A Tight Approximation for an EOQ Model with Supply Disruptions

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Abstract

We consider a continuous-review inventory model for a firm that faces deterministic demand but whose supplier experiences random disruptions. The supplier experiences "wet" and "dry" (operational and disrupted) periods whose durations are exponentially distributed. The firm follows an EOQ-like policy during wet periods but may not place orders during dry periods; any demands occurring during dry periods are lost if the firm does not have sufficient inventory to meet them.

This paper introduces a simple but effective approximation for this model that maintains the tractability of the classical EOQ and permits analysis similar to that typically performed for the EOQ. We provide analytical and numerical bounds on the approximation error in both the cost function and the optimal order quantity. We prove that the optimal power-of-two policy has a worst-case error bound of 6%. Finally, we demonstrate numerically that the results proved for the approximate cost function hold, at least approximately, for the original exact function.

Keywords: inventory, supply disruptions, EOQ, approximations, power-of-two policies

1 Introduction

Despite the careful attention paid to inventory planning in a supply chain, supply disruptions are inevitable. Disruptions may come from a variety of sources, including labor actions, machine breakdowns, and natural or man-made disasters. Recent high-profile events—including hurricanes Katrina and Rita in 2005 (Barrionuevo and Deutsch 2005), the west-coast port lockout in 2002 (Greenhouse 2002), and the Taiwan earthquake in 1999 (Burrows 1999)—have called attention to the impact of major disruptions on supply chain operations. Just as important, however, are smaller-scale disruptions that occur more frequently. For example, Wal-Mart's emergency operations center receives a distress call from one of its stores or distribution centers nearly every day (Leonard 2005). The model presented in this paper is applicable to either large or small disruptions, provided that the disruption and recovery rates are reasonably stationary over time.

Firms have a range of strategies for managing disruptions (see, e.g., Tomlin 2006). Our focus in this paper is on the use of inventory to mitigate the impact of disruptions. Inventory managers who ignore the risk of supply disruptions will encounter excess costs when disruptions occur, in the form of stockout costs, expediting costs, and loss of goodwill. On the other hand, disruptions at a given location are typically relatively infrequent, so holding too much extra inventory is costly, as well. An effective inventory policy should strike a balance between protecting against stockouts during disruptions and maintaining low inventory levels.

We examine a model for setting order quantities in a continuous-review inventory system managed by a retailer who faces deterministic demand and random supply disruptions. (We use the term "retailer" throughout, though of course the model applies equally well to other types of firms.) The durations of the supplier's "wet" and "dry" (operational and disrupted) periods are exponentially distributed. Orders cannot be placed during dry periods, and demands occurring during dry periods are lost if the retailer does not have sufficient inventory to meet them. We refer to this problem as the *economic order quantity with disruptions* (EOQD). The EOQD was first introduced by Parlar and Berkin (1991), whose model was shown by Berk and Arreola-Risa (1994) to be incorrect. Although Berk and Arreola-Risa's corrected model can be optimized numerically using efficient line-search techniques, it cannot be solved in closed form, as ours can.

Closed-form solutions are attractive for two main reasons. First, they allow researchers to develop analytical results that are unattainable for models that must be solved numerically. For example, classical results about the EOQ model, such as the equality of the average ordering and holding costs at optimality, the famous sensitivity analysis result, and the impact of changes in the problem parameters on the optimal solution, depend on the availability of closed-form expressions for the optimal order quantity and its cost.

Second, simple models such as the EOQ and EOQD are rarely implemented as standalone models; rather, they serve as building blocks for richer and more complex models. Formulations of the more complex models often require closed-form expressions for the simple models. For example, Roundy's celebrated bound for power-of-2 policies in a one-warehouse, multi-retailer system (Roundy 1985) depends on having a closed-form expression for the optimal EOQ cost. Similarly, a recent joint location–inventory model (Daskin, Coullard and Shen 2002, Shen, Coullard and Daskin 2003) embeds the cost of the optimal (Q, R) inventory policy into the objective function of a facility location model. Since no closed-form expression is known for this cost, they approximate it using the EOQ cost plus the cost of safety stock, for which the optimal costs are known. Their approximation obviates the need for explicit inventory variables and permits a compact formulation and an effective algorithm. A similar approach is taken by Qi, Shen and Snyder (2008), who embed a variant of the approximate model presented in this paper into a location–inventory framework with unreliable suppliers; see Section 5 below for more details.

This paper makes the following contributions to the literature on inventory management under the threat of supply disruptions. We present a cost function that closely approximates the EOQD cost function of Berk and Arreola-Risa (1994). Our approximate cost function is convex and can be solved in closed form. We prove analytical error bounds on the approximate solution and its cost (versus the exact model). We demonstrate that the approximation shares several important properties with the classical EOQ model, proving a simple linear relationship between the optimal order quantity and cost, monotonicity and convexity properties of the optimal cost with respect to the inputs, a simple sensitivity-analysis formula, and a worst-case bound of 6% for power-of-two policies. Finally, we perform an extensive numerical study to demonstrate the quality of the approximation, identify instances in which the approximation is likely to perform poorly, and demonstrate that many of our analytical results hold, at least approximately, for the original, exact model.

The remainder of this paper is structured as follows. In Section 2, we provide a review of the literature on inventory models with supply disruptions. In Section 3, we introduce the model, our approximate cost function, and its optimal solution. We prove analytical bounds on the approximation error in the cost function and the optimal solution in Section 4 and additional properties in Section 5. In Section 6, we discuss sensitivity analysis and power-of-two policies.

Our computational results are detailed in Section 7. Finally, in Section 8, we draw conclusions from our analysis and suggest future research directions. Proofs of all lemmas, theorems, etc. are provided in the Appendix.

2 Literature Review

Supply uncertainty takes the form of either *yield uncertainty*, in which supply is always available but the quantity delivered is a random variable (see, e.g., Yano and Lee 1995), or *disruptions*, in which the supplier experiences failures during which it cannot provide any product. This paper is concerned with disruptions. (Disruptions may be considered as a special case of random yield in which the yield variable is Bernoulli; however, most random yield models assume continuous random variables and are not immediately applicable to disruptions.)

The earliest paper to consider supply disruptions seems to be that of Meyer, Rothkopf and Smith (1979), who consider a production facility facing constant, deterministic demand. The facility has a capacitated storage buffer, and the production process is subject to stochastic failures and repairs. The goal of the paper is not to optimize the system but to compute the percentage of time that demands are met. The optimization of such finite-production-rate systems has been considered by a number of subsequent authors (e.g., Hu 1995, Moinzadeh and Aggarwal 1997, Liu and Cao 1999, Abboud 2001).

Parlar and Berkin (1991) introduce the first of a series of models that incorporate supply disruptions into classical inventory models. They study the EOQD: an EOQ-like system in which the supplier experiences intermittent failures. Demands are lost if the retailer has insufficient inventory to meet them during supplier failures. The retailer follows a zero-inventory ordering (ZIO) policy. Their cost function was shown to be incorrect in two respects by Berk and Arreola-Risa (1994), who propose a corrected cost function. It is their function that we approximate in this paper.

Weiss and Rosenthal (1992) derive the optimal ordering quantity for a similar EOQ-based system in which a disruption to either supply or demand is possible at a single point in the future. This point is known but the disruption duration is random. Parlar and Perry (1995) extend the EOQD by relaxing the ZIO assumption, by making the time between order attempts a decision variable (assuming a non-zero cost to ascertain the state of the supplier), and by considering both random and deterministic yields. (The ZIO assumption was also considered by Bielecki and Kumar (1988), who found that, under certain modeling assumptions, a ZIO policy may be optimal even in the face of supply disruptions, countering the common view that if any uncertainty exists, it is optimal to hold some safety stock to buffer against it.) Parlar and Perry (1996) consider the EOQD with one, two, or multiple suppliers and non-zero reorder points. They show that if the number of suppliers is large, the problem reduces to the classical EOQ. The suppliers are non-identical with respect to reliability but identical with respect to price, so as long as at least one supplier is active, the retailer does not care which one it orders from. Gürler and Parlar (1997) generalize the two-supplier model by allowing more general failure and repair processes. They present asymptotic results for large order quantities.

Given the complexities introduced by supply disruptions, only a few papers have considered stochastic demand, as well. Gupta (1996) formulates a (Q, R)-type model with Poisson demand and exponential wet and dry periods. Parlar (1997) studies a similar but more general model than Gupta—for example, allowing for stochastic lead times—but formulates an approximate cost function. Mohebbi (2003, 2004) extends Gupta's model to consider compound Poisson demand and stochastic lead times; he derives expressions for the inventory level distribution and expected cost, both of which must be evaluated numerically except in the special case in which demand sizes are exponentially distributed. Chao (1987) and Chao, et al. (1989) consider stochastic demand for electric utilities with market disruptions and solve the problem using stochastic dynamic programming.

Periodic-review inventory models with supply disruptions have received somewhat less attention in the literature than their continuous-review counterparts. Arreola-Risa and DeCroix (1998) develop exact expressions for (s, S) models with supplier disruptions but use numerical optimization since analytical solutions cannot be obtained. Song and Zipkin (1996) present a model in which the availability of the supplier, while random, is partially known to the decision maker. They prove that a state-dependent base-stock policy is optimal (for linear order costs) and solve the model using dynamic programming. Tomlin (2006) explores a range of strategies for coping with supply disruptions, including the use of inventory, routine dual sourcing, and emergency dual sourcing; he characterizes settings in which each strategy is optimal. Tomlin and Snyder (2006) consider a "threat-advisory" system in which the disruption risk is non-stationary and the firm has some indication of the current threat level; they examine the benefit of such a system and the effect that it has on the optimal disruption-management strategy.

A special case of Tomlin's (2006) model is a periodic-review base-stock system with supply disruptions and deterministic demand. Tomlin provides a simple, intuitive formula for the optimal base-stock level for this system; this formula is also closely related to a formula by Güllü, Onol and Erkip (1997). Schmitt, Snyder and Shen (2007) prove several properties of this system and provide an approximation for such systems with stochastic demand.

Chopra, Reinhardt and Mohan (2007) consider a newsvendor facing both supply disruptions and yield uncertainty in a single-period setting. They examine the error inherent in "bundling" the two sources of supply risk; i.e., acting as though the disruptions are simply a manifestation of yield uncertainty. Schmitt and Snyder (2007) extend their analysis to the infinite-horizon case and show that the effect of bundling can be quite different in single-period and infinite-horizon settings.

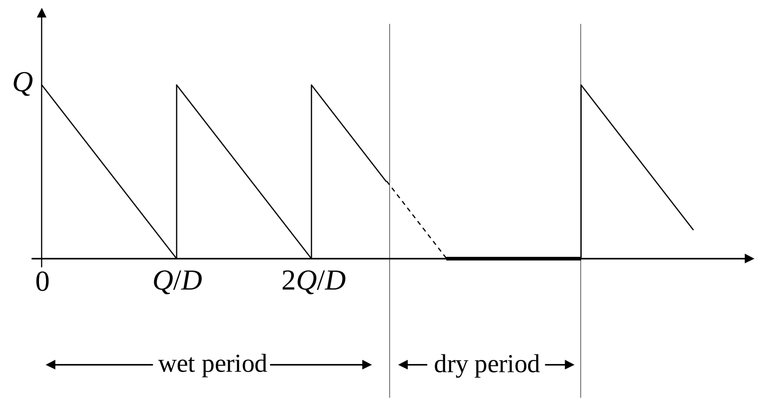
Most of the papers cited in this section propose a numerical approach for optimizing their cost functions—few are solved in closed form. In contrast, the approximate cost function proposed in this paper may be solved in closed form, and as a consequence, a number of analytical results may be derived for it. Our model has been extended by several authors, including Heimann and Waage (2006), who relax the ZIO assumption; Ross, Rong and Snyder (2008), who consider non-stationary demand and disruption parameters; Qi, Shen and Snyder (2007), who consider disruptions at the retailer as well as the supplier; and Qi et al. (2008), who use the model of Qi et al. (2007) in a joint location–inventory context.

3 Model Formulation

3.1 Original Model

Consider an EOQ model under continuous review with fixed ordering cost K, holding cost h per unit per year, and constant, deterministic demand rate D units per year. (Without loss of generality we assume that the time unit is one year.) Suppose that the supplier is not perfectly reliable—that it functions normally for a certain duration (called a "wet period") and then shuts down for a certain duration (a "dry period"). During dry periods, no orders can be placed, and if the retailer runs out of inventory during a dry period, all demands observed until the beginning of the next wet period are lost, with a stockout cost of p per lost sale. The durations of both wet and dry periods are exponentially distributed, with rates λ and μ, respectively. Every order placed by the retailer is for the same quantity, Q, orders are only placed when the

Figure 1: EOQ inventory curve with disruptions.



inventory level reaches 0, and orders placed during wet periods are received immediately (there is no lead time). The goal of the model is to choose Q to minimize the expected annual cost. We refer to this problem as the *economic order quantity with disruptions* (EOQD).

