i, i+1, \ldots, j\}$. Define $T_{[i,j]}$ to be the tableau obtained by applying the jeu de taquin algorithm (see, e.g., [21, Sec. 1.2], [55, Ch. 7]) to the subtableau of T consisting of all boxes holding entries belonging to the interval $[i, j]$, and by subtracting $i - 1$ from the resulting tableau. For example, if

$$T = \begin{matrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{matrix},$$

then we have $T_{[1,3]} = \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}$ and $T_{[2,4]} = \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}$. We will denote the tableau $T_{[1,j]}$ by $T_{[j]}$. We define the chain order on tableaux by declaring $T \trianglelefteq_C U$ if for all $1 \leq i \leq j \leq n$ we have

$$\mathrm{sh}(T_{[i,j]}) \preceq \mathrm{sh}(U_{[i,j]}).$$

By considering the condition corresponding to $i = 1$ and $j = n$, we see that $T \trianglelefteq_C U$ implies $\mathrm{sh}(T) \preceq \mathrm{sh}(U)$.

Melnikov demonstrated the following relationship between the Kazhdan-Lusztig preorders and the chain order [43, Sec. 3.6, 4.7]. (See also [43, Sec. 1.9], [57, Thm. 3.7].)

Lemma 4.5. *If $w \leq_R v$ then $Q(v) \trianglelefteq_C Q(w)$; if $w \leq_L v$ then $P(v) \trianglelefteq_C P(w)$.*

The tableaux $\{Q(w)_{[i,j]} \,|\, w \in S_n\}$ have a natural interpretation in terms of the one-line notation of permutations. Define $w_{[i,j]}$ to be the element of S_{j-i+1} obtained by rearranging the letters $1, \ldots, j-i+1$ so that their relative order matches that of the letters of the word $w_i \cdots w_j$. Then we have the following result of Schützenberger. (See [30, Thm. 5.1.4 C].)

Lemma 4.6. *For any $w \in S_n$ and indices $1 \leq i \leq j \leq n$, we have*

$$\mathrm{sh}(w_{[i,j]}) = \mathrm{sh}(Q(w)_{[i,j]}).$$

Now define $S_{[i,j]}$ to be the parabolic subgroup of S_n generated by $\{s_i, \ldots, s_{j-1}\}$. In terms of one-line notation, $S_{[i,j]}$ consists of all permutations $w_1 \cdots w_n$ in S_n satisfying $w_k = k$ for $k = 1, \ldots, i-1, j+1, \ldots, n$. The restriction of the left-right preorder to $S_{[i,j]}$ (from S_n) is related to the shapes of the tableaux $\{Q(w)_{[i,j]} \,|\, w \in S_n\}$ as follows.

Lemma 4.7. *Permutations u, v in $S_{[i,j]}$ satisfy $v \leq_{LR} u$ (in S_n) if and only if they satisfy* $\mathrm{sh}(u_{[i,j]}) \preceq \mathrm{sh}(v_{[i,j]})$*.*

Proof. Recall that we have

$$\mathrm{sh}(u) = \mathrm{sh}(P(u)) = \mathrm{sh}(Q(u)), \qquad \mathrm{sh}(v) = \mathrm{sh}(P(v)) = \mathrm{sh}(Q(v)).$$

Since u, v belong to $S_{[i,j]}$, each of the four tableaux above contains the numbers $1, \ldots, i-1, j+1, \ldots, n$ in its first column. Removing these numbers from the tableaux and applying jeu de taquin, we obtain the shapes

$$\mathrm{sh}(u_{[i,j]}) = \mathrm{sh}(P(u)_{[i,j]}) = \mathrm{sh}(Q(u)_{[i,j]}), \qquad \mathrm{sh}(v_{[i,j]}) = \mathrm{sh}(P(v)_{[i,j]}) = \mathrm{sh}(Q(u)_{[i,j]}),$$

each of which differs from the corresponding original shape only by the subtraction of 1 from each of the first $i-1$ parts and last $n-j$ parts. It is therefore easy to see that we have $\mathrm{sh}(u_{[i,j]}) \preceq \mathrm{sh}(v_{[i,j]})$ if and only if $\mathrm{sh}(u) \preceq \mathrm{sh}(v)$, which by (3.1) is equivalent to $v \leq_{LR} u$. $\square$

Studying connections between representations of Coxeter groups, parabolic subgroups, and Hecke algebras defined in terms of unequal parameters, Geck [23] defined families of parabolic analogs of the Kazhdan-Lusztig preorders on S_n. Each preorder is parametrized by a subset $J \subset \{s_1, \ldots, s_{n-1}\}$ of generators of S_n. Define $\leq_{L,J}$, the *relative left preorder parametrized by* J to be the transitive closure of the relation $\lessdot_{L,J}$ defined by $v \lessdot_{L,J} u$ if $C'_v(q)$ appears with nonzero coefficient in the Kazhdan-Lusztig expansion of $T_w C'_u(q)$ for some $w \in W_J$. Similarly, define a *relative right preorder* $\leq_{R,J}$ to be the transitive closure of the relation $\lessdot_{R,J}$ defined by $v \lessdot_{R,J} u$ if $C'_v(q)$ appears with nonzero coefficient in the Kazhdan-Lusztig expansion of $C'_u(q)T_w$ for some $w \in W_J$. Finally, define a *relative left-right preorder* $\leq_{LR,J}$ to be the transitive closure of the relation $\lessdot_{LR,J}$ defined by $v \lessdot_{LR,J} u$ if $v \lessdot_{L,J} u$ or $v \lessdot_{R,J} u$.

It is easy to see that the relative preorder inequalities $v \leq_{L,J} u$, $v \leq_{R,J} u$, $v \leq_{LR,J} u$ imply the ordinary preorder inequalities $v \leq_L u$, $v \leq_R u$, $v \leq_{LR} u$, respectively, and that we have

$$(v \leq_{L,J} u \text{ or } v \leq_{R,J} u) \Rightarrow v \leq_{LR,J} u \Rightarrow v \leq_{LR} u \Rightarrow \mathrm{sh}(u) \preceq \mathrm{sh}(v). \tag{4.1}$$

A more surprising implication concerns factorizations of the form $v = v'v''$ where v' belongs to W_J and v'' is Bruhat-minimal in the coset $W_J v''$. (Given J and v, such a factorization necessarily exists and is unique. See, e.g., [6, Prop. 2.4.4].) In particular, Geck showed the following [23, Prop. 4.4].

Proposition 4.8. *Fix generators $J \subset \{s_1, \ldots, s_{n-1}\}$ and let $v, w \in S_n$ factor uniquely as $v = v'v''$, $w = w'w''$, where v', w' belong to W_J and v'', w'' are Bruhat-minimal in $W_J v''$, $W_J w''$, respectively. Then $w \leq_{L,J} v$ implies that $w' \leq_{LR,J} v'$.*

This result allows us to construct certain S_n bimodules as follows.

Lemma 4.9. *Fix indices $i \leq j \leq n$, and let $S_{[i,j]}$ and S_n act on $\mathbb{C}[S_n]$ by left and right multiplication, respectively. Then for each partition λ of $j-i+1$, the space*

$$W = W(i,j,\lambda) \underset{\mathrm{def}}{=} \mathrm{span}\{C'_u(1) \,|\, \mathrm{sh}(u_{[i,j]}) \succeq \lambda\} \subset \mathbb{C}[S_n]$$

is a sub-bimodule of $\mathbb{C}[S_n]$.

Proof. Fix $\lambda \vdash j-i+1$ and define W as above. To see that $S_{[i,j]}W \subset W$, choose $z \in S_{[i,j]}$ and $v \in S_n$ satisfying $\mathrm{sh}(v_{[i,j]}) \succeq \lambda$. Expand $zC'_v(1)$ with respect to the Kazhdan-Lusztig basis of $\mathbb{C}[S_n]$ and let $w \in S_n$ be a permutation for which $C'_w(1)$ appears with nonzero coefficient in this expansion. Letting J be the set of generators $\{s_i, \ldots, s_{j-1}\}$, we have by definition that $w \leq_{L,J} v$.