A typical inventory curve is pictured in Figure 1. Note that the inventory position never becomes negative since unmet demands are lost.

The EOQD was first formulated by Parlar and Berkin (1991), whose expected cost function was shown by Berk and Arreola-Risa (1994) to be incorrect in two respects. Berk and Arreola-Risa derive the following corrected expression for the expected annual cost as a function of Q:

$$g_0(Q) = \frac{K + hQ^2/2D + Dp\beta_0(Q)/\mu}{Q/D + \beta_0(Q)/\mu} \tag{1}$$

where

$$\beta_0(Q) = \frac{\lambda}{\lambda + \mu}\left(1 - e^{-(\lambda+\mu)Q/D}\right) \tag{2}$$

is the probability that the supplier is in a dry period when the retailer's inventory level reaches 0. We will often suppress the argument Q in $\beta_0(Q)$ when it is clear from the context.

The first-order condition $dg_0/dQ = 0$ cannot be solved in closed form because it has the functional form

$$\alpha_1 Q^2 + \alpha_2 Q + \alpha_3 + (\alpha_4 Q^2 + \alpha_5 Q + \alpha_6)e^{-\alpha_7 Q} = 0,$$

for suitable constants α_i, for which no closed-form solution is readily available. (The first-order condition is written out explicitly in equation (17) in our Appendix.) Moreover, Berk and Arreola-Risa prove that $g_0(Q)$ is unimodal (i.e., quasiconvex), but it is not known whether it is convex.

3.2 Assumptions

Before introducing our approximation to (1), we impose three mild assumptions on the problem parameters. First, we assume that all costs and other problem parameters are non-negative. Second, we assume that $\lambda < \mu$, that is, wet periods last longer on average than dry periods.

Third, we assume that $\sqrt{2KDh} < pD$. If there were no disruptions, this model would reduce to the classical EOQ model, whose optimal annual cost is well known to equal $\sqrt{2KDh}$ (see, e.g., Zipkin 2000). Therefore $\sqrt{2KDh}$ is a lower bound on the optimal cost of the system with disruptions. One feasible solution for the EOQD is for the retailer never to place an order and instead to stock out on every demand; the annual cost of this strategy is pD. Therefore, the assumption that $\sqrt{2KDh} < pD$ is meant to prohibit the situation in which it is more expensive to serve demands than to lose them.

For convenience, we define $g_E(Q) = \frac{KD}{Q} + \frac{hQ}{2}$, the classical EOQ cost function.

3.3 Approximation

We propose approximating Berk and Arreola-Risa's cost function by replacing $\beta_0(Q)$ with

$$\beta = \frac{\lambda}{\lambda + \mu} r \tag{3}$$

for a constant $0 < r \leq 1$. The resulting approximate cost function is

$$g(Q) = \frac{K + hQ^2/2D + Dp\beta/\mu}{Q/D + \beta/\mu} = \frac{h\mu Q^2/2 + KD\mu + D^2 p\beta}{Q\mu + \beta D}. \tag{4}$$

Note that the functional form of this cost function,

$$\frac{aQ^2 + b}{cQ + d}, \tag{5}$$

is similar to that of the EOQ cost function, $\frac{aQ^2+b}{cQ}$. This similarity in structure gives rise to many of the EOQ-like properties derived in Sections 5 and 6. Indeed, many of the results in this paper hold (with appropriate modifications) for any cost function of the form given in (5).

The first term in $\beta_0(Q)$, $\lambda/(\lambda + \mu)$, is the steady-state probability that the supplier is in a dry period, while the second term, $1 - \exp(-(\lambda + \mu)Q/D)$, accounts for the knowledge that when the inventory level hits 0, we were in a wet period as recently as Q/D time units ago. Our approximation replaces this exponential term by a constant r that is independent of Q. In the special case in which $r = 1$, the approximation ignores the recent history of the system state and assumes that the system is already in steady state when each order attempt is made.

In general, one should set r close to 1 if the Markov process that governs disruptions and recoveries reaches steady state quickly relative to Q/D (the time between order attempts), and to a smaller value otherwise. (By "steady-state" we mean that the probability of the system being in a given state at time $t + \Delta t$ is roughly equal to the steady-state probability, and is roughly independent of the system state at time t.) The Markov process reaches steady state quickly relative to Q/D if state transitions occur frequently (i.e., if λ and/or μ are large) or if Q is large or D is small.

Ideally, one would set $r = 1 - \exp(-(\lambda + \mu)Q_0/D)$, where Q_0 is the optimal order quantity for the exact model (i.e., Q_0 minimizes $g_0(Q)$), but of course this is not practical since Q_0 is not known *a priori*. In Section 7.2.1, we test a range of r values and find that $r = 1.0$ is quite robust, performing well for a wide range of instances. If λ and μ are small or D is large, or if Q is likely to be small because K is small or h is large, then one might use a smaller value of r (or a larger value in the opposite case).

A slightly more sophisticated approach would set $r = 1 - \exp(-(\lambda + \mu)Q/D)$ using a value of Q obtained using some heuristic procedure, for example, using the EOQ model. Alternately, one could set r to some initial value, say 1.0, then use the optimal Q^* given in Theorem 2 below to obtain a more accurate value for r. However, the disadvantage of letting r depend on the parameter values is that it may destroy some of the theoretical properties (e.g., convexity/concavity with respect to the parameters) proved below. In addition, algorithms that depend on a closed-form expression for Q^* may not accommodate the extra step of computing r endogenously. For example, the model by Qi et al. (2008) requires the optimal inventory cost to be concave with respect to the demand D, which is computed endogenously; r must be a constant and may not also be a function of this endogenous D.

We suggest using $r = 1.0$ in general, and deviating from this value only if Q is likely to be very small relative to D or if transitions between wet and dry states occur very infrequently.

Although Berk and Arreola-Risa assume exponentially distributed wet and dry period durations, other distributions would yield similar cost functions, with the term $1-\exp(-(\lambda+\mu)Q/D)$ replaced by a distribution-specific term. Our approximation is applicable to these cases, as well, with the quality of the approximation determined by the rate with which the system approaches steady-state.

One would expect that as the supplier's reliability improves, the EOQD begins to resemble the EOQ more and more closely. In particular, as λ gets small or μ gets large (so that wet periods last much longer than dry periods), g approaches the classical EOQ cost function, as Proposition 1 demonstrates. The proof is omitted; it follows from the fact that as $\lambda/\mu \to 0$, $\beta \to 0$.

Proposition 1

$$\lim_{\lambda/\mu \to 0} g(Q) = g_E(Q),$$

where $g_E(Q) = \frac{KD}{Q} + \frac{hQ}{2}$ *is the classical EOQ cost function.*

The same result holds for Berk and Arreola-Risa's g_0, though it does not hold for Parlar and Berkin's original (incorrect) cost function.

3.4 Optimal Solution

In this section we show that our approximate cost function g is convex and provide a closed-form solution for the optimal value of Q, denoted Q^*. All proofs are given in the Appendix.

Theorem 2 *(a)* $g(Q)$ *is convex in* Q

(b) The value of Q *that minimizes* $g(Q)$ *is given by*

$$Q^* = \frac{\sqrt{(\beta D h)^2 + 2h\mu(KD\mu + D^2 p\beta)} - \beta D h}{h\mu}. \tag{6}$$

Note that Q^* can be rewritten as

$$Q^* = \sqrt{\frac{2KD}{h} + a^2 + b} - a$$

for appropriate constants a and b, emphasizing the relationship between Q^* and the optimal order quantity for the classical EOQ, $\sqrt{2KD/h}$.

4 Accuracy of Approximation

4.1 Accuracy of Cost Function

In this section, we discuss the accuracy of g as an approximation for g_0. Our first result provides a simple characterization of the instances in which $g(Q)$ overestimates $g_0(Q)$, i.e., in which the approximation is conservative.

Proposition 3 *(a)* $g(Q) \geq g_0(Q)$ *if and only if* either $\beta \geq \beta_0(Q)$ *and* $g_E(Q) \leq Dp$ or $\beta \leq \beta_0(Q)$ *and* $g_E(Q) \geq Dp$*. Equality holds if and only if* $\beta = \beta_0(Q)$ *or* $g_E(Q) = Dp$*.*

(b) $g(Q^*) \geq g_0(Q^*)$ *if and only if* $\beta \geq \beta_0(Q^*)$. *Equality holds if and only if* $\beta = \beta_0(Q^*)$.

(Note that if $r = 1$, then $\beta > \beta_0(Q)$ for all Q, simplifying the assumptions in the "if and only if" statements.) The condition in part (a) of Proposition 3 holds for any Q for which it is cheaper for the firm to use an order quantity of Q than to stock out on every demand. Typically, this encompasses quite a wide range of Q values. Part (b) of the proposition confirms that the optimal Q is in the critical range.

Next we show that $g(Q)$ does not deviate from $g_0(Q)$ by too much by proving a worst-case bound on the magnitude of the error. This bound holds for the case of $Q = Q^*$; part (b) of the theorem also provides another, sometimes tighter, bound for this case.

Theorem 4 *(a) For all* $Q > 0$ *such that* $g_E(Q) < Dp$,

$$\frac{|g(Q) - g_0(Q)|}{g_0(Q)} < \frac{|\beta - \beta_0(Q)|}{\beta_0(Q)} \left[1 - \frac{g_E(Q)}{Dp}\right] < \frac{|\beta - \beta_0(Q)|}{\beta_0(Q)}.$$

(b) If $g_E(Q^*) < Dp$, *then*

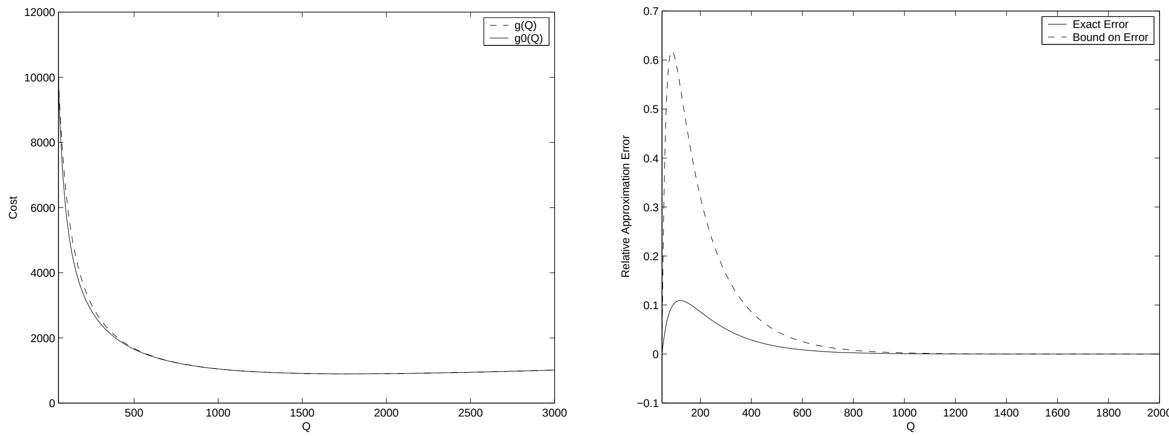
$$\frac{|g(Q^*) - g_0(Q^*)|}{g_0(Q^*)} < \min\left\{\frac{|\beta - \beta_0(Q^*)|}{\beta_0(Q^*)} \left[1 - \frac{g_E(Q^*)}{Dp}\right], \frac{|\beta - \beta_0(Q^*)|}{\beta + \beta_0(Q^*)}\right\} < 1.$$

(c) Either bound in the min *in part (b) may prevail.*

The bound in Theorem 4(a) does not have a fixed worst-case value, since $\beta_0(Q) \to 0$ as $(\lambda + \mu)Q/D \to 0$. Theorem 4(b) does establish a fixed worst-case bound of 1 on the approximation error for $g(Q^*)$. However, for reasonable values of the parameters, both bounds are much smaller, as demonstrated numerically in Section 7.2.3. Although part (c) of the theorem states that either bound in part (b) may attain the minimum, instances in which the second bound prevails appear to be extremeley rare: It happend in none of the 10200 instances tested in Section 7.2.3.

Typically, g approximates g_0 very tightly for small Q. The approximation weakens somewhat as Q increases but tightens again quickly as Q continues to increase. Figure 2(a) plots the curves g and g_0 and Figure 2(b) plots the approximation error $(g(Q) - g_0(Q))/g_0(Q)$ and the bound $\frac{\beta - \beta_0}{\beta_0}\left[1 - \frac{g_E(Q)}{Dp}\right]$ as functions of Q for $K = 500,\ h = 0.5,\ p = 10,\ D = 1000,\ \lambda = 1,\ \mu = 5,\ r = 1$. As Q increases, the error increases to a maximum of 10%, then quickly decreases virtually to 0. The approximation error is 1% for $Q = 575$ and decreases thereafter. By the time $Q = Q^* = 1793$, the approximation error is 4.0×10^{-6}. When $Q \approx 39950$ (not pictured), the point at which $g_E(Q) = Dp$, $g(Q) - g_0(Q)$ equals 0 and then becomes very slightly negative as Q continues to increase, as predicted by Proposition 3.

Figure 2: Accuracy of approximation. (a) g_0 (solid curve) and g (dashed curve) vs. Q. (b) Actual (solid curve) and bound (dashed curve) on approximation error vs. Q.



(a) (b)

4.2 Accuracy of Optimal Solution

In this section we examine the gap between Q^* and the quantity Q_0 that minimizes $g_0(Q)$. The next proposition demonstrates that $Q^* \geq Q_0$ in the special case in which $r = 1$; Theorem 6 then establishes a bound on the gap between Q^* and Q_0 for all r, under a certain condition regarding g_0.

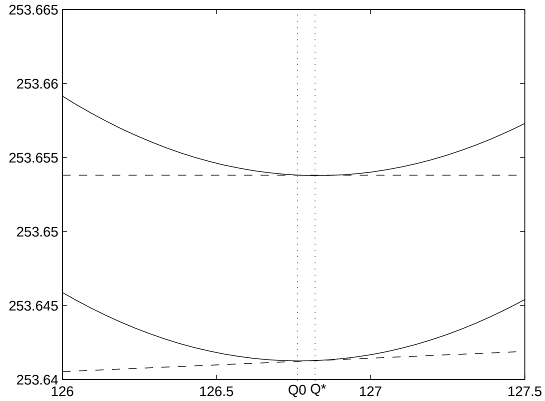
Proposition 5 *If $r = 1$, then $Q^* > Q_0$, where Q_0 is the value of Q that minimizes $g_0(Q)$.*

For $r < 1$, there appears to be no simple characterization of the cases in which $Q^* > Q_0$. For example, the condition under which $g(Q) \geq g_0(Q)$ in Proposition 3, $\beta \geq \beta_0(Q)$ and $g_E(Q) \leq Dp$, does not work here—one can find instances that satisfy this condition even though for some, $Q^* > Q_0$, and for others, $Q^* < Q_0$.

The next theorem provides an upper bound on the approximation error in the optimal solutions, but it relies on the second derivative of g_0 being positive at Q^* and the third derivative of g_0 being negative on the range $[Q_0, Q^*]$. The sign of the second derivative is not known (since g_0 is known to be quasiconvex but not necessarily convex), nor is that of the third derivative. If the derivatives happen to have the correct signs, then the bound holds; otherwise the bound is likely to hold approximately, since g approximates g_0 closely in this range and the derivatives of g *do* have the correct signs: $d^2g/dQ^2 > 0$ everywhere by Theorem 2(a), and

$$\frac{d^3g}{dQ^3} = -\frac{3D\mu^2(h\beta^2 D + 2\mu^2 K + 2\mu Dp\beta)}{(Q\mu + \beta D)^4} < 0$$

Figure 3: g and g_0 near their minima, with tangents at $Q = Q^*$. (Upper curve = g, lower curve = g_0.)



so $d^3g/dQ^3 < 0$ everywhere.

In what follows, the notation $[Q_0, Q^*]$ should be taken to mean $[Q^*, Q_0]$ if $Q^* < Q_0$.

Theorem 6 *If* $\frac{d^2g_0}{dQ^2} > 0$ *at* $Q = Q^*$ *and* $\frac{d^3g_0}{dQ^3} < 0$ *everywhere on the range* $[Q_0, Q^*]$*, then*

$$\frac{|Q^* - Q_0|}{Q^*} \leq \frac{|g_0'(Q^*)|}{Q^* g_0''(Q^*)}$$

where $g_0'(Q^*) = \left.\frac{dg_0}{dQ}\right|_{Q=Q^*}$ *and* $g_0''(Q^*) = \left.\frac{d^2g_0}{dQ^2}\right|_{Q=Q^*}$.

$g_0'(Q^*)$ and $g_0''(Q^*)$ are too cumbersome to write out explicitly here, but they can be computed simply by differentiating g_0 and plugging (6) in for Q. In general, the bound provided by Theorem 6 tends to be small since $g'(Q^*) = 0$ and typically $g_0(Q) \approx g(Q)$ in the neighborhood near Q^*. Figure 3 depicts g (upper curve) and g_0 (lower curve) near their minima, along with tangent lines for both curves at $Q = Q^*$. Note that the tangent line to g_0 is nearly horizontal.

4.3 Use as Heuristic

It is natural to think of Q^* as a heuristic solution for the EOQD in cases for which the lack of closed-form solution for Q_0 makes it impractical to compute it exactly. Theorem 7 presents a bound on the relative error that results from using Q^* instead of Q_0 when the exact cost function g_0 prevails. It applies to the special case in which $r = 1$ only. Bounds are also available for $r < 1$ but they are more mathematically cumbersome. The bound is subject to the assumption made in Theorem 6.

Theorem 7 *Let $\theta \equiv g_0'(Q^*)/g_0''(Q^*)$. If $r = 1$ and if the assumptions of Theorem 6 hold, then*

$$\frac{g_0(Q^*) - g_0(Q_0)}{g_0(Q_0)} \leq \frac{h\mu\theta(2Q^* - \theta)/2 - D^2 p\beta_0(-\theta)\left[1 - \frac{\beta_0(Q^*)}{\beta}\right]}{h\mu(Q^* - \theta)^2/2 + KD\mu + D^2 p\beta_0(Q^* - \theta)}.$$

We argued in Section 4.2 that, typically, $\theta \approx 0$, so the numerator of the bound in Theorem 7 is generally small while the denominator is several orders of magnitude larger. Therefore, the error resulting from using Q^* as a heuristic solution tends to be quite small. Numerical confirmation of this claim can be found in Section 7.2.5.

5 Properties of Optimal Solution

Having established the validity of g as an approximation for g_0, we now set g_0 aside and examine properties of g itself. We first compare the optimal order quantity and cost for the (approximate) EOQD to those of the classical EOQ quantity and cost. Then we show that g exhibits several properties that mirror the behavior of the classical EOQ model. In Section 6, we will show that the approximate EOQD lends itself to sensitivity analysis and the analysis of power-of-two policies.

Proposition 8 establishes that the cost of a given order quantity Q under the (approximate) EOQD model is greater than that of the EOQ under the same Q for reasonable values of Q, i.e., those for which Q results in a cost that is less than the cost of stocking out on every demand. Part (b) of the proposition also verifies that Q^* has this property.

Proposition 8 *(a) For all $Q > 0$, $g_E(Q) < g(Q)$ if and only if $g_E(Q) < Dp$.*

(b) $g_E(Q^*) < g(Q^*)$.

The next proposition demonstrates that Q^* [$g(Q^*)$] is larger than the optimal EOQ solution [cost], and that the difference between them may be arbitrarily large.

Proposition 9 *Let $Q_E = \sqrt{2KD/h}$ be the optimal EOQ solution and $g_E(Q_E) = \sqrt{2KDh}$ its cost. Then*

(a) $Q^* > Q_E$

(b) For any $M \in \mathbb{R}$, there exist values of the problem parameters such that

$$(Q^* - Q_E)/Q_E > M.$$

(c) $g(Q^*) > g_E(Q_E)$

(d) For any $M \in \mathbb{R}$*, there exist values of the problem parameters such that* $(g(Q^*) - g_E(Q_E))/g_E(Q_E) > M$*.*

The implication of Proposition 9 is that ignoring disruptions in the EOQ can lead to serious errors, and the EOQ solution may perform poorly when supply is uncertain; we demonstrate this numerically in Section 7.3.

Recall that the optimal Q in the classical EOQ model is $\sqrt{2KD/h}$ and the corresponding cost is $\sqrt{2KDh}$; that is, the optimal cost equals h times the optimal order quantity. The same holds for $g(Q)$:

Theorem 10 $g(Q^*) = hQ^*$

The next theorem establishes monotonicity and convexity properties of the optimal cost with respect to the demand and cost parameters.

Theorem 11 *(a) The optimal cost* $g(Q^*)$ *is an increasing, strictly concave function of* h*,* p*,* K*, and* D*.*

(b) The optimal order quantity Q^* *is a decreasing, strictly convex function of* h *and an increasing, strictly concave function of* D*,* p*, and* K*.*

We have been unable to prove, but our numerical experience supports, the following conjecture:

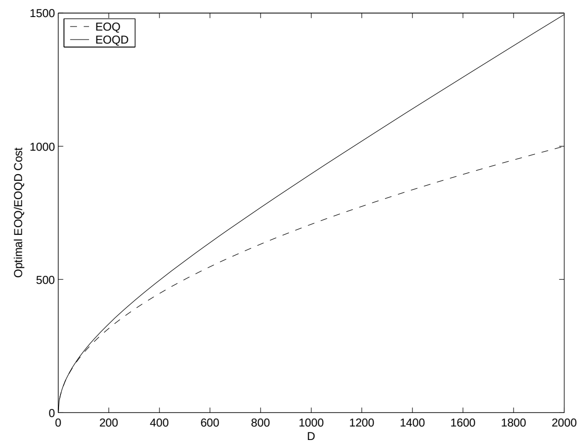
Conjecture 12 *The optimal cost* $g(Q^*)$ *is an increasing, strictly concave function of* λ *and a decreasing, strictly convex function of* μ*.*

In light of Theorem 10, Conjecture 12 would also imply that Q^* is increasing and concave in λ and decreasing and convex in μ.

The concavity of the optimal cost with respect to D is useful in several contexts. For example, Qi et al. (2008) formulate a joint location–inventory model with supply disruptions; the approximate inventory cost at each facility is calculated in closed form using an extension of Theorem 2. Translated into our notation and simplifying some of their assumptions, their objective function contains terms of the following form, one for each facility:

$$\frac{1}{\mu}\left[\sqrt{\left(\beta h \sum_{i=1}^{n} D_i Y_i\right)^2 + 2h\mu\left(K\mu \sum_{i=1}^{n} D_i Y_i + p\beta\left(\sum_{i=1}^{n} D_i Y_i\right)^2\right)} - \beta h \sum_{i=1}^{n} D_i Y_i\right], \qquad (7)$$

Figure 4: Optimal EOQD and EOQ costs as functions of D.



where $i = 1, \ldots, n$ are the customers, D_i is the (mean) demand of customer i, and $Y_i = 1$ if customer i is assigned to the facility, 0 otherwise. (7) is simply equal to $g(Q^*) = hQ^*$, with the demand determined endogenously based on the decision variables Y_i. A similar approach is used by Daskin et al. (2002) for a location–inventory model without disruptions; their term is based on the EOQ rather than the EOQD. Daskin et al.'s (2002) Lagrangian relaxation algorithm for the location–inventory model is nearly as efficient as similar algorithms for classical location models such as the uncapacitated fixed-charge location problem (UFLP), and it relies critically on the objective function being concave with respect to the demand served by each facility. The algorithms of both Qi et al. (2008) and Daskin et al. (2002) work only because (a) the approximate inventory cost can be expressed in closed form, and (b) the cost is a concave function of the demand.

As it happens, the EOQD cost function is "less concave" (more linear) than that of the EOQ with respect to D (see Figure 4) since we can re-write $g(Q^*)$ using suitable constants as

$$g(Q^*) = \sqrt{aD^2 + 2KDh} - cD \approx (\sqrt{a} - c)D.$$

The implication of this is that economies of scale are less strong in the EOQD than in the EOQ. In the context of the location–inventory model of Qi et al. (2008), this means that consolidation of facilities becomes a less attractive strategy as supply uncertainty increases, since the benefits of consolidation are partially offset by the increased supply uncertainty inherent in reducing the supply base.

6 Sensitivity Analysis and Power-of-Two Policies

In this section, we derive an expression to compare the cost of an arbitrarily chosen Q to that of the optimal Q (paralleling similar results for the EOQ model) as well as bounds on the cost of the optimal power-of-two ordering policy.

6.1 Sensitivity to Q

It is well known (see, e.g., Zipkin 2000) that if Q_E is the optimal solution to the classical EOQ model, then the ratio of the cost of an arbitrary Q to that of Q_E is given by

$$\epsilon\left(\frac{Q_E}{Q}\right), \tag{8}$$

where $\epsilon(x) = (x + 1/x)/2$ is the so-called EOQ error function. We now prove a similar result for g.