Now factor v, w uniquely as $v = v'v''$, $w = w'w''$, with $v', w' \in S_{[i,j]}$ and v'', w'' Bruhat-minimal in $S_{[i,j]}v''$ and $S_{[i,j]}w''$. By Proposition 4.8 and (4.1), the condition $w \leq_{L,J} v$ implies that $w' \leq_{LR,J} v'$ and therefore that $\mathrm{sh}(v') \preceq \mathrm{sh}(w')$. By Lemma 4.7, the shapes of the permutations $v'_{[i,j]}$ and $w'_{[i,j]}$ are related by $\mathrm{sh}(v'_{[i,j]}) \preceq \mathrm{sh}(w'_{[i,j]})$. Furthermore, by the minimality of v'' and w'' we now have that

$$\mathrm{sh}(w_{[i,j]}) = \mathrm{sh}(w'_{[i,j]}) \succeq \mathrm{sh}(v'_{[i,j]}) = \mathrm{sh}(v_{[i,j]}) \succeq \lambda.$$

Thus, $C'_w(1)$ belongs to W.

To see that $WS_n \subset W$, choose v as before, let $z \in S_n$ be arbitrary, and let w be a permutation for which $C'_w(1)$ appears with nonzero coefficient in the Kazhdan-Lusztig expansion of $C'_v(1)z$. By the definition of the right preorder on S_n we thus have $w \leq_R v$. By Lemma 4.5, this implies that $Q(v) \trianglelefteq_C Q(w)$. Now by Lemma 4.6 and the definition of the chain order, we have that

$$\mathrm{sh}(w_{[i,j]}) = \mathrm{sh}(Q(w)_{[i,j]}) \succeq \mathrm{sh}(Q(v)_{[i,j]}) = \mathrm{sh}(v_{[i,j]}) \succeq \lambda,$$

and again $C'_w(1)$ belongs to W. □

The bimodules $\{W(i,j,\lambda) \,|\, \lambda \vdash j-i+1\}$ of $\mathbb{C}[S_n]$ expose a relationship between column (or row) repetition in a matrix and the vanishing of dual canonical basis elements. Given an $n \times n$ matrix A and a subinterval $[i,j]$ of $[n]$, we may consider rows $i, \ldots, j$ of A to be a multiset of vectors having n components. Define the partition $\mu_{[i,j]}(A)$ of $j-i+1$ to be the decreasing sequence of multiplicities in this multiset. Similarly, we may consider columns $i, \ldots, j$ of A to be a multiset and will define the partition $\nu_{[i,j]}(A)$ to be the decreasing sequence of multiplicities in this multiset. For example, given the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and the intervals $[6], [5], [2, 6], [3, 5]$, we have

$$\begin{aligned} \mu_{[6]}(A) &= 411, & \nu_{[6]}(A) &= 321, \\ \mu_{[5]}(A) &= 311, & \nu_{[5]}(A) &= 221, \\ \mu_{[2,6]}(A) &= 311, & \nu_{[2,6]}(A) &= 311, \\ \mu_{[3,5]}(A) &= 21, & \nu_{[3,5]}(A) &= 111. \end{aligned}$$

We may now strengthen Proposition 4.3 by stating specific patterns of repetition among rows and columns of a matrix which cause a dual canonical basis element to vanish on that matrix.

Theorem 4.10. *Fix a permutation $w \in S_n$ and indices $1 \leq i \leq j \leq n$. Then for each $n \times n$ matrix A satisfying $\mathrm{sh}(w_{[i,j]}) \not\leq \mu_{[i,j]}(A)$ or $\mathrm{sh}(w^{-1}_{[i,j]}) \not\leq \nu_{[i,j]}(A)$ we have $\mathrm{Imm}_w(A) = 0$.*

Proof. Let $S_{[i,j]}$ permute rows of A and let $H \subset S_{[i,j]}$ be the stabilizer of A. Then H is conjugate in $S_{[i,j]}$ to a Young subgroup $H' \subset S_{[i,j]}$ of S_n of the form

$$H' \cong S_1^{i-1} \times S_\mu \times S_1^{n-j},$$

where $\mu = \mu_{[i,j]}(A)$ is a partition of $j - i + 1$. Now consider the $\mathbb{C}[S_n]$ element

$$t = \sum_{v \in S_n} a_{1,v_1} \cdots a_{n,v_n} v,$$

which factors as

$$t = \Big(\sum_{v \in H} v \Big) g = y \Big(\sum_{v \in H'} v \Big) y^{-1} g$$

for some $y \in S_{[i,j]}$ and $g \in \mathbb{C}[S_n]$. Letting u be the longest element of H', we may rewrite this factorization as $yC'_u(1)y^{-1}g$. Defining the subspace $W = W(i, j, \mu)$ of $\mathbb{C}[S_n]$ as in Lemma 4.9, we see that t belongs to W.

On the other hand, we may use the inversion formula (3.2) to expand t in terms of the Kazhdan-Lusztig basis as

$$t = \sum_{v \in S_n} \mathrm{Imm}_v(A) C'_v(1).$$

Since t belongs to W, we must have $\mathrm{Imm}_v(A) = 0$ for each permutation v not satisfying $\mathrm{sh}(v_{[i,j]}) \succeq \mu = \mu_{[i,j]}(A)$.

Observing that $\nu_{[i,j]}(A) = \mu_{[i,j]}(A^\top)$ and applying the above argument to w^{-1} and $A^\top$, we see that the condition $\mathrm{sh}(w^{-1}_{[i,j]}) \not\preceq \nu_{[i,j]}(A)$ implies that $\mathrm{Imm}_{w^{-1}}(A^\top) = 0$. By Proposition 3.1, this gives the desired result. □

Theorem 4.10 can be quite easy to apply in the case that $i = 1$ and $j = n$. For example, consider the permutation and matrix

$$w = 1324, \qquad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{bmatrix}. \tag{4.2}$$

We have $\mathrm{Imm}_{1324}(A) = 0$, since

$$\mathrm{sh}(1324) = 211 \not\preceq \mu_{[4]}(A) = 22.$$

Analogous to Theorem 4.10 is the following result for bitableaux.

Proposition 4.11. *Fix a permutation $w \in S_n$ and an index $j \leq n$. Then for each $n \times n$ matrix A satisfying $\mathrm{sh}(w_{[j]}) \not\preceq \mu_{[j]}(A)$ or $\mathrm{sh}(w^{-1}_{[j]}) \not\preceq \nu_{[j]}(A)$, we have $R_w(A) = 0$.*

Proof. Define $\lambda = \mathrm{sh}(w_{[j]})$, $\mu = \mu_{[j]}(A)$ and assume that $\lambda \not\preceq \mu$. Create a tableau T of shape μ by placing the indices $1, \ldots, j$ so that each row i contains the indices of μ_i equal rows of A. By Lemma 4.6, $Q(w)_{[j]}$ is a tableau of shape λ containing $1, \ldots, j$. We may therefore apply the well-known Dominance Lemma (see, e.g., [52, Lem. 2.2.4]) to the tableaux T and $Q(w)_{[j]}$ to deduce that there exists a pair (k, ℓ) of indices which appear together in some row of T and also in some column of $Q(w)_{[j]}$. Since $Q(w)_{[j]}$ is a subtableau of $Q(w)$, the indices also appear together in some column of $Q(w)$. Since this column of $Q(w)$ corresponds to a minor which is a factor of $R_w(x)$ and since rows k and ℓ of A are equal, we have $R_w(A) = 0$.

As in the proof of the previous proposition, we use the fact that $\nu_{[j]}(A) = \mu_{[j]}(A^\top)$ and apply the above argument to w^{-1} and $A^\top$. We then see that the condition $\mathrm{sh}(w^{-1}_{[j]}) \not\preceq \nu_{[j]}(A)$ implies that $R_{w^{-1}}(A^\top) = 0$. By Proposition 2.2, this gives the desired result. □

Since Proposition 4.11 is completely analogous to the special case $i = 1$ of Theorem 4.10, we may apply it to our example in (4.2) and deduce that $R_{1324}(A) = 0$. On the other hand, we remark that Proposition 4.11 lacks inequalities of the forms $\mathrm{sh}(w_{[i,j]}) \not\preceq \mu_{[i,j]}(A)$ and $\mathrm{sh}(w^{-1}_{[i,j]}) \not\preceq \nu_{[i,j]}(A)$ for general i, which appear in Theorem 4.10. Indeed, the corresponding generalization of Proposition 4.11 is false. For example, consider the permutation and matrix

$$w = 213, \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

We have $R_{213}(A) = 1$, in spite of the conditions

$$\mathrm{sh}(213_{[2,3]}) = \mathrm{sh}(213^{-1}_{[2,3]}) = 11 \not\geq \mu_{[2,3]}(A) = \nu_{[2,3]}(A) = 2$$

which imply that $\mathrm{Imm}_{213}(A) = 0$.