Theorem 13 *Let $Q > 0$ be any order quantity. Then*

$$\frac{g(Q)}{g(Q^*)} = \epsilon\left(\frac{Q^*}{Q}\right) - \left[\epsilon\left(\frac{Q^*}{Q}\right) - 1\right]\frac{\beta D}{Q\mu + \beta D}. \tag{9}$$

Since $\epsilon(x) \geq 1$ for all $x > 0$, the expression given in (9) is smaller than that in (8), i.e., the (approximate) EOQD cost function is flatter around its optimum than that of the classical EOQ. The two expressions are closer (i.e., the second term in (9) is smaller) when $(\lambda+\mu)Q/D$ is large. (See Section 3.3 for further interpretation of this condition.) This is because $(\lambda + \mu)Q/D = Q\lambda r/\beta D < Q\mu/\beta D$, so when $(\lambda + \mu)Q/D$ is large, $Q\mu/\beta D$ is even larger, in which case $\beta D/(Q\mu + \beta D)$ is small. As $(\lambda + \mu)Q/D$ decreases, the second term in (9) increases and the cost function becomes flatter.

6.2 Power-of-Two Policies

In our analysis thus far, we have treated the order quantity, Q, as the decision variable. But we could have formulated an equivalent model in which the order interval (call it T) is the decision variable. As in the classical EOQ model, placing orders of size Q means placing orders every Q/D years (during wet periods), so $T = Q/D$. Then the expected annual cost can be expressed as a function of T as follows:

$$f(T) = g(TD) = \frac{h\mu DT^2/2 + K\mu + Dp\beta}{T\mu + \beta}.$$

It is straightforward to show that $f(T)$ is strictly convex and that the optimal value of T is given by

$$T^* = \frac{Q^*}{D} = \frac{\sqrt{(\beta h)^2 + 2h\mu\left(\frac{K\mu}{D} + p\beta\right)} - \beta h}{h\mu}. \tag{10}$$

which has cost $f(T^*) = g(Q^*) = hQ^*$.

Following Muckstadt and Roundy (1993), we define a *power-of-two* policy to be one in which the order interval is restricted to be a power-of-two multiple of some base time period T_B; that is, $T = 2^k T_B$ for some $k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. T_B is fixed.

Our analysis parallels the classical analysis by first deriving lower and upper bounds on the optimal $2^k T_B$ and then proving that the cost of each endpoint is less than or equal to $1.06 f(T^*)$. Since f is convex, the optimal power-of-two cost is guaranteed to be less than or equal to this value.

By the convexity of f, the optimal k is the smallest k that satisfies

$$\begin{aligned}
& f\left(2^k T_B\right) \leq f\left(2^{k+1} T_B\right) \\
\iff & \frac{\frac{h\mu D}{2}\left(2^k T_B\right)^2 + K\mu + Dp\beta}{2^k T_B \mu + \beta} \leq \frac{\frac{h\mu D}{2}\left(2^{k+1} T_B\right)^2 + K\mu + Dp\beta}{2^{k+1} T_B \mu + \beta} \\
\iff & \frac{h\mu D}{2}\left(2^k T_B\right)^2 \left(\frac{1}{2^k T_B \mu + \beta} - \frac{4}{2^{k+1} T_B \mu + \beta}\right) \leq \\
& \qquad (K\mu + Dp\beta)\left(\frac{1}{2^{k+1} T_B \mu + \beta} - \frac{1}{2^k T_B \mu + \beta}\right) \\
\iff & \frac{h\mu D}{2}\left(2^k T_B\right)^2 \left(2^{k+1} T_B \mu + 3\beta\right) \geq \mu(K\mu + Dp\beta)\left(2^k T_B\right) \\
\iff & h\mu D\left(2^k T_B\right)^2 + \frac{3}{2}\beta h D\left(2^k T_B\right) - (K\mu + Dp\beta) \geq 0
\end{aligned} \tag{11}$$

Viewed as a function of $2^k T_B$, the expression on the left-hand side of (11) has two real roots, one positive and one negative. Since $2^k T_B \geq 0$, inequality (11) holds if and only if $2^k T_B$ is greater than or equal to the positive root; that is,

$$\begin{aligned}
\implies 2^k T_B \geq & \frac{-\frac{3}{2}\beta h D + \sqrt{\left(\frac{3}{2}\beta h D\right)^2 + 4(h\mu D)(K\mu + Dp\beta)}}{2(h\mu D)} \\
= & \frac{3}{4} \cdot \frac{-\beta h + \sqrt{(\beta h)^2 + \frac{16}{9} h\mu\left(\frac{K\mu}{D} + p\beta\right)}}{h\mu}
\end{aligned}$$

We also know that the optimal k satisfies

$$f\left(2^{k-1} T_B\right) \geq f\left(2^k T_B\right).$$

Using similar reasoning as above, this implies that

$$2^k T_B \leq \frac{3}{2} \cdot \frac{-\beta h + \sqrt{(\beta h)^2 + \frac{16}{9} h\mu \left(\frac{K\mu}{D} + p\beta \right)}}{h\mu}.$$

We have now proved the following result:

Lemma 14 *Let*

$$\hat{T} = \frac{\sqrt{(\beta h)^2 + \frac{16}{9} h\mu \left(\frac{K\mu}{D} + p\beta \right)} - \beta h}{h\mu}. \tag{12}$$

The k yielding the optimal power-of-two policy satisfies

$$\frac{3}{4}\hat{T} \leq 2^k T_B \leq \frac{3}{2}\hat{T}.$$

By the convexity of f, the cost of the optimal power-of-two policy is no more than the maximum of the costs of the two endpoints specified in Lemma 14. In fact, the two endpoints have the same cost, and that cost is no more than $3\sqrt{2}/4$ times the cost of the optimal (general) policy, as stated in the next lemma. Note that the same bound applies to the classical EOQ; see, e.g., Muckstadt and Roundy (1993).

Lemma 15 *Let $\hat{T}$ be defined as in Lemma 14. Then*

$$\frac{f\left(\frac{3}{4}\hat{T}\right)}{f(T^*)} = \frac{f\left(\frac{3}{2}\hat{T}\right)}{f(T^*)} \leq \frac{3\sqrt{2}}{4} \approx 1.06.$$

Therefore, we have now proved:

Theorem 16 *If $2^k T_B$ is the optimal power-of-two order interval, then*

$$\frac{f\left(2^k T_B\right)}{f(T^*)} \leq \frac{3\sqrt{2}}{4} \approx 1.06.$$

It is not known whether the bound in Theorem 16 is tight, though we suspect it is: In our computational tests in Section 7.4, we found an instance that is only 0.00004 less than $3\sqrt{2}/4$. On the other hand, the results in that section suggest that the actual error is closer to 2% on average.

Table 1: Problem parameters for benchmark data sets.

Instance	h	K	p	D
1	0.8	30	12.96	540
2	15.0	10	40.00	14
3	6.5	175	12.50	2000
4	2.0	50	25.00	200
5	45.0	4500	440.49	2319
6	5.0	300	50.00	3000
7	0.0132	20	0.34	1000
8	5.0	28	80.00	520
9	0.005	12	0.12	3120
10	3.6	12000	65.73	8000

7 Computational Results

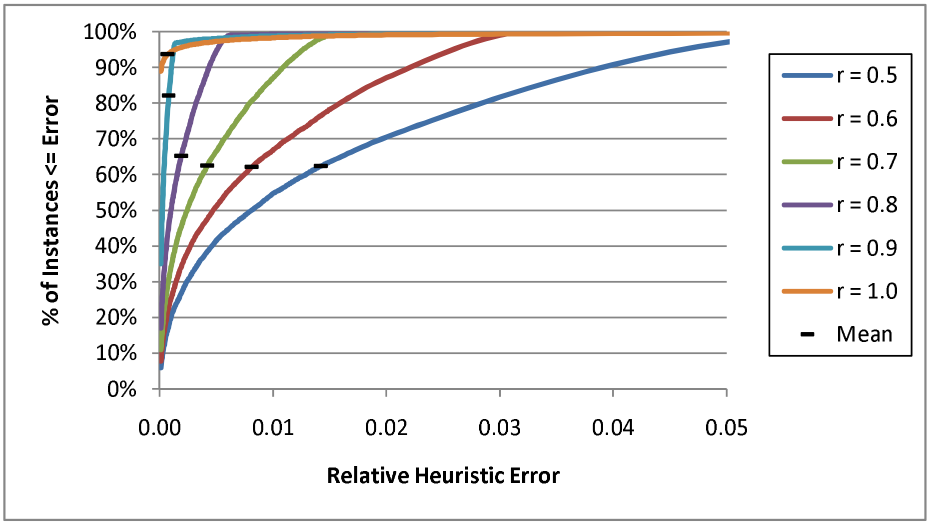
7.1 Experimental Design

We tested our model using 200 benchmark and 10,000 randomly generated data sets. The benchmark sets consisted of 10 values of each of the parameters h, K, p, and D, shown in Table 1. These problem instances were adapted from sample problems for the (Q, R) model (which uses the same cost parameters as the EOQD) contained in several production and inventory textbooks. For each benchmark problem, we considered 5 values for λ (0.5, 1, 4, 8, and 12) and 4 values for μ (2λ, 4λ, 10λ, and 20λ), resulting in 200 instances. The random instances were generated by drawing parameters from the following distributions:

- $K \sim U[0, 1000]$
- $h \sim U[0, 250]$
- $p \sim U[\max\{h, 250\}, 1000]$
- $D \sim U[0, 1000]$
- $\lambda \sim U[0.5, 12]$
- $\mu \sim U[2\lambda, 20\lambda]$

The bounds were chosen so that the first two assumptions in Section 3.2 (non-negative parameters and $\lambda < \mu$) are always satisfied. Any instance that did not satisfy the third assumption ($\sqrt{2KDh} < pD$) was discarded and re-sampled. Our bounds also ensure $h < p$, though this assumption is not necessary for the results presented in this paper. For each instance (benchmark and random), we computed Q^* using equation (6) and found Q_0 using MATLAB's `fminsearch` function.

Figure 5: Percentage of instances within a given heuristic error.



The sections that follow first present results on the accuracy of the approximation, then results relating to analytical properties of the approximate model, and finally results confirming that the insights and results proven for the approximate function hold, at least approximately, for the exact function.

7.2 Approximation Error

7.2.1 Heuristic Error

We first test the quality of our approximation by evaluating the error that results from using Q^* as a heuristic solution, measured as $(g_0(Q^*) - g_0(Q_0))/g_0(Q_0)$. Table 2 displays the mean and maximum heuristic error for several values of r, as well as the fraction of instances whose relative error is less than a given value, for the random and benchmark instances separately, and then for the 10,200 instances as a whole. (For example, if $r = 0.5$, then 14.5% of benchmark instances have relative errors less than 0.001, 58.0% have relative errors less than 0.01, etc.) Figure 5 displays these results graphically, plotting the heuristic error on the x-axis and the percentage of instances with no more than that error on the y-axis, for each value of r. The mean error for each r-value is indicated with a hatch mark. The closer a curve is to the top-left corner of the graph, the better the approximation.

Based on these results, we recommend $r = 1.0$ for most instances, since (a) it has the smallest mean error, (b) 99.7% of instances have errors less than 5%, and (c) the special case

Table 2: Heuristic error: $(g_0(Q^*) - g_0(Q_0))/g_0(Q_0)$.

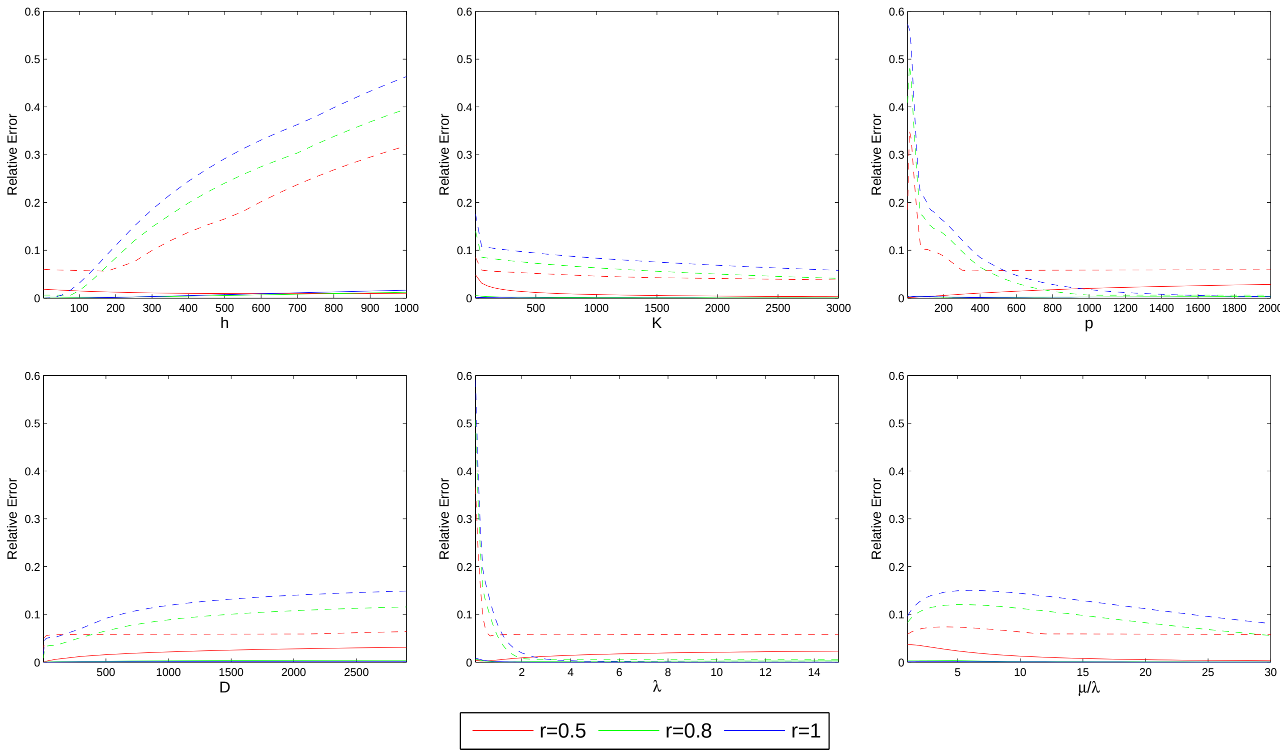
Problem Type	Measure	$r = 0.5$	$r = 0.6$	$r = 0.7$	$r = 0.8$	$r = 0.9$	$r = 1.0$
Benchmark	Mean	0.0121	0.0071	0.0041	0.0025	0.0019	0.0021
	Max	0.0574	0.0699	0.0817	0.0928	0.1034	0.1134
	% <0.001	0.1450	0.1900	0.2200	0.2950	0.4500	0.8100
	% <0.01	0.5800	0.7050	0.8850	0.9650	0.9650	0.9650
	% <0.02	0.7400	0.9000	0.9850	0.9850	0.9700	0.9650
	% <0.05	0.9850	0.9950	0.9900	0.9900	0.9900	0.9850
	% <0.10	1.0000	1.0000	1.0000	1.0000	0.9950	0.9950
Random	Mean	0.0143	0.0081	0.0042	0.0019	0.0008	0.0007
	Max	0.1474	0.1661	0.1861	0.2062	0.2254	0.2439
	% <0.001	0.0575	0.0752	0.1074	0.1673	0.3480	0.8913
	% <0.01	0.5467	0.6679	0.8704	0.9926	0.9891	0.9835
	% <0.02	0.7045	0.8707	0.9966	0.9947	0.9936	0.9918
	% <0.05	0.9712	0.9994	0.9989	0.9988	0.9981	0.9973
	% <0.10	0.9998	0.9998	0.9998	0.9997	0.9996	0.9993
Overall	Mean	0.0142	0.0081	0.0042	0.0019	0.0008	0.0007
	Max	0.1474	0.1661	0.1861	0.2062	0.2254	0.2439
	% <0.001	0.0592	0.0775	0.1096	0.1698	0.3500	0.8897
	% <0.01	0.5474	0.6686	0.8707	0.9921	0.9886	0.9831
	% <0.02	0.7052	0.8713	0.9964	0.9945	0.9931	0.9913
	% <0.05	0.9715	0.9993	0.9987	0.9986	0.9979	0.9971
	% <0.10	0.9998	0.9998	0.9998	0.9997	0.9995	0.9992

$r = 1$ has an intuitive interpretation (see Section 3.3) and theoretical properties not available for other r values (e.g., Proposition 5, Theorem 7). Smaller values of r tend to perform better in the worst case but worse on average.

To explore this effect further—and to identify characteristics of instances for which our approximation performs poorly—we systematically varied each of the six parameters and calculated the mean and maximum heuristic error, over the first 1000 random instances, for $r = 0.5,\ 0.8,\ 1.0$. The results are plotted in Figure 6. Note that we include ranges of each parameter that fall outside the ranges given in Section 7.1 in order to "stress" our assumptions about the normal range of parameter values and to test the quality of our approximation outside this range. In the figure, each point on a solid [dashed] curve represents the mean [maximum] heuristic error, over the first 1000 random instances, when a given parameter is set to the value on the x-axis and all other parameters remain at their original (randomly generated) values.

The mean errors are generally very small—less than 1% for most r values and parameter values. The mean cost is usually smallest for $r = 1$ (blue curve), but $r = 1$ also has the largest maximum cost. In general, the heuristic error for a fixed value of r increases as Q/D decreases; that is, as K or p decrease or as h increases (Theorem 11), as λ decreases or μ increases (Conjecture 12), or as D increases (since Q increases slower than linearly with D by

Figure 6: Mean (solid lines) and maximum (dashed lines) heuristic error as parameter values vary.



Theorem 11 and therefore Q/D decreases with D). This is because smaller order quantities warrant smaller values of r. Actually, as μ increases, the error first increases due to the decreasing order size and then decreases due to the increase in the term $(\lambda+\mu)Q/D$ in β_0. The sharp increases as p or λ approach 0 are due to the fact that the costs themselves are small in this range, and therefore the relative error becomes more pronounced (although the absolute error may be small).

The most troubling aspect of Figure 6 is the steady increase in the maximum error as h increases (although the mean error increases much more slowly). This is caused by decreasing order quantities and could be remedied by using a smaller value of r. For $h < 250$, our suggested r-value of 1.0 works reasonably well in the worst case, with a maximum error of roughly 15%, but it performs more poorly as h increases to 1000. On the other hand, h-values greater than 250 are somewhat out of proportion with our data sets, since we use a maximum value of 1000 for both K and p, and typically h is much smaller than both of these parameters.

Certainly, it is possible to construct instances for which our approximation performs poorly, but such instances appear to be the exception rather than the rule. Moreover the analysis above

Table 3: Accuracy of β approximation: $(\beta - \beta_0(Q^*))/\beta_0(Q^*)$.

λ	μ/λ	Benchmark Mean	Benchmark Max	Benchmark #	Random Mean	Random Max	Random #	Overall Mean	Overall Max	Overall #
0.5	2	0.0755	0.3811	10	0.1365	0.6455	13	0.1100	0.6455	23
0.5	4	0.0682	0.3392	10	0.1098	0.3871	37	0.1009	0.3871	47
0.5	10	0.0461	0.2484	10	0.0781	0.3164	79	0.0745	0.3164	89
0.5	20	0.0209	0.1269	10	0.0570	0.2744	62	0.0519	0.2744	72
1	2	0.0264	0.1561	10	0.0299	0.1893	90	0.0296	0.1893	100
1	4	0.0228	0.1388	10	0.0281	0.2358	324	0.0279	0.2358	334
1	10	0.0108	0.0770	10	0.0170	0.1988	662	0.0169	0.1988	672
1	20	0.0025	0.0198	10	0.0147	0.2791	465	0.0145	0.2791	475
4	2	0.0011	0.0090	10	0.0040	0.0593	159	0.0038	0.0593	169
4	4	0.0006	0.0054	10	0.0026	0.0537	688	0.0025	0.0537	698
4	10	<0.0001	0.0003	10	0.0011	0.0482	1337	0.0010	0.0482	1347
4	20	<0.0001	<0.0001	10	0.0006	0.0465	875	0.0006	0.0465	885
8	2	<0.0001	0.0006	10	0.0003	0.0059	179	0.0002	0.0059	189
8	4	<0.0001	0.0002	10	0.0002	0.0110	731	0.0002	0.0110	741
8	10	<0.0001	<0.0001	10	<0.0001	0.0185	1642	<0.0001	0.0185	1652
8	20	<0.0001	<0.0001	10	<0.0001	0.0018	947	<0.0001	0.0018	957
12	2	<0.0001	<0.0001	10	<0.0001	0.0017	102	<0.0001	0.0017	112
12	4	<0.0001	<0.0001	10	<0.0001	0.0036	408	<0.0001	0.0036	418
12	10	<0.0001	<0.0001	10	<0.0001	0.0044	715	<0.0001	0.0044	725
12	20	<0.0001	<0.0001	10	<0.0001	0.0015	485	<0.0001	0.0015	495
Total		0.0137	0.3811	200	0.0050	0.6455	10000	0.0052	0.6455	10200

can provide guidelines to determine *a priori* whether the approximation will perform well for a given instance.

For the remainder of Section 7, we use $r = 1.0$ in all tests.

7.2.2 Accuracy of β

We next examine $(\beta - \beta_0(Q^*))/\beta_0(Q^*)$, since our results rely on β being a good approximation for $\beta_0(Q)$, particularly at $Q = Q^*$. Table 3 provides the mean and maximum values of $(\beta - \beta_0(Q^*))/\beta_0(Q^*)$ for the benchmark and random problems. For the benchmark problems, the λ and μ/λ values listed are exact, while for the random problems they represent the following ranges: $\lambda \in [0.5, 0.75), [0.75, 2.5), [2.5, 6), [6, 10), [10, 12]$ and $\mu/\lambda \in [2, 3), [3, 7), [7, 15), [15, 20]$. (This interpretation also holds for all tables below.)

These results validate our assertion in Section 3.3 that β is a good approximation for β_0, since the mean error across all instances is only 0.52%. As expected, the approximation is worse for smaller values of λ and μ and improves substantially as λ and μ increase. This trend persists throughout our computational study.

Table 4: Accuracy of cost function at Q^*: $(g(Q^*) - g_0(Q^*))/g_0(Q^*)$ (bounds and actual).

		Benchmark				Random				Overall			
		Actual		Bound		Actual		Bound		Actual		Bound	
λ	μ/λ	Mean	Max	Mean	Max	Mean	Max	Mean	Max	Mean	Max	Mean	Max
0.5	2	0.0248	0.1158	0.0332	0.1601	0.0455	0.1710	0.0581	0.2440	0.0365	0.1710	0.0473	0.2440
0.5	4	0.0237	0.1130	0.0306	0.1450	0.0417	0.1467	0.0499	0.1622	0.0379	0.1467	0.0458	0.1622
0.5	10	0.0133	0.0697	0.0213	0.1105	0.0294	0.1171	0.0365	0.1366	0.0276	0.1171	0.0348	0.1366
0.5	20	0.0042	0.0213	0.0100	0.0597	0.0197	0.0994	0.0265	0.1207	0.0175	0.0994	0.0242	0.1207
1	2	0.0094	0.0550	0.0124	0.0724	0.0119	0.0655	0.0144	0.0865	0.0117	0.0655	0.0142	0.0865
1	4	0.0079	0.0491	0.0108	0.0649	0.0113	0.0882	0.0135	0.1055	0.0112	0.0882	0.0134	0.1055
1	10	0.0026	0.0187	0.0053	0.0371	0.0063	0.0803	0.0082	0.0904	0.0063	0.0803	0.0082	0.0904
1	20	0.0004	0.0023	0.0012	0.0098	0.0051	0.1132	0.0071	0.1225	0.0050	0.1132	0.0070	0.1225
4	2	0.0004	0.0034	0.0005	0.0045	0.0016	0.0219	0.0020	0.0288	0.0015	0.0219	0.0019	0.0288
4	4	0.0002	0.0015	0.0003	0.0027	0.0010	0.0244	0.0013	0.0261	0.0010	0.0244	0.0013	0.0261
4	10	<0.0001	<0.0001	<0.0001	0.0001	0.0004	0.0197	0.0005	0.0236	0.0004	0.0197	0.0005	0.0236
4	20	<0.0001	<0.0001	<0.0001	<0.0001	0.0002	0.0211	0.0003	0.0227	0.0002	0.0211	0.0003	0.0227
8	2	<0.0001	0.0002	<0.0001	0.0003	0.0001	0.0027	0.0001	0.0030	<0.0001	0.0027	0.0001	0.0030
8	4	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0043	0.0001	0.0054	<0.0001	0.0043	0.0001	0.0054
8	10	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0089	<0.0001	0.0092	<0.0001	0.0089	<0.0001	0.0092
8	20	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0008	<0.0001	0.0009	<0.0001	0.0008	<0.0001	0.0009
12	2	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0007	<0.0001	0.0009	<0.0001	0.0007	<0.0001	0.0009
12	4	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0017	<0.0001	0.0018	<0.0001	0.0017	<0.0001	0.0018
12	10	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0018	<0.0001	0.0022	<0.0001	0.0018	<0.0001	0.0022
12	20	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0006	<0.0001	0.0008	<0.0001	0.0006	<0.0001	0.0008
Total		0.0043	0.1158	0.0063	0.1601	0.0019	0.1710	0.0024	0.2440	0.0019	0.1710	0.0025	0.2440

7.2.3 Accuracy of $g(Q^*)$

Table 4 provides the mean and maximum approximation error in the cost function at Q^* for the benchmark and random instances. It lists the actual approximation error, $(g(Q^*) - g_0(Q^*))/g_0(Q^*)$, and the minimum of the two bounds given in Theorem 4(b).