One more easy fact about the zero set of a standard bitableau is the following.

Proposition 4.12. *If* $\mathrm{rank}(A) < k$ *and* w *has an increasing subsequence of length* k*, then* $R_w(A) = 0$.

Proof. If w has an increasing subsequence of length k, then $P(w)$ has at least k rows. It follows that $R_w(x)$ has a factor which is a minor of size at least $k \times k$. On the other hand, if $\mathrm{rank}(A) < k$, then every $k \times k$ (or larger) minor of A vanishes. □

In Corollary 6.3 we will state an analog of Proposition 4.12 for dual canonical basis elements.

5. Triangularity of transition matrices

Our results in Section 4 show that standard bitableaux and dual canonical basis elements have similar vanishing properties. It is therefore natural to hope that suitable orderings of the two bases leads to a transition matrix having a particularly nice form. This hope seems even more natural if one considers the results in [22], [32], [33], [40], which relate bases constructed from Kazhdan-Lusztig polynomials to others constructed from Young tableaux. Indeed we will show in Theorem 5.8 - Corollary 5.12 that the transition matrix relating the dual canonical basis and bitableau basis is an infinite direct sum of unitriangular transition matrices, each of which corresponds to a component $\mathcal{A}_{M,N}$ of $\mathbb{C}[x]$.

Let us begin by considering the immanant space and coefficients $\{d_{u,v} \,|\, u, v \in S_n\}$ defined by

$$R_v(x) = \sum_{u \in S_n} d_{u,v} \mathrm{Imm}_u(x). \tag{5.1}$$

Recall that each minor of x belongs to the dual canonical basis and therefore that each bitableau is a product of dual canonical basis elements. By (3.4), the coefficients in (5.1) are thus nonnegative integers. Furthermore, we have the following.

Lemma 5.1. *The coefficients in Equation (5.1) satisfy* $d_{u,v} = d_{u^{-1},v^{-1}}$.

Proof. Using (5.1) to expand $R_{v^{-1}}(x^\top)$ in terms of the Kazhdan-Lusztig immanants, and applying Proposition 3.1, we have

$$R_{v^{-1}}(x^\top) = \sum_{u^{-1} \in S_n} d_{u^{-1},v^{-1}} \mathrm{Imm}_{u^{-1}}(x^\top) = \sum_{u^{-1} \in S_n} d_{u^{-1},v^{-1}} \mathrm{Imm}_u(x).$$

By Proposition 2.2 this expression is also equal to $R_v(x)$. We may therefore compare coefficients above to those in (5.1) to obtain the desired equality. □

To prove more facts about the coefficients in (5.1), we will use the vanishing properties stated in Section 4 and Greene's results on decreasing subsequences in permutations. We will also consider a partial order on standard tableaux which we call *iterated dominance of tableaux* (and which is sometimes called *row dominance of tableaux* in the literature, e.g., in [9]). Given two standard tableaux T, U having n boxes, we define $T \trianglelefteq_I U$ if for $j = 1, \dots, n$ we have

$$\mathrm{sh}(T_{[j]}) \preceq \mathrm{sh}(U_{[j]}),$$

where $T_{[j]}$ and $U_{[j]}$ are the subtableaux of T and U consisting of all boxes holding entries less than or equal to j. By considering the condition corresponding to $j = n$, we see that $T \trianglelefteq_I U$ implies $\mathrm{sh}(T) \preceq \mathrm{sh}(U)$. By considering the conditions corresponding to $i = 1$ in the definition of the chain order, we also see that $T \trianglelefteq_C U$ implies $T \trianglelefteq_I U$. Thus we have

$$T \trianglelefteq_C U \Rightarrow T \trianglelefteq_I U \Rightarrow \mathrm{sh}(T) \preceq \mathrm{sh}(U). \tag{5.2}$$

Figure 5.2 shows the iterated dominance order on standard tableaux having four boxes. Ehresmann [19] showed the iterated dominance order on standard tableaux of a fixed shape to be equivalent to the Bruhat order on the column reading words of those tableaux. (See also [39, Thm. 3.8].)

Among all standard tableaux of a fixed shape $\lambda = (\lambda_1, \dots, \lambda_\ell)$, there is a unique maximal tableau and a unique minimal tableau with respect to iterated dominance. In particular, define $U(\lambda)$ to be the unique standard tableau of shape λ in which all entries in row i are less than all entries in row $i+1$, for $i = 1, \dots, \ell - 1$. For example

$$U(421) = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & & & \end{matrix}\ , \qquad U(3221) = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & 7 & \\ 8 & & \end{matrix}\ . \tag{5.3}$$

Observation 5.2. *Fix a partition $\lambda \vdash n$. Each standard tableau T of shape λ satisfies $U(\lambda^\top)^\top \trianglelefteq_I T \trianglelefteq_I U(\lambda)$. Furthermore we have $U(\lambda) \trianglelefteq_I U(\mu)$ if and only if $\lambda \preceq \mu$.*

The expansion of nonstandard bitableaux in the bitableau basis is sometimes described by defining *iterated dominance of bitableaux* to be the componentwise iterated dominance of tableaux. That is, we define $(T{:}U)(x) \trianglelefteq_I (T'{:}U')(x)$ if $T \trianglelefteq_I T'$ and

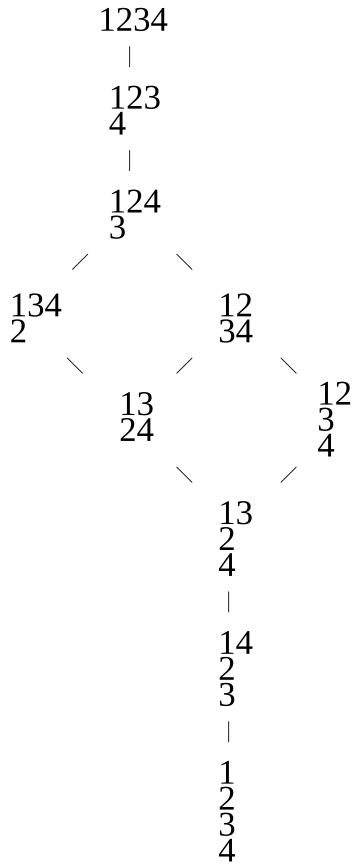

FIGURE 5.1. The iterated dominance order on standard Young tableaux having four boxes

$U \trianglelefteq_I U'$. In particular, Clausen used iterated dominance of bitableaux to strengthen results in [10, Thm. 2.1], [11, Thm. 5], [15, Thm. 3] as follows [8, Thm. 4.5].

Theorem 5.3. *For any column-strict injective bitableau $(T{:}U)(x)$, we have*

$$(T{:}U)(x) \in (T'{:}U')(x) + \mathrm{span}_{\mathbb{Z}}\{(P{:}Q)(x) \,|\, (P{:}Q)(x) \triangleleft_I (T'{:}U')(x)\}, \tag{5.4}$$

where T', U' are standard tableaux obtained from T, U by sorting entries within rows.

Combining the Robinson-Schensted column correspondence with iterated dominance of bitableaux allows us to define a related poset on S_n by declaring $u \leq_I v$ if $(P(u){:}Q(u))(x) \trianglelefteq_I (P(v){:}Q(v))(x)$. We will refer to this poset as *iterated dominance of permutations*. It is clear that we have $u \leq_I v$ if and only if $u^{-1} \leq_I v^{-1}$. By Lemma 4.5 and Equation (5.2), we have

$$(w \leq_R v \text{ and } w \leq_L v) \Rightarrow (P(v) \trianglelefteq_C P(w) \text{ and } Q(v) \trianglelefteq_C Q(w)) \Rightarrow v \leq_I w.$$

Figure 5.2 shows the iterated dominance order on S_4.