Table 4 demonstrates that the approximation provided by g is quite tight at $Q = Q^*$. The approximate cost function differs from the exact function at Q^* by an average of 0.43% for the benchmark instances and 0.19% for the random instances, with theoretical bounds of 0.63% and 0.24%, on average, respectively. These errors are significantly smaller than the worst-case bound of 1 given in Theorem 4(b). Moreover, the actual error was less than 0.1% for 85.4% of the 10200 instances tested and less than 1% for 95.0% of the instances.

In every instance tested, the first term in the minimum in Theorem 4(b) is smaller than the second. However, this is not true in general; see Theorem 4(c).

7.2.4 Accuracy of Q^*

Table 5 lists the actual approximation error and the theoretical bounds (from Theorem 6) for $(Q^* - Q_0)/Q^*$ for the benchmark and random problems. We tested the assumptions stipulated

Table 5: Accuracy of optimal solution: $(Q^* - Q_0)/Q^*$ (bounds and actual).

		Benchmark				Random				Overall			
		Actual		Bound		Actual		Bound		Actual		Bound	
λ	μ/λ	Mean	Max	Mean	Max	Mean	Max	Mean	Max	Mean	Max	Mean	Max
0.5	2	0.1459	0.6558	0.2201	1.1692	0.2678	0.8163	0.4561	2.1752	0.2148	0.8163	0.3535	2.1752
0.5	4	0.1195	0.5259	0.1793	0.9687	0.2348	0.9039	0.3458	1.8420	0.2103	0.9039	0.3103	1.8420
0.5	10	0.0594	0.2607	0.0734	0.3719	0.1541	0.5259	0.2040	1.0146	0.1435	0.5259	0.1893	1.0146
0.5	20	0.0196	0.0788	0.0210	0.0880	0.0982	0.5469	0.1313	0.9139	0.0872	0.5469	0.1160	0.9139
1	2	0.0589	0.3435	0.0713	0.4372	0.0756	0.3962	0.0880	0.5322	0.0739	0.3962	0.0863	0.5322
1	4	0.0431	0.2569	0.0519	0.3306	0.0673	0.4860	0.0797	0.7563	0.0665	0.4860	0.0789	0.7563
1	10	0.0135	0.0841	0.0145	0.0936	0.0356	0.4851	0.0412	0.7024	0.0352	0.4851	0.0408	0.7024
1	20	0.0022	0.0118	0.0022	0.0120	0.0278	0.5859	0.0333	1.0871	0.0273	0.5859	0.0327	1.0871
4	2	0.0028	0.0230	0.0028	0.0236	0.0106	0.1326	0.0112	0.1485	0.0102	0.1326	0.0108	0.1485
4	4	0.0011	0.0106	0.0012	0.0107	0.0068	0.1472	0.0071	0.1678	0.0067	0.1472	0.0070	0.1678
4	10	<0.0001	0.0003	<0.0001	0.0003	0.0025	0.1101	0.0026	0.1236	0.0025	0.1101	0.0025	0.1236
4	20	<0.0001	<0.0001	<0.0001	<0.0001	0.0013	0.1220	0.0014	0.1378	0.0013	0.1220	0.0014	0.1378
8	2	0.0002	0.0019	0.0002	0.0019	0.0009	0.0193	0.0009	0.0197	0.0008	0.0193	0.0008	0.0197
8	4	<0.0001	0.0003	<0.0001	0.0003	0.0007	0.0281	0.0007	0.0290	0.0007	0.0281	0.0007	0.0290
8	10	<0.0001	<0.0001	<0.0001	<0.0001	0.0003	0.0545	0.0003	0.0580	0.0003	0.0545	0.0003	0.0580
8	20	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0061	<0.0001	0.0061	<0.0001	0.0061	<0.0001	0.0061
12	2	<0.0001	0.0002	<0.0001	0.0002	0.0002	0.0058	0.0002	0.0058	0.0002	0.0058	0.0002	0.0058
12	4	<0.0001	<0.0001	<0.0001	<0.0001	0.0002	0.0121	0.0002	0.0123	0.0002	0.0121	0.0002	0.0123
12	10	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0127	<0.0001	0.0129	<0.0001	0.0127	<0.0001	0.0129
12	20	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0049	<0.0001	0.0049	<0.0001	0.0049	<0.0001	0.0049
Total		0.0233	0.6558	0.0319	1.1692	0.0108	0.9039	0.0132	2.1752	0.0110	0.9039	0.0136	2.1752

Table excludes 3 random instances that violate assumptions of Theorem 6.

in Theorem 6 concerning the derivatives of g_0 numerically and found that all instances satisfied them except for 3 random instances. These instances have been omitted from the table.

For the benchmark problems, the mean error in Q^* is 2.3%, with a mean theoretical bound of 3.2%. The corresponding values for random instances are 1.1% and 1.3%. The error is less than 0.1% for 74.6% of all instances tested (benchmark and random) and less than 1% for 87.2%. The error decreases substantially as λ and μ increase. Note that larger errors in Q^* are not necessarily indicative of larger errors in the cost, since the cost function is flat around its optimum. A better indicator is the heuristic error, which we explore further in the next section.

7.2.5 Use as Heuristic

Table 6 lists the mean and maximum error (actual and bound) that results from using Q^* as a heuristic solution in place of Q_0, as discussed in Section 4.3. This table is a more detailed version of the "$r = 1.0$" column of Table 2. The 3 instances omitted from Table 5 are omitted from this table as well. Clearly, Q^* is an extremely effective solution for the exact cost function: The actual error is 0.07% on average and is less than 1% for 89.0% of the instances tested.

Table 6: Accuracy of Q^* as heuristic solution for g_0: $(g_0(Q^*) - g_0(Q_0))/g_0(Q_0)$ (bounds and actual).

		Benchmark				Random				Overall			
		Bound		Actual		Bound		Actual		Bound		Actual	
λ	μ/λ	Mean	Max	Mean	Max	Mean	Max	Mean	Max	Mean	Max	Mean	Max
0.5	2	0.0175	0.1134	<0.0001	1.4868	0.0418	0.2250	<0.0001	7.8757	0.0312	0.2250	<0.0001	7.8757
0.5	4	0.0133	0.0914	1.3976	13.0957	0.0294	0.2439	<0.0001	7.9284	0.0260	0.2439	0.0522	13.0957
0.5	10	0.0036	0.0270	0.0725	0.4561	0.0138	0.1002	0.6134	27.7035	0.0126	0.1002	0.5526	27.7035
0.5	20	0.0004	0.0028	0.0152	0.0673	0.0089	0.0920	0.3131	8.8907	0.0077	0.0920	0.2717	8.8907
1	2	0.0041	0.0303	0.0796	0.5732	0.0043	0.0405	0.0875	0.8718	0.0043	0.0405	0.0867	0.8718
1	4	0.0026	0.0212	0.0505	0.3665	0.0040	0.0699	0.0929	2.6619	0.0040	0.0699	0.0917	2.6619
1	10	0.0003	0.0029	0.0106	0.0716	0.0018	0.0651	0.0416	2.0608	0.0017	0.0651	0.0411	2.0608
1	20	<0.0001	<0.0001	0.0015	0.0081	0.0016	0.1132	0.0178	17.5752	0.0016	0.1132	0.0175	17.5752
4	2	<0.0001	0.0002	0.0018	0.0154	0.0002	0.0057	0.0079	0.1165	0.0002	0.0057	0.0075	0.1165
4	4	<0.0001	<0.0001	0.0008	0.0071	0.0001	0.0074	0.0049	0.1428	0.0001	0.0074	0.0049	0.1428
4	10	<0.0001	<0.0001	<0.0001	0.0002	<0.0001	0.0046	0.0018	0.0990	<0.0001	0.0046	0.0018	0.0990
4	20	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0055	0.0010	0.1139	<0.0001	0.0055	0.0010	0.1139
8	2	<0.0001	<0.0001	0.0001	0.0012	<0.0001	0.0002	0.0006	0.0130	<0.0001	0.0002	0.0005	0.0130
8	4	<0.0001	<0.0001	<0.0001	0.0002	<0.0001	0.0003	0.0004	0.0196	<0.0001	0.0003	0.0004	0.0196
8	10	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0012	0.0002	0.0421	<0.0001	0.0012	0.0002	0.0421
8	20	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0041	<0.0001	<0.0001	<0.0001	0.0041
12	2	<0.0001	<0.0001	<0.0001	0.0001	<0.0001	<0.0001	0.0001	0.0037	<0.0001	<0.0001	0.0001	0.0037
12	4	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0001	0.0082	<0.0001	<0.0001	0.0001	0.0082
12	10	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0087	<0.0001	<0.0001	<0.0001	0.0087
12	20	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	0.0033	<0.0001	<0.0001	<0.0001	0.0033
Total		0.0021	0.1134	0.0500	13.0957	0.0007	0.2439	0.0138	27.7035	0.0007	0.2439	0.0145	27.7035

Table excludes 3 random instances that violate assumptions of Theorem 6.

7.3 Comparison to EOQ

We proved in Proposition 9 that the optimal solution to the (approximate) EOQD, Q^*, is greater than or equal to the optimal EOQ solution, Q_E. Table 7 provides empirical evidence demonstrating the magnitude of the difference. The table lists the mean and maximum (over the 10200 instances) relative difference between Q^* and Q_E. By Theorem 10, this is also equal to the relative difference between $g(Q^*)$ and the optimal EOQ cost. The table also lists the "ignorance cost" of applying the EOQ model instead of the EOQD: the relative increase in cost if the classical EOQ model is applied when supply uncertainty exists, computed as $(g(Q_E) - g(Q^*))/g(Q^*)$.

The EOQ and EOQD solutions can differ radically, and the cost of using the EOQ model instead of the EOQD can be quite large. On average, the EOQD order quantity is 121% larger than the EOQ order quantity, and the difference reaches 14938% for one instance. In addition, using the EOQ solution can be quite costly if supply uncertainty exists: the EOQ quantity yields a cost 33% larger than the optimal EOQD cost, on average, and reaches over 2000% for some instances.

Table 7: Comparison to EOQ solution.

		Benchmark				Random				Overall			
		$\frac{Q^*-Q_E}{Q_E}$		$\frac{g(Q_E)-g(Q^*)}{g(Q^*)}$		$\frac{Q^*-Q_E}{Q_E}$		$\frac{g(Q_E)-g(Q^*)}{g(Q^*)}$		$\frac{Q^*-Q_E}{Q_E}$		$\frac{g(Q_E)-g(Q^*)}{g(Q^*)}$	
λ	μ/λ	Mean	Max	Mean	Max	Mean	Max	Mean	Max	Mean	Max	Mean	Max
0.5	2	6.0309	19.1206	1.1160	2.7772	10.8358	31.5462	1.8143	4.6770	8.7467	31.5462	1.5107	4.6770
0.5	4	3.1932	10.5675	0.8640	2.8708	5.2314	22.5438	1.3515	3.0025	4.7978	22.5438	1.2478	3.0025
0.5	10	1.1191	4.1703	0.3173	1.4965	2.9531	26.5143	0.9236	6.1261	2.7471	26.5143	0.8554	6.1261
0.5	20	0.4274	1.8094	0.0929	0.5643	1.5066	11.7376	0.4603	4.1228	1.3567	11.7376	0.4093	4.1228
1	2	4.2244	13.6729	1.0107	2.9829	8.6177	126.1182	1.6112	5.3209	8.1783	126.1182	1.5511	5.3209
1	4	2.1314	7.3427	0.6252	2.4102	4.9718	149.3780	1.2142	13.1484	4.8868	149.3780	1.1965	13.1484
1	10	0.6913	2.7473	0.1759	0.9483	1.6166	25.2210	0.4934	10.4383	1.6029	25.2210	0.4886	10.4383
1	20	0.2466	1.1140	0.0425	0.2888	0.8488	10.9227	0.2323	4.4214	0.8361	10.9227	0.2283	4.4214
4	2	1.9071	6.6524	0.5629	2.2521	3.6588	25.9486	1.0844	7.1422	3.5552	25.9486	1.0535	7.1422
4	4	0.8662	3.3396	0.2335	1.1842	2.4758	81.7990	0.7377	13.3535	2.4527	81.7990	0.7305	13.3535
4	10	0.2374	1.0766	0.0402	0.2748	0.8264	43.1420	0.2254	10.6368	0.8220	43.1420	0.2240	10.6368
4	20	0.0741	0.3687	0.0063	0.0495	0.3708	6.2644	0.0800	2.5202	0.3675	6.2644	0.0792	2.5202
8	2	1.2253	4.5125	0.3522	1.6173	2.7964	24.0691	0.8464	5.2964	2.7133	24.0691	0.8203	5.2964
8	4	0.5252	2.1648	0.1228	0.7102	1.4677	31.2644	0.4445	7.8018	1.4549	31.2644	0.4401	7.8018
8	10	0.1319	0.6319	0.0163	0.1214	0.4937	26.8260	0.1211	10.4354	0.4915	26.8260	0.1205	10.4354
8	20	0.0389	0.1989	0.0020	0.0165	0.3401	74.0756	0.0825	27.6804	0.3369	74.0756	0.0817	27.6804
12	2	0.9319	3.5577	0.2553	1.2685	2.5581	42.3550	0.7312	7.4394	2.4129	42.3550	0.6887	7.4394
12	4	0.3855	1.6533	0.0806	0.5009	1.1568	18.2377	0.3529	7.4506	1.1384	18.2377	0.3464	7.4506
12	10	0.0921	0.4528	0.0091	0.0702	0.4234	12.7753	0.1022	5.7310	0.4188	12.7753	0.1009	5.7310
12	20	0.0264	0.1365	0.0010	0.0082	0.2194	11.6338	0.0491	4.8965	0.2155	11.6338	0.0481	4.8965
Total		1.2253	19.1206	0.2963	2.9829	1.2075	149.3780	0.3268	27.6804	1.2079	149.3780	0.3262	27.6804

Since the EOQD approaches the EOQ as λ decreases or μ increases (Proposition 1), the difference between the EOQ and EOQD solutions decreases as λ decreases or μ increases, as does the "ignorance cost."

7.4 Power-of-Two Policies

For each instance, we computed the optimal power-of-two policy using $T_B = 1/52$ (1 week) by enumerating $k = \ldots, -2, -1, 0, 1, 2, \ldots$. Table 8 lists the mean and maximum of the value of $f(2^{k^*}T_B)/f(T^*)$, where k^* is the optimal value of k.

The cost of the optimal power-of-two policy is, on average, 1.020 times that of the optimal policy in our tests. As predicted by Theorem 16, the increase in cost is less than the bound of $3\sqrt{2}/4 \approx 1.06066$ for every instance tested. Moreover, this bound appears to be tight, since an increase of 1.06062 was attained by one instance.

7.5 Implications for Exact Function

The accuracy of our approximation, proven in Section 4 and demonstrated numerically in Section 7.2, suggests that the analytical results proven for g also apply to g_0, at least approximately.

Table 8: Power-of-two policies: $f(2^{k^*}T_B)/f(T^*)$.

λ	μ/λ	Benchmark Mean	Benchmark Max	Random Mean	Random Max	Overall Mean	Overall Max
0.5	2	1.0161	1.0377	1.0118	1.0381	1.0137	1.0381
0.5	4	1.0175	1.0567	1.0192	1.0474	1.0188	1.0567
0.5	10	1.0206	1.0465	1.0180	1.0575	1.0183	1.0575
0.5	20	1.0235	1.0484	1.0160	1.0577	1.0171	1.0577
1	2	1.0216	1.0426	1.0209	1.0527	1.0210	1.0527
1	4	1.0162	1.0426	1.0186	1.0588	1.0186	1.0588
1	10	1.0177	1.0408	1.0197	1.0597	1.0197	1.0597
1	20	1.0168	1.0542	1.0190	1.0595	1.0190	1.0595
4	2	1.0210	1.0499	1.0187	1.0578	1.0188	1.0578
4	4	1.0136	1.0329	1.0199	1.0600	1.0198	1.0600
4	10	1.0170	1.0545	1.0199	1.0600	1.0199	1.0600
4	20	1.0237	1.0590	1.0203	1.0605	1.0203	1.0605
8	2	1.0258	1.0559	1.0200	1.0573	1.0203	1.0573
8	4	1.0256	1.0563	1.0198	1.0598	1.0198	1.0598
8	10	1.0220	1.0575	1.0198	1.0603	1.0198	1.0603
8	20	1.0215	1.0598	1.0197	1.0604	1.0197	1.0604
12	2	1.0142	1.0317	1.0204	1.0586	1.0198	1.0586
12	4	1.0206	1.0498	1.0195	1.0601	1.0195	1.0601
12	10	1.0249	1.0585	1.0214	1.0603	1.0215	1.0603
12	20	1.0201	1.0601	1.0205	1.0606	1.0205	1.0606
Total		1.0200	1.0601	1.0199	1.0606	1.0199	1.0606

In this section we present numerical evidence confirming these results.

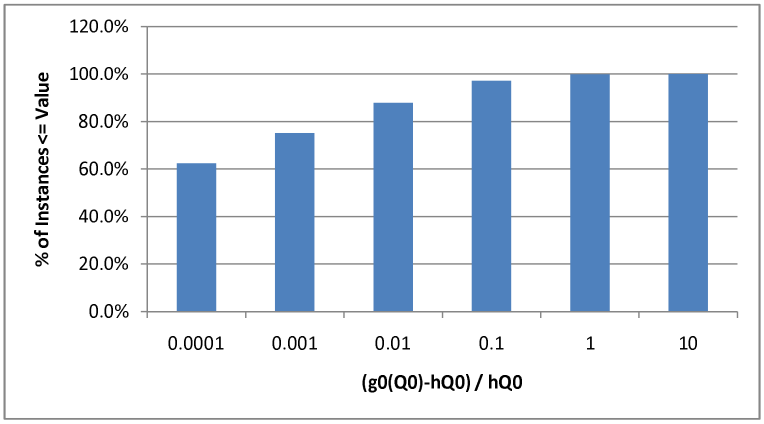
We first examine whether $g_0(Q_0) \approx hQ_0$ (recall that $g(Q^*) = hQ^*$ by Theorem 10). Figure 7 contains a histogram for $(g_0(Q_0) - hQ_0)/hQ_0$ and demonstrates that $g_0(Q_0) \neq hQ_0$—in fact, the two are close for the majority of instances (the relative difference is less than 0.001 for 75.2% of instances) but the relative difference can be in excess of 1. On the other hand, for every instance, $g_0(Q_0)$ is greater than or equal to hQ_0 (or is very slightly less, within the margin of error of the optimization procedure), leading us to make the following conjecture:

Conjecture 17 $g_0(Q_0) \geq hQ_0$.

Next, we examine the error that results from using the EOQ solution instead of Q_0. We performed an analysis similar to that in Table 7 but using g_0 and Q_0 in place of g and Q^*. The results suggest that, as in Section 7.3, the EOQ solution can perform quite poorly if disruptions are possible: The average increase in cost was 21.1% across all instances, with a maximum of 566.6%. (Detailed tabular results are omitted due to space considerations.)

We also tested whether the sensitivity analysis result for g in Theorem 13 holds approximately for g_0. For each instance, we calculated $g_0(\gamma Q_0)$ for $\gamma \in \{0.7, 0.8, 0.9, 1.1, 1.2, 1.3\}$, and then calculated the right-hand side of (9) using Q_0 and $\beta_0(\gamma Q_0)$ in place of Q^* and β. Let

Figure 7: Histogram: $(g_0(Q_0) - hQ_0)/hQ_0$.



$\psi = (\text{RHS} - g_0(\gamma Q_0))/g_0(\gamma Q_0)$, where RHS is the right-hand side of (9). If Theorem 13 holds *exactly* for g_0, then $\psi = 0$.

We omit detailed results here but summarize them as follows: $\psi \leq 0.01\%$ for 72.0% of all instances tested, $\psi \leq 0.1\%$ for 86.9%, and $\psi \leq 1\%$ for 97.8%. The result is slightly less accurate (i.e., ψ is slightly larger) as Q moves farther from Q_0. The largest value of ψ we found was 5.4%, which suggests that Theorem 13 does not hold exactly for g_0, although it appears to hold approximately for the vast majority of instances. In addition, we found $\psi \geq 0$ for every instance (to within the margin of error for the optimization), leading us to make the following conjecture:

Conjecture 18 *The right-hand side of* (9) *always overestimates the true error.*

Proposition 3 lends support to this conjecture, since it implies that g_0 is flatter around its optimum than g is.

Finally, we examine power-of-two policies for the exact cost function g_0. Here, we make the following conjecture:

Conjecture 19 *Theorem 16 holds exactly for g_0; that is, the cost of the optimal power-of-two solution under g_0 is at most $3\sqrt{2}/4$ times $g_0(Q_0)$.*

To test this conjecture, we determined the optimal power-of-two solution using $T_B = 1/52$ as described in Section 7.4, substituting g_0 and Q_0 for g and Q^*. We found that the ratio $f(2^{k^*}T_B)/f(T^*)$ attained a mean value of 1.019 and a maximum value, over all 10200 instances, of $1.06062 < 3\sqrt{2}/4$, providing numerical evidence for Conjecture 19. The conjecture is reasonable in light of Conjecture 18, since the earlier conjecture implies that g_0 is less sensitive to deviations from the optimal Q than g is.

8 Conclusions

In this paper, we presented a simple approximation for an EOQ model with disruptions (EOQD). Our approximation is quite tight, especially when the order cycle time is long relative to the duration of wet and/or dry periods. We presented a closed-form solution to our model and provided theoretical and numerical bounds on the error in the cost, the optimal solution, and the optimality error resulting from using the approximate solution as a heuristic for the exact one. We then introduced a number of analytical properties of our exact model, showing that it behaves like the EOQ in several important ways and deriving sensitivity analysis and power-of-two results that mirror those for the EOQ. On the other hand, we proved that, although the cost functions are similar, the EOQ solution may be a poor substitute for the EOQD solution; thus, ignoring supply uncertainty when it exists can be very costly. We also demonstrated numerically that the analytical results that we proved for the approximate function also hold, at least approximately, for the exact function.

Interest in supply chain models with supply disruptions has been growing steadily in recent years. A number of papers have appeared in the literature that incorporate supply disruptions into classical inventory models. Unfortunately, the introduction of supply uncertainty often destroys the tractability of otherwise simple models, forcing a numerical solution. Although these models are interesting in their own right, their impact is amplified when researchers can obtain analytical results and insights from them or embed them into more complex models (e.g., the multi-echelon supply chain design models of Qi and Shen (2007) and Qi et al. (2008)). The lack of closed-form solutions often makes both goals difficult to attain. We expect the formulation of approximations to other inventory and supply chain models with disruptions to be an active area of future research.

Acknowledgments

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Appendix: Proofs

Proof of Theorem 2. The reader can verify that

$$\frac{dg}{dQ} = \frac{\frac{h\mu^2}{2}Q^2 + \beta Dh\mu Q - (KD\mu + D^2 p\beta)\mu}{(Q\mu + \beta D)^2} \tag{13}$$

$$\frac{d^2g}{dQ^2} = \frac{D\mu(h\beta^2 D + 2\mu^2 K + 2\mu D p\beta)}{(Q\mu + \beta D)^3} \tag{14}$$

Since all terms in d^2g/dQ^2 are positive, g is convex, proving part (a). To prove part (b), note that

$$\frac{dg}{dQ} = 0 \iff \frac{h\mu^2}{2}Q^2 + \beta Dh\mu Q - (KD\mu + D^2p\beta)\mu = 0$$

$$\iff Q = \frac{-\beta Dh \pm \sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)}}{h\mu}$$

using the quadratic formula. Clearly, using the $+$ sign in the $\pm$ yields a positive value of Q while using the $-$ sign yields a negative value. □

Before proving the remaining results, we introduce two lemmas and the proof of Theorem 10 (out of order), all of which are used in subsequent proofs.