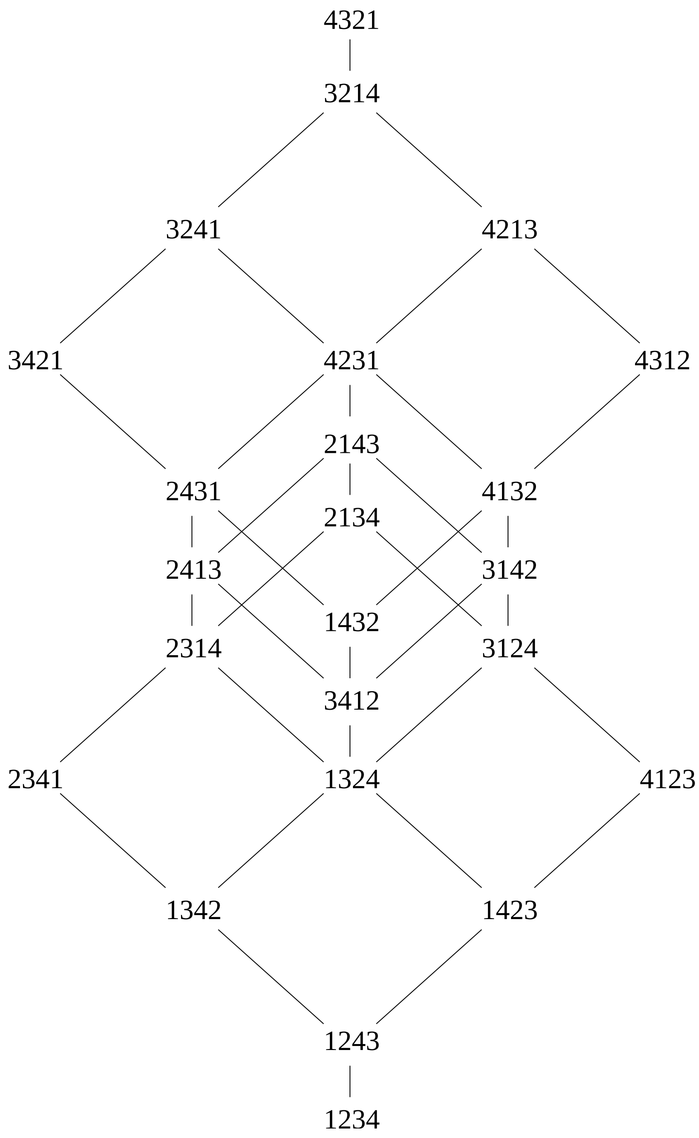

FIGURE 5.2. The iterated dominance order on S_4

By Observation 5.2, it is clear that among all permutations of shape $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, there is a unique maximal permutation and a unique minimal permutation in iterated dominance. Specifically, these are the permutations $w(\lambda)$ and $u(\lambda)$ satisfying

$$P(w(\lambda)) = Q(w(\lambda)) = U(\lambda), \qquad P(u(\lambda)) = Q(u(\lambda)) = U(\lambda^\top)^\top. \tag{5.5}$$

We remark that one may compute $w(\lambda)$ by reading the entries of $U(\lambda)$ from right to left in rows $1, \ldots, \ell$. For example if $\lambda = 431$, then we may use the tableau $U(431)$ in (5.3) to obtain $w(\lambda) = 43217658$.

Observation 5.4. *Fix a partition $\lambda \vdash n$ and define permutations $u(\lambda)$ and $w(\lambda)$ as above. Then each permutation v of shape λ satisfies*

$$u(\lambda) \leq_I v \leq_I w(\lambda).$$

Furthermore, each of the conditions $u(\lambda) \leq_I u(\mu)$ and $w(\lambda) \leq_I w(\mu)$ is equivalent to $\lambda \preceq \mu$.

Proof. Apply Observation 5.2 to Equation (5.5). □

The iterated dominance order on S_n is closely related to decreasing subsequences within the one-line notation of a permutation. Fix a permutation $w = w_1 \cdots w_n$ in S_n. Following Greene [24], call a subsequence $\sigma = w_{i_1} w_{i_2} \cdots w_{i_q}$ of $w_1 \cdots w_n$ a *p-decreasing subsequence* of w if, as a set, it can be partitioned as

$$\sigma = \sigma_1 \uplus \cdots \uplus \sigma_p$$

where each block σ_i corresponds to a decreasing subsequence of w. Greene related p-decreasing subsequences and partitions as follows [24, Thm. 3.1].

Theorem 5.5. *The length of the longest p-decreasing subsequence of w is equal to the sum of the lengths of the first p rows of* $\mathrm{sh}(w)$*.*

Let $\eta_{p,r}(w)$ be the length of the longest p-decreasing subsequence of $w_1 \cdots w_r$. Then we have

$$\eta_{p,r}(w) = \mathrm{sh}(w_{[r]})_1 + \cdots + \mathrm{sh}(w_{[r]})_p. \tag{5.6}$$

Since $\mathrm{sh}(w_{[r]}) = \mathrm{sh}(Q(w)_{[r]})$ by Lemma 4.6, it follows that $\eta_{p,r}(w)$ also is equal to the number of entries less than or equal to r in the first p rows of $Q(w)$. Thus we may restate the definition of iterated dominance in S_n as follows.

Proposition 5.6. *Permutations $u, v \in S_n$ satisfy $u \leq_I v$ if and only if we have $\eta_{p,r}(u) \leq \eta_{p,r}(v)$ and $\eta_{p,r}(u^{-1}) \leq \eta_{p,r}(v^{-1})$ for all indices $p \leq r \leq n$.*

Proof. We have $Q(u) \trianglelefteq_I Q(v)$ if and only if $\mathrm{sh}(Q(u)_{[r]}) \preceq \mathrm{sh}(Q(v)_{[r]})$ for $r = 1, \ldots, n$. These conditions may be restated as

$$\mathrm{sh}(Q(u)_{[r]})_1 + \cdots + \mathrm{sh}(Q(u)_{[r]})_p \leq \mathrm{sh}(Q(v)_{[r]})_1 + \cdots + \mathrm{sh}(Q(v)_{[r]})_p$$

for all $p \leq r \leq n$, or equivalently as $\eta_{p,r}(u) \leq \eta_{p,r}(v)$ for all $p \leq r \leq n$. Similarly, we have $P(u) \trianglelefteq_I P(v)$ if and only if $\eta_{p,r}(u^{-1}) \leq \eta_{p,r}(v^{-1})$ for all $p \leq r \leq n$. □

Greene's result on decreasing subsequences is also related to the vanishing properties described in Theorem 4.10 and Proposition 4.11. In particular, it aids in the construction of matrices on which desired basis elements vanish. Such matrices can then be used to obtain information about coefficients appearing in expressions of the forms

$$\mathrm{Imm}_f(x) = \sum_u c_u R_u(x) = \sum_u c'_u \mathrm{Imm}_u(x). \tag{5.7}$$

(See, e.g., [49, Cor. 3.5-3.6], [51, Prop. 2.3], [53, Thm. 4.4-4.5].)

We will use Theorem 5.5 to construct matrices as follows. Fix $w = w_1 \cdots w_n$ in S_n and suppose that for some indices $p \leq q \leq r$, the longest p-decreasing subsequence $\sigma = \sigma_1 \uplus \cdots \uplus \sigma_p$ of $w_1 \cdots w_r$ has length q. Let $I_1, \ldots, I_p$ be the (disjoint) sets of indices corresponding to the positions in w of the p decreasing subsequences, and let $J_1, \ldots, J_p$ be the (disjoint) sets of components of these decreasing subsequences. Let M be the permutation matrix of w ($m_{i,j} = 1$ if $w_i = j$) and construct a matrix $B = B(w, \sigma_1, \ldots, \sigma_p, I_1, \ldots, I_p)$ by replacing the submatrices $M_{I_1,J_1}, \ldots, M_{I_p,J_p}$ with matrices of all ones.

For example, consider the permutation $w = 7135246$ and numbers $p = 2$, $r = 6$. A 2-decreasing subsequence of $w_1 \cdots w_6 = 713524$ has length at most $q = 5$, and one such subsequence consists of the five letters 73524 in positions 13456. We therefore define $\sigma_1 = 732$, $\sigma_2 = 54$, $I_1 = 135$, $I_2 = 46$, $J_1 = 237$, $J_2 = 45$. Letting M be the permutation matrix of w, we construct the matrix $B = B(w, \sigma_1, \sigma_2, I_1, I_2)$ by replacing the $(135, 237)$ and $(46, 45)$ submatrices of M with matrices of ones:

$$M = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & \mathbf{1} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

While the number q defined above does not play an important role in the definition of the matrix B, it does play an important role in the description of certain basis elements which vanish on B.

Lemma 5.7. *Let w, p, q, r, $\sigma_1, \ldots, \sigma_p$, $I_1, \ldots, I_p$ satisfy the conditions stated after (5.7) and define $B = B(w, \sigma_1, \ldots, \sigma_p, I_1, \ldots, I_p)$. Then we have*

(1) If $\eta_{p,r}(u) < q$ then $R_u(B) = \mathrm{Imm}_u(B) = 0$.
(2) $\mathrm{Imm}_w(B) = 1$.
(3) If $u \not\leq w$ then $\mathrm{Imm}_u(B) = 0$.

Proof. (1). Suppose that $\eta_{p,r}(u) < q$ and let $\lambda = \mathrm{sh}(u_{[r]})$, $\mu = \mu_{[r]}(B)$. By (5.6) we have $\lambda_1 + \cdots + \lambda_p < q$, and by the construction of B we have $\mu_1 + \cdots + \mu_p = q$. Thus $\lambda \not\geq \mu$, and by Theorem 4.10 and Proposition 4.11, we have $\mathrm{Imm}_u(B) = R_u(B) = 0$.

(2) - (3). Recall the definition

$$\mathrm{Imm}_u(B) = \sum_{v \geq u} (-1)^{\ell(v)-\ell(u)} Q_{u,v}(1) b_{1,v_1} \cdots b_{n,v_n}.$$

Our use of the decreasing sequences $\sigma_1, \dots, \sigma_p$ in the construction of B ensures that we have $b_{1,v_1} \cdots b_{n,v_n} = 0$ for each permutation $v \not\leq w$. Thus if $u \not\leq w$, then each permutation v appearing in the above sum also satisfies $v \not\leq w$ and thus each term in the sum is zero. On the other hand, if $u = w$, then exactly one term in the sum is nonzero:

$$\mathrm{Imm}_w(B) = (-1)^0 Q_{w,w}(1) b_{1,w_1} \cdots b_{n,w_n} = b_{1,w_1} \cdots b_{n,w_n} = 1.$$

□

Now let us define the matrix $D = (d_{u,v})$ of coefficients from (5.1) by ordering the dual canonical and bitableau bases of $\mathcal{A}_{[n],[n]}$ according to an arbitrary linear extension of the iterated dominance order on S_n. Using Lemma 5.7, we can now show that D is triangular.

Theorem 5.8. *We have $d_{u,v} = 0$ whenever $u \not\leq_I v$.*

Proof. First we claim that $d_{u,v} = 0$ whenever $Q(u) \not\leq_I Q(v)$. Assume the contrary. Then by Proposition 5.6 there exist permutations u, v and numbers p, r satisfying $\eta_{p,r}(u) > \eta_{p,r}(v)$ and $d_{u,v} > 0$. Let q be the maximum index for which the set $\{u \,|\, \eta_{p,r}(u) = q, d_{u,v} > 0\}$ is nonempty, and let w be a Bruhat-minimal element of this set. Thus $d_{w,v} > 0$ and $Q(w) \not\leq_I Q(v)$. Now write

$$R_v(x) = \sum_{\substack{u \\ \eta_{p,r}(u)<q}} d_{u,v}\mathrm{Imm}_u(x) + d_{w,v}\mathrm{Imm}_w(x) + \sum_{\substack{u \not\leq w \\ \eta_{p,r}(u)=q}} d_{u,v}\mathrm{Imm}_u(x). \tag{5.8}$$

Since $\eta_{p,r}(w) = q$, we can choose a p-decreasing subsequence $\sigma = \sigma_1 \uplus \cdots \uplus \sigma_p$ of $w_1 \cdots w_r$ which has length q. Defining a matrix $B = B(w, \sigma_1, \dots, \sigma_p, I_1, \dots, I_p)$ as before Lemma 5.7 and substituting $x = B$ into Equation (5.8), we may apply Lemma 5.7 to obtain the contradiction

$$0 = 0 + d_{w,v} \cdot 1 + 0.$$

Now we claim that $d_{u,v} = 0$ whenever $P(u) \not\leq_I P(v)$. Suppose that u satisfies $P(u) \not\leq_I P(v)$ and observe that $P(u) = Q(u^{-1})$ and $P(v) = Q(v^{-1})$. Thus Lemma 5.1 and the above argument imply that $d_{u,v} = d_{u^{-1},v^{-1}} = 0$. Since $u \not\leq_I v$ if and only if $Q(u) \not\leq_I Q(v)$ or $P(u) \not\leq_I P(v)$, it follows that $d_{u,v} = 0$ for all $u \not\leq_I v$. □

Thus Equation (5.1) becomes

$$R_v(x) = \sum_{u \leq_I v} d_{u,v}\mathrm{Imm}_u(x). \tag{5.9}$$

Furthermore, the diagonal entries of D are equal to one.

Proposition 5.9. *We have* $d_{u,u} = 1$ *for all* $u \in S_n$.

Proof. Each Kazhdan-Lusztig immanant $\mathrm{Imm}_v(x)$ belongs to $\mathbb{Z}[x]$. Thus by Theorem 2.1 there exist integers $\{e_{u,v} \,|\, u, v \in S_n\}$ such that

$$\mathrm{Imm}_v(x) = \sum_{u \in S_n} e_{u,v} R_u(x)$$

for all v. The matrix $E = (e_{u,v})$ of these coefficients is clearly inverse to $D = (d_{u,v})$, and both matrices are triangular if we order rows and columns by any linear extension of iterated dominance of S_n. Since $\det(D)$ and $\det(E)$ are both integers, they are both ± 1.

Recalling that products of minors expand nonnegatively (3.4) in the dual canonical basis, we see that D has nonnegative integer entries, and therefore that all diagonal entries $d_{u,u}$ are equal to 1, as are all diagonal entries $e_{u,u} = d_{u,u}$ of E. □

It would be interesting to characterize the pairs (u, v) for which $d_{u,v}$ is strictly positive. The following partial result addresses entries in certain columns of D.

Proposition 5.10. *Fix a partition* $\lambda \vdash n$ *and define* $w = w(\lambda)$ *as in Equation (5.5). Then we have* $d_{v,w} > 0$ *for all* $v \leq_I w$.

Proof. Choose a permutation $v \leq_I w$. By [51, Lem. 4.1, Thm. 4.2], the immanant $R_w(x) - R_v(x)$ is equal to a nonnegative linear combination of immanants of the form

$$(T{:}U)(x) \cdot (\det(x_{I,J}) \det(x_{I',J'}) - \det(x_{K,L}) \det(x_{K',L'})), \tag{5.10}$$

where the subsets $I, I', \ldots, L, L'$ of $[n]$ satisfy

$$I \cap I' = J \cap J' = K \cap K' = L \cap L' = \emptyset,$$
$$I \cup I' = K \cup K', \quad J \cup J' = L \cup L',$$

and the tableaux T, U have entries in $[n] \smallsetminus (I \cup I')$ and $[n] \smallsetminus (J \cup J')$, respectively. By [49, Cor. 4.6], each of the above differences of products of minors is equal to a nonnegative linear combination of Kazhdan-Lusztig immanants of the submatrix $x_{I \cup I', J \cup J'}$. Thus each term (5.10) is a product of dual canonical basis elements and therefore by (3.4) is equal to a nonnegative linear combination of Kazhdan-Lusztig immanants of x, as is $R_w(x) - R_v(x)$.

On the other hand, the coefficient of $\mathrm{Imm}_v(x)$ in $R_w(x) - R_v(x)$ is $d_{v,w} - 1$. Thus we have $d_{v,w} > 0$. □

Evaluating Equation 5.9 at matrices of the form $x_{L,M}$ and applying Theorem 5.8, Proposition 5.9, and Equation (3.3), we see that for all pairs (L, M) of r-element multisets of $[n]$, the bitableau and dual canonical bases of $\mathcal{A}_{L,M}$ are related by unitriangular transition matrices.

Corollary 5.11. *Fix a pair (L, M) of r-element multisets of $[n]$ and define $S \subset S_r$ to be the set of all Bruhat-maximal elements of double cosets of the form $W_{\iota(L)}wW_{\iota(M)}$. Then for each permutation $v \in S$ we have*

$$R_v(x_{L,M}) = \mathrm{Imm}_v(x_{L,M}) + \sum_{\substack{u \in S \\ u <_I v}} d_{u,v}^{L,M} \mathrm{Imm}_u(x_{L,M}),$$

where all coefficients $\{d_{u,v}^{L,M} \,|\, u, v \in S\}$ are nonnegative integers.

Since $\{\mathrm{Imm}_v(x_{L,M}) \,|\, v$ Bruhat maximal in $W_{\iota(L)}vW_{\iota(M)}\}$ is a basis of $\mathcal{A}_{L,M}$, Corollary 5.11 implies that $\{R_v(x_{L,M}) \,|\, v$ Bruhat maximal in $W_{\iota(L)}vW_{\iota(M)}\}$ is a basis as well. In addition, we may use permutations of the form $w(\lambda)$ defined in Equation (5.5) to state the following sufficient conditions for positivity of the above coefficients.

Corollary 5.12. *Fix a pair (L, M) of r-element multisets of $[n]$ and define S as in Corollary 5.11. If an element $v \in S$ has the form $v = w(\lambda)$ for some partition $\lambda \vdash r$, then for all $u \in S$ satisfying $u \leq_I v$ we have $d_{u,v}^{L,M} > 0$.*

Proof. Specialize Equation (5.9) at $x_{L,M}$ and apply Proposition 5.10. □

6. Filtrations of the immanant space

It would be interesting to partition the bitableau basis or the dual canonical basis into blocks according to various properties of the basis elements and to study the subspaces spanned by each block. On the other hand, grouping basis elements by a natural property often leads to nested subsets. For instance, while Maschke's Theorem guarantees that the S_n-module $\mathcal{A}_{[n],[n]}$ defined by

$$s_i \circ \mathrm{Imm}_f(x) \underset{\mathrm{def}}{=} \mathrm{Imm}_f(s_i x) \tag{6.1}$$

may be decomposed as a direct sum of irreducible S_n-modules, neither the dual canonical basis nor the bitableau basis can be partitioned into blocks so that the subspace spanned by each block is an S_n-submodule of $\mathcal{A}_{[n],[n]}$. Nevertheless, it is possible to use either of these bases to form a filtration of nested S_n-submodules of $\mathcal{A}_{[n],[n]}$, as Clausen did in [9, Thm. 8.1 (a)]. Another particularly simple filtration, studied in [51], is coarser than that of Clausen. Let $S_{n,k}$ denote the subset of permutations $w \in S_n$ for which the one-line notation $w_1 \cdots w_n$ contains no decreasing subsequence of length $k + 1$. Equivalently, $S_{n,k}$ consists of all permutations in S_n whose shapes have k or fewer columns. Then we have the following [51, Thm. 2.4].

Theorem 6.1. *We have*

$$\mathrm{span}\{\mathrm{Imm}_w(x) \,|\, w \in S_{n,k}\} = \mathrm{span}\{R_w(x) \,|\, w \in S_{n,k}\}.$$

These subspaces, each of which is an S_n-module (see Corollary 6.5), form a filtration

$$\begin{aligned}(6.2)\quad 0 \subset \operatorname{span}\{\mathrm{Imm}_w(x) \,|\, w \in S_{n,1}\} \subset \operatorname{span}\{\mathrm{Imm}_w(x) \,|\, w \in S_{n,2}\} \subset \cdots \\ \subset \operatorname{span}\{\mathrm{Imm}_w(x) \,|\, w \in S_{n,n}\} = \mathcal{A}_{[n],[n]}\end{aligned}$$

of $\mathcal{A}_{[n],[n]}$. Note that $\operatorname{span}\{\mathrm{Imm}_w(x) \,|\, w \in S_{n,1}\}$ is just the one-dimensional space $\operatorname{span}\{\det(x)\}$. The next space has dimension $\frac{1}{n+1}\binom{2n}{n}$ and was studied in [49], where the elements $\{\mathrm{Imm}_w(x) \,|\, w \in S_{n,2}\}$ were related to the Temperley-Lieb algebra. For progress on the description of basis elements of $\operatorname{span}\{\mathrm{Imm}_w(x) \,|\, w \in S_{n,3}\}$, see [46].

Strengthening the result in Theorem 6.1, we will consider other families of subspaces which more finely decompose the immanant space $\mathcal{A}_{[n],[n]}$. We will characterize each subspace as the span of certain bitableaux, the span of certain dual canonical basis elements, and the set of all immanants vanishing on certain matrices. Because subspaces in each family are partially ordered by inclusion, we will call the families *partial filtrations.*

One partial filtration of $\mathcal{A}_{[n],[n]}$ consists of the subspaces $\{\mathcal{U}_v \,|\, v \in S_n\}$, defined by

$$\mathcal{U}_v = \mathcal{U}_v(x) = \operatorname{span}\{R_u(x) \,|\, u \leq_I v\}. \tag{6.3}$$

It is clear that we have $\mathcal{U}_v \subset \mathcal{U}_w$ if and only if $v \leq_I w$. Thus the partial ordering of these spaces is isomorphic to the iterated dominance order on S_n. While it is not in general true that $\mathcal{U}_v$ is an S_n-submodule of $\mathcal{A}_{[n],[n]}$, the elements of this subspace have certain common properties which make the definition of the subspace quite natural. In particular, each subspace $\mathcal{U}_v$ may be characterized in terms of Kazhdan-Lusztig immanants or vanishing properties as follows.

Theorem 6.2. *Fix a permutatition $v \in S_n$ and a function $f : S_n \to \mathbb{C}$. Then the following are equivalent.*

(1) $\mathrm{Imm}_f(x)$ *belongs to* $\mathcal{U}_v = \operatorname{span}\{R_u(x) \,|\, u \leq_I v\}$.
(2) $\mathrm{Imm}_f(x)$ *belongs to* $\operatorname{span}\{\mathrm{Imm}_u(x) \,|\, u \leq_I v\}$.
(3) $\mathrm{Imm}_f(A) = 0$ *for all matrices A in the set*

$$V \underset{\text{def}}{=} \{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \,|\, \mu_{[i]}(A) \not\preceq \mathrm{sh}(v_{[i]}) \text{ or } \nu_{[i]}(A) \not\preceq \mathrm{sh}(v^{-1}_{[i]}) \text{ for some } i\}.$$

Proof. ($1 \Leftrightarrow 2$) Follows immediately from Theorem 5.8 and Proposition 5.9.

($1 \Rightarrow 3$) Follows immediately from Theorem 4.10 and Proposition 4.11.

($3 \Rightarrow 1$) Assume that $\mathrm{Imm}_f(A) = 0$ for all matrices $A \in V$ and that $\mathrm{Imm}_f(x)$ does not belong to $\mathcal{U}_v$. Writing

$$\mathrm{Imm}_f(x) = \sum_{u \in S_n} c_u \mathrm{Imm}_u(x), \tag{6.4}$$

we therefore have that $c_u > 0$ for some $u \not\leq_I v$, and there exist numbers p, r such that $\eta_{p,r}(u) > \eta_{p,r}(v)$ or $\eta_{p,r}(u^{-1}) > \eta_{p,r}(v^{-1})$.

Suppose first that we have $\eta_{p,r}(u) > \eta_{p,r}(v)$. Following the proof of Theorem 5.8, let $q > \eta_{p,r}(v)$ be the maximum index for which the set $\{u \,|\, c_u > 0, \eta_{p,r}(u) = q\}$ is nonempty, let w be a Bruhat-minimal element of this set, and write

$$\mathrm{Imm}_f(x) = \sum_{\substack{u \\ \eta_{p,r}(u)<q}} c_u \mathrm{Imm}_u(x) + c_w \mathrm{Imm}_w(x) + \sum_{\substack{u \not\leq w \\ \eta_{p,r}(u)=q}} c_u \mathrm{Imm}_u(x). \tag{6.5}$$

Now define a matrix $B = B(w, \sigma_1, \dots, \sigma_p, I_1, \dots, I_p)$ as before Lemma 5.7. Since the first p parts of $\mu_{[r]}(B)$ sum to q, which is greater than the sum $\eta_{p,r}(v)$ of the first p parts of $\mathrm{sh}(v_{[r]})$, we have $\mu_{[r]}(B) \not\preceq \mathrm{sh}(v_{[r]})$. Thus, B belongs to V and we have $\mathrm{Imm}_f(B) = 0$. Substituting $x = B$ in Equation (6.5), we therefore may apply Lemma 5.7 to obtain $c_w = 0$.

Now suppose that we have $\eta_{p,r}(u^{-1}) > \eta_{p,r}(v^{-1})$. Let $q > \eta_{p,r}(v^{-1})$ be the maximum index for which the set $\{u \,|\, c_u > 0, \eta_{p,r}(u^{-1}) = q\}$ is nonempty, let w be a Bruhat-minimal element of this set, and use Proposition 3.1 to write the variation

$$\mathrm{Imm}_f(x) = \sum_{\substack{u \\ \eta_{p,r}(u^{-1})<q}} c_u \mathrm{Imm}_{u^{-1}}(x^\top) + c_w \mathrm{Imm}_{w^{-1}}(x^\top) + \sum_{\substack{u \\ u^{-1} \not\leq w^{-1} \\ \eta_{p,r}(u^{-1})=q}} c_u \mathrm{Imm}_{u^{-1}}(x^\top) \tag{6.6}$$

of (6.5). Now define a matrix $B = B(w^{-1}, \sigma_1, \dots, \sigma_p, I_1, \dots, I_p)$ as before Lemma 5.7. Since the first p parts of $\mu_{[r]}(B)$ sum to q, which is greater than the sum $\eta_{p,r}(v^{-1})$ of the first p parts of $\mathrm{sh}(v^{-1}_{[r]})$, we have $\nu_{[r]}(B^\top) \not\preceq \mathrm{sh}(v^{-1}_{[r]})$. Thus, $B^\top$ belongs to V and we have $\mathrm{Imm}_f(B^\top) = 0$. Substituting $x = B^\top$ in Equation (6.6), we may again apply Lemma 5.7 to obtain the contradiction $c_w = 0$. ☐

As a consequence, we see that dual canonical basis elements vanish on matrices of sufficiently low rank.

Corollary 6.3. *If* $\mathrm{rank}(A) < k$ *and* w *has an increasing subsequence of length* k*, then* $\mathrm{Imm}_w(A) = 0$*.*

Proof. Suppose w has an increasing subsequence of length k. Then $\mathrm{sh}(w)$ has at least k rows. By Theorem 6.2, $\mathrm{Imm}_w(x)$ is equal to a linear combination of standard bitableaux $R_v(x)$ satisfying $v \leq_I w$ and therefore $\mathrm{sh}(v) \preceq \mathrm{sh}(w)$. For each of these basis elements, $\mathrm{sh}(v)$ has at least k rows, and Proposition 4.12 therefore implies that we have $R_v(A) = 0$. ☐

For certain permutations v, the result of Theorem 6.2 can be simplified considerably. In particular, for each partition $\lambda \vdash n$, let $w(\lambda)$ be the permutation defined in Equation (5.5), and denote the space $\mathcal{U}_{w(\lambda)}$ by $\mathcal{U}_\lambda$. These subspaces $\{\mathcal{U}_\lambda \,|\, \lambda \vdash n\}$ thus form a partial filtration of $\mathcal{A}_{[n],[n]}$ which is coarser than that defined in (6.3). It is clear that we have $\mathcal{U}_\lambda \subset \mathcal{U}_\mu$ if and only if $\lambda \preceq \mu$. Thus the partial ordering of these spaces is isomorphic to the dominance order. By Theorem 5.3, it is clear that $\mathcal{U}_\lambda$ is equal to the span of all injective bitableaux of shape dominated by or equal to λ. (See also [9, Sec. 3], [10, Sec. 2], and references there.) The following result gives several alternative characterizations of this space.

Theorem 6.4. *Fix a partition $\lambda \vdash n$ and a function $f : S_n \to \mathbb{C}$. Then the following are equivalent.*

(1) $\mathrm{Imm}_f(x)$ *belongs to* $\mathcal{U}_\lambda$.
(2) $\mathrm{Imm}_f(x)$ *belongs to* $\mathrm{span}\{R_v(x) \,|\, \mathrm{sh}(v) \preceq \lambda\}$.
(3) $\mathrm{Imm}_f(x)$ *belongs to* $\mathrm{span}\{\mathrm{Imm}_v(x) \,|\, \mathrm{sh}(v) \preceq \lambda\}$.
(4) $\mathrm{Imm}_f(A) = 0$ *for all $n \times n$ matrices A satisfying $\mu_{[n]}(A) \not\preceq \lambda$ or $\nu_{[n]}(A) \not\preceq \lambda$.*

Proof. ($1 \Leftrightarrow 2 \Leftrightarrow 3$) Follows immediately from Observation 5.4 and Theorem 6.2.

($1 \Rightarrow 4$) Follows immediately from Theorem 4.10 and Proposition 4.11.

($4 \Rightarrow 1$) Assume that $\mathrm{Imm}_f(A) = 0$ for all $n \times n$ matrices A satisfying $\mu_{[n]}(A) \not\preceq \lambda$ or $\nu_{[n]}(A) \not\preceq \lambda$ and suppose that $\mathrm{Imm}_f(x)$ does not belong to $\mathcal{U}_\lambda$. By Theorem 6.2, there exists a matrix A and an index i such that $\mathrm{Imm}_f(A) \neq 0$ and we have

$$\mu_{[i]}(A) \not\preceq \mathrm{sh}(w_{[i]}) \text{ or } \nu_{[i]}(A) \not\preceq \mathrm{sh}(w_{[i]}) = \mathrm{sh}(w_{[i]}^{-1}).$$

Suppose that $\mu_{[i]}(A) \not\preceq \mathrm{sh}(w_{[i]})$, and let j be the least index for which

$$k \underset{\mathrm{def}}{=} \mu_{[i]}(A)_1 + \cdots + \mu_{[i]}(A)_j$$

is greater than $\mathrm{sh}(w_{[i]})_1 + \cdots + \mathrm{sh}(w_{[i]})_j$. Clearly we have $\mu_{[n]}(A)_1 + \cdots + \mu_{[n]}(A)_j \geq k$. On the other hand, all of the numbers $k, \ldots, n$ appear after row j in the tableau $U(\lambda)$. This implies that $\mu_{[n]}(A) \not\preceq \mathrm{sh}(w) = \lambda$ and therefore that $\mathrm{Imm}_f(A) = 0$, a contradiction.

We must therefore have $\nu_{[i]}(A) \not\preceq \mathrm{sh}(w_{[i]})$. Now redefine j and k so that j is the least index for which

$$k \underset{\mathrm{def}}{=} \nu_{[i]}(A)_1 + \cdots + \nu_{[i]}(A)_j > \mathrm{sh}(w_{[i]})_1 + \cdots + \mathrm{sh}(w_{[i]})_j.$$

As before, we clearly have $\nu_{[n]}(A)_1 + \cdots + \nu_{[n]}(A)_j \geq k$, while the numbers $k, \ldots, n$ appear after row j in the tableau $U(\lambda)$. This implies that $\nu_{[n]}(A) \not\preceq \mathrm{sh}(w) = \lambda$, and again we have the contradiction $\mathrm{Imm}_f(A) = 0$. □

Since the action (6.1) of S_n maps an injective bitableau to another injective bitableau of the same shape, we may use Theorem 5.3 to see that each space $\mathcal{U}_\lambda$ is an S_n-module. Furthermore, we see from Corollary 6.3 that for any partition λ having k parts, all elements of $\mathcal{U}_\lambda$ vanish on matrices of rank less than or equal to $k-1$. The converse is not true. (See Proposition 6.6 and [44, p. 83].)

Now we return to the filtration (6.2). The set $S_{n,k}$ induces a subposet of the iterated dominance order on S_n, and this subposet has a unique maximal element w. Letting $\lambda(n, k)$ be the unique partition of n having $\lfloor \frac{n}{k} \rfloor$ parts equal to k and at most one part equal to the residue of n modulo k, we have that w is equal to the permutation $w(\lambda(n, k))$ defined in Equation (5.5). Thus the filtration (6.2) is coarser still than that defined immediately before Theorem 6.4, and does consist entirely of S_n-modules. The following result follows easily from Theorems 6.2-6.4 and provides a converse to [51, Prop. 2.2].

Corollary 6.5. *Fix an index $k \leq n$ and a function $f : S_n \to \mathbb{C}$. Then the following are equivalent.*

(1) $\mathrm{Imm}_f(x)$ *belongs to* $\mathrm{span}\{R_v(x) \,|\, v \in S_{n,k}\} = \mathrm{span}\{\mathrm{Imm}_v(x) \,|\, v \in S_{n,k}\}$.
(2) $\mathrm{Imm}_f(x)$ *belongs to* $\mathcal{U}_{w(\lambda(n,k))} = \mathcal{U}_{\lambda(n,k)}$.
(3) $\mathrm{Imm}_f(A) = 0$ *for all $n \times n$ matrices A having $k+1$ equal rows or $k+1$ equal columns.*

Again using Corollary 6.3, we have that all elements of $\mathcal{U}_{\lambda(n,k)}$ vanish on matrices of rank less than $\lceil \frac{n}{k} \rceil$.

The characterizations of subspaces of immanants in Theorem 6.2-Corollary 6.5 can aid in relating elements of $\mathcal{A}_{[n],[n]}$ to the bitableau and dual canonical bases, and in describing their zero sets. As an application, consider the *irreducible character immanants* defined [34, p. 81] by

$$\mathrm{Imm}_\lambda(x) \underset{\mathrm{def}}{=} \sum_{w \in S_n} \chi^\lambda(w) x_{1,w_1} \cdots x_{n,w_n},$$

where $\chi^\lambda : S_n \to \mathbb{C}$ is the irreducible S_n-character corresponding to λ. (See, e.g., [52].) Merris [44, Cor. 4] proved that $\mathrm{Imm}_\lambda(A) = 0$ whenever $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and A has more than λ_1 equal rows (or columns). We strengthen this result as follows.

Proposition 6.6. *Fix $\lambda \vdash n$ and an $n \times n$ matrix A. If $\mu_{[n]}(A) \not\preceq \lambda$ or $\nu_{[n]}(A) \not\preceq \lambda$, then* $\mathrm{Imm}_\lambda(A) = 0$*.*

Proof. Let $\epsilon^\lambda : S_n \to \mathbb{C}$ be the S_n-character induced from the sign character of any Young subgroup S_λ of S_n. (See, e.g., [52].) Merris and Watkins showed [45, Eq. (27)] that Littlewood's results [34, Sec. 6.5] imply the identity

$$\mathrm{Imm}_{\epsilon^\mu}(x) = \sum_T (T{:}T)(x),$$

where the sum is over all column-strict injective tableaux T of shape $\mu^\top$. Thus by Theorem 5.3, $\mathrm{Imm}_{\epsilon^\mu}(x)$ belongs to $\mathcal{U}_{\mu^\top}$. Expressing irreducible characters in terms of the induced characters, we have

$$\chi^\lambda = \sum_{\mu \succeq \lambda^\top} K^{-1}_{\mu,\lambda^\top} \epsilon^\mu, \qquad \mathrm{Imm}_\lambda(x) = \sum_{\mu \succeq \lambda^\top} K^{-1}_{\mu,\lambda^\top} \mathrm{Imm}_{\epsilon^\mu}(x),$$

where $\{K^{-1}_{\mu,\lambda^\top} \mid \lambda, \mu \vdash n\}$ are the *inverse Kostka numbers*. (See [3].) Thus $\mathrm{Imm}_\lambda(x)$ belongs to $\mathcal{U}_\lambda$ and the result now follows from Theorem 6.4. $\square$

Since the inverse Kostka numbers satisfy $K^{-1}_{\lambda,\lambda^\top} = 1$, the proof of Proposition 6.6 implies that $\mathrm{Imm}_\lambda(x)$ belongs to $\mathcal{U}_\mu$ only if $\lambda \preceq \mu$. We remark also that Merris proved in [44, Cor. 5], [45, Thm. 8] that $\mathrm{Imm}_\lambda(A) = 0$ whenever $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mathrm{rank}(A) < \ell$. The second of these proofs is essentially the same as that of Proposition 4.12.

The characterizations of subspaces of immanants in Theorem 6.2-Corollary 6.5 also expose properties of the multigrading (2.1) of $\mathbb{C}[x]$. Given subsets I, J, of $[n]$, define $\overline{I} = [n] \smallsetminus I$ and $\overline{J} = [n] \smallsetminus J$. If $|I| = |J| = k$, then each of the rings $\mathbb{C}[x_{I,J}]$, $\mathbb{C}[x_{\overline{I},\overline{J}}]$ has an immanant space, and it is clear that multiplication in $\mathbb{C}[x]$ satisfies

$$\mathcal{A}_{[k],[k]}(x_{I,J})\mathcal{A}_{[n-k],[n-k]}(x_{\overline{I},\overline{J}}) \subset \mathcal{A}_{[n],[n]}(x).$$

Furthermore, the dominance filtrations of these three immanant spaces enjoy a similar relationship which generalizes the result in [51, Cor. 3.1]. Given partitions λ, μ, define the partition $\lambda + \mu$ to be the componentwise sum $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$.

Proposition 6.7. *Let I, J be k-element subsets of $[n]$ and fix partitions $\lambda \vdash k$, $\mu \vdash (n-k)$. Then we have*

$$\mathcal{U}_\lambda(x_{I,J})\mathcal{U}_\mu(x_{\overline{I},\overline{J}}) \subset \mathcal{U}_{\lambda+\mu}(x).$$

Proof. Observe that each element of $\mathcal{U}_\lambda(x_{I,J})$ is a linear combination of products of minors whose sizes are given by a partition $\kappa \succeq \lambda^\top$ of k, while each element of $\mathcal{U}_\mu(x_{\overline{I},\overline{J}})$ is a linear combination of products of minors whose sizes are given by a partition $\nu \succeq \mu^\top$ of $n-k$. It follows that each element $\mathrm{Imm}_f(x)$ of $\mathcal{U}_\lambda(x_{I,J})\mathcal{U}_\mu(x_{\overline{I},\overline{J}})$ is a linear combination of products of minors whose sizes are a multiset union $\rho = \kappa \Cup \nu$ for some κ, ν satisfying the above conditions. Since $I \cap \overline{I}$ and $J \cap \overline{J}$ are both empty, each such product of minors is an injective bitableau.

Furthermore, the conditions $\kappa \succeq \lambda^\top$ and $\nu \succeq \mu^\top$ imply that each partition ρ occurring above satisfies $\rho \succeq \lambda^\top \Cup \mu^\top$, or equivalently, $\rho^\top \preceq \lambda + \mu$. Thus we have

$$\mathrm{Imm}_f(x) \in \mathrm{span}\{(T{:}U)(x) \mid \mathrm{sh}(T) \preceq \lambda + \mu\} = \mathcal{U}_{\lambda+\mu}(x).$$

$\square$

We remark that multiplication of elements in the above spaces sometimes yields a factorization of the form

$$\mathrm{Imm}_u(x_{I,J})\mathrm{Imm}_v(x_{\overline{I},\overline{J}}) = \mathrm{Imm}_w(x). \tag{6.7}$$

These factorizations have been characterized in [51, Thm. 3.2].

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