Lemma 20 $(Q^*)^2 = \frac{2D}{h\mu}(K\mu + Dp\beta - \beta hQ^*)$

Proof. Follows from setting the right-hand side of (13) to 0. □

Proof of Theorem 10.

$$\begin{aligned}
\frac{g(Q^*)}{Q^*} &= \frac{KD\mu + \frac{h\mu}{2}\frac{2D}{h\mu}(K\mu + Dp\beta - \beta hQ^*) + D^2p\beta}{Q^*(Q^*\mu + \beta D)} \text{ (using Lemma 20)} \\
&= \frac{2(KD\mu + D^2p\beta) - \beta DhQ^*}{Q^*(Q^*\mu + \beta D)} \\
&= \frac{(Q^*)^2h\mu + \beta DhQ^*}{Q^*(Q^*\mu + \beta D)} \text{ (using Lemma 20 again)} \\
&= \frac{Q^*h\mu + \beta Dh}{Q^*\mu + \beta D} \\
&= h
\end{aligned}$$

□

Lemma 21 $\sqrt{2KDh} < g(Q^*) < Dp$

Proof. By assumption,

$$\begin{aligned}
&\sqrt{2KDh} < Dp \\
\implies &2\beta h\mu D\sqrt{2KDh} + 2KDh\mu^2 + (\beta Dh)^2 < 2\beta h\mu D^2 p + 2KDh\mu^2 + (\beta Dh)^2 \\
\implies &\left(\mu\sqrt{2KDh} + \beta Dh\right)^2 < (\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta) \\
\implies &\sqrt{2KDh} < \frac{\sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)} - \beta Dh}{\mu} \\
&= hQ^* = g(Q^*)
\end{aligned}$$

by Theorem 10. Similarly,

$$
\begin{aligned}
& Dp > \sqrt{2KDh} \\
\implies\ & (Dp\mu)^2 + 2D^2p\mu\beta h + (\beta Dh)^2 > 2KDh\mu^2 + 2D^2p\mu\beta h + (\beta Dh)^2 \\
\implies\ & (Dp\mu + \beta Dh)^2 > (\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta) \\
\implies\ & Dp > \frac{\sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)} - \beta Dh}{\mu} \\
& = hQ^* = g(Q^*).
\end{aligned}
$$

□

Proof of Proposition 3.

(a) The reader can verify that

$$
\begin{aligned}
g(Q) - g_0(Q) &= \frac{(h\mu Q^2/2 + KD\mu + D^2p\beta)(Q\mu + \beta_0 D) - (h\mu Q^2/2 + KD\mu + D^2p\beta_0)(Q\mu + \beta D)}{(Q\mu + \beta D)(Q\mu + \beta_0 D)} \\
&= \frac{(\beta - \beta_0)D\mu(DpQ - KD - hQ^2/2)}{(Q\mu + \beta D)(Q\mu + \beta_0 D)}. \qquad (15)
\end{aligned}
$$

Now, (15) is non-negative iff $\beta - \beta_0$ and $DpQ - KD - hQ^2/2$ have the same sign. But $DpQ - KD - hQ^2/2 \geq 0$ iff $g_E(Q) \leq Dp$. Therefore, $g(Q) \geq g_0(Q)$ iff *either* $\beta \geq \beta_0$ and $g_E(Q) \leq Dp$ *or* $\beta \leq \beta_0$ and $g_E(Q) \geq Dp$, and equality holds iff $\beta = \beta_0$ or $g_E(Q) = Dp$.

(b) By part (a), it suffices to show that $g_E(Q^*) < Dp$. This is immediate from Proposition 8, below, and Lemma 21. (Note that, although Proposition 8 appears after this proposition, its proof does not rely on this or any subsequent results.) □

Proof of Theorem 4.

(a)

$$
g(Q) - g_0(Q) = \frac{(\beta - \beta_0)D\mu(DpQ - KD - hQ^2/2)}{(Q\mu + \beta D)(Q\mu + \beta_0 D)}
$$

(see (15)). Therefore

$$
\begin{aligned}
\frac{|g(Q) - g_0(Q)|}{g_0(Q)} &= \left| \frac{(\beta - \beta_0)D\mu(DpQ - KD - hQ^2/2)}{(Q\mu + \beta D)(h\mu Q^2/2 + KD\mu + D^2p\beta_0)} \right| \\
&= \left| \frac{(\beta - \beta_0)D\mu(DpQ - KD - hQ^2/2)}{(Q\mu + \beta D)(KD\mu + h\mu Q^2/2)} \frac{KD\mu + h\mu Q^2/2}{KD\mu + h\mu Q^2/2 + D^2p\beta_0} \right| \\
&= \left| \frac{(\beta - \beta_0)(Dp - KD/Q - hQ/2)}{(Q\mu/D + \beta)(KD/Q + hQ/2)} \frac{KD/Q + hQ/2}{KD/Q + hQ/2 + D^2p\beta_0/\mu Q} \right| \\
&= \left| \frac{(\beta - \beta_0)(Dp - g_E(Q))}{(Q\mu/D + \beta)g_E(Q)} \frac{g_E(Q)}{g_E(Q) + D^2p\beta_0/\mu Q} \right| \\
&= \left| \frac{\beta - \beta_0}{(Q\mu/D + \beta)(1 + D^2p\beta_0/\mu Q g_E(Q))} \left[\frac{Dp}{g_E(Q)} - 1 \right] \right| \\
&< \frac{|\beta - \beta_0|}{(Q\mu/D)(D^2p\beta_0/\mu Q g_E(Q))} \left[\frac{Dp}{g_E(Q)} - 1 \right] \quad \text{(because } Dp/g_E(Q) > 1\text{)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{|\beta - \beta_0|}{\beta_0} \frac{g_E(Q)}{Dp} \left[\frac{Dp}{g_E(Q)} - 1 \right] \\
&= \frac{|\beta - \beta_0|}{\beta_0} \left[1 - \frac{g_E(Q)}{Dp} \right] \\
&< \frac{|\beta - \beta_0|}{\beta_0}
\end{aligned}
$$

(b) The first term in the minimization follows from (a). To prove the second:

$$
\begin{aligned}
g(Q^*) - g_0(Q^*) &= hQ^* - \frac{h\mu(Q^*)^2/2 + KD\mu + D^2 p\beta_0}{Q^*\mu + \beta_0 D} \text{ (by Theorem 10)} \\
&= \frac{\frac{h\mu}{2}(Q^*)^2 + \beta_0 DhQ^* - KD\mu - D^2 p\beta_0}{Q^*\mu + \beta_0 D} \\
&= \frac{D(K\mu + Dp\beta - \beta hQ^*) + \beta_0 DhQ^* - KD\mu - D^2 p\beta_0}{Q^*\mu + \beta_0 D} \text{ (by Lemma 20)} \\
&= \frac{(\beta - \beta_0)(Dp - hQ^*)D}{Q^*\mu + \beta_0 D}
\end{aligned}
$$

Then, since $Dp - hQ^* > 0$ by Lemma 21 and Theorem 10,

$$
\begin{aligned}
\frac{|g(Q^*) - g_0(Q^*)|}{g_0(Q^*)} &= \frac{|\beta - \beta_0|(Dp - hQ^*)D}{Q^*\mu + \beta_0 D} \cdot \frac{Q^*\mu + \beta_0 D}{h\mu(Q^*)^2/2 + KD\mu + D^2 p\beta_0} \\
&= \frac{|\beta - \beta_0|(Dp - hQ^*)D}{KD\mu + D(K\mu + Dp\beta - \beta hQ^*) + D^2 p\beta_0} \text{ (by Lemma 20)} \\
&= \frac{|\beta - \beta_0|(Dp - hQ^*)D}{2KD\mu + \beta_0 D^2 p + \beta D(Dp - hQ^*)} \\
&= \frac{|\beta - \beta_0|}{\beta + \frac{\beta_0 Dp}{Dp - hQ^*} + \frac{2K\mu}{Dp - hQ^*}} \\
&\leq \frac{|\beta - \beta_0|}{\beta + \beta_0} \\
&< 1 \text{(by the triangle inequality.)}
\end{aligned}
$$

(c) First let $K = 10$, $h = p = 1$, $D = 50$, $\lambda = 1$, $\mu = 2$, $r = 1$. This instance satisfies the assumptions from Section 3.2, as well as the assumption from part (b) of the theorem since $Q^* = 35.2875$, $g_E(Q^*) = 31.8131$, and $Dp = 50$. In addition, $\beta = 0.3333$ and $\beta_0(Q^*) = 0.2932$. The bounds are then equal to

$$
\begin{aligned}
\frac{|\beta - \beta_0(Q^*)|}{\beta_0(Q^*)} \left[1 - \frac{g_E(Q^*)}{Dp} \right] &= \frac{0.3333 - 0.2932}{0.2932} \left[1 - \frac{31.8131}{50} \right] = 0.0498 \\
\frac{\beta - \beta_0(Q^*)}{\beta + \beta_0(Q^*)} &= \frac{0.3333 - 0.2932}{0.3333 + 0.2932} = 0.0640,
\end{aligned}
$$

so the first bound prevails.

Now let $D = 100$ but keep all other parameters as-is. Again, the assumptions from Section 3.2 are satisfied, as is the assumption from part (b) ($Q^* = 58.2407$, $g_E(Q^*) = 46.2905 < 100 = Dp$).

Then $\beta = 0.3333$, $\beta_0(Q^*) = 0.2752$, and

$$\frac{\beta - \beta_0(Q^*)}{\beta_0(Q^*)}\left[1 - \frac{g_E(Q^*)}{Dp}\right] = \frac{0.3333 - 0.2752}{0.2752}\left[1 - \frac{46.2905}{100}\right] = 0.1133$$

$$\frac{\beta - \beta_0(Q^*)}{\beta + \beta_0(Q^*)} = \frac{0.3333 - 0.2752}{0.3333 + 0.2752} = 0.0954,$$

so the second bound prevails. □

Proof of Proposition 5. The first derivative of g_0 is given by[1]

$$\begin{aligned}\frac{dg_0}{dQ} =& \frac{1}{\left[Q\mu + \beta D\left(1 - e^{-(\lambda+\mu)Q/D}\right)\right]^2}\left(\frac{h\mu^2}{2}Q^2 + \beta Dh\mu Q - KD\mu^2 - D^2p\beta\mu\right.\\ &+ \left.\left[-\frac{h\lambda\mu}{2}Q^2 - \beta Dh\mu Q - KD\lambda\mu + D^2p\beta\mu + pD\lambda\mu Q\right]e^{-(\lambda+\mu)Q/D}\right)\end{aligned} \tag{16}$$

The first-order condition is satisfied if the numerator is 0. The numerator can be rewritten as

$$\frac{h\mu^2}{2}Q^2 + \beta Dh\mu Q - KD\mu^2 - D^2p\beta\mu \tag{17a}$$

$$+\left[-\frac{h\mu^2}{2}Q^2 - \beta Dh\mu Q + KD\mu^2 + D^2p\beta\mu\right. \tag{17b}$$

$$\left.+\frac{h\mu}{2}(\mu - \lambda)Q^2 - KD\mu(\mu + \lambda) + pD\lambda\mu Q\right]e^{-(\lambda+\mu)Q/D} \tag{17c}$$

Now suppose that $Q = Q^*$; we will show that (17) is positive. The expression in (17a) is the first-order condition for g (see (13)) and that in (17b) is its negative, so when $Q = Q^*$, both (17a) and (17b) equal 0. Using Lemma 20, (17c) can be rewritten as

$$\begin{aligned}&\left[\frac{h\mu}{2}\frac{2D}{h\mu}(K\mu + Dp\beta - \beta hQ^*)(\mu - \lambda) - KD\mu(\mu + \lambda) + pD\lambda\mu Q^*\right]e^{-(\lambda+\mu)Q^*/D}\\ =& \left[\beta D(\mu - \lambda)(Dp - hQ^*) + \lambda\mu(Q^*Dp - 2KD)\right]e^{-(\lambda+\mu)Q^*/D}\end{aligned}$$

The first term inside the brackets is positive since $\mu - \lambda > 0$ by assumption and $Dp - hQ^* > 0$ by Theorem 10 and Lemma 21. The second term is positive since, again using Theorem 10 and Lemma 21,

$$Q^*h > \sqrt{2KDh} \implies (Q^*)^2h > 2KD$$

and

$$Dp > Q^*h \implies Q^*Dp > (Q^*)^2h > 2KD.$$

Therefore (17), and hence dg_0/dQ, is positive when $Q = Q^*$. Since g_0 is quasiconvex (Proposition 2(b) in Berk and Arreola-Risa (1994)), it must attain its minimum to the left of Q^*. Therefore $Q_0 < Q^*$, as desired. □

[1]Note: The first-order condition given in Proposition 2(c) of Berk and Arreola-Risa contains an error: the first term on the second line should read $-\frac{C_h DQ^2}{2D}$ instead of $-\frac{C_h \lambda Q^2}{2D}$. Translated into our notation, the corrected expression is the numerator of (16) above.

Proof of Theorem 6.

Since $\frac{d^3 g_0}{dQ^3} < 0$, g_0' is concave on $[Q_0, Q^*]$. Therefore

$$\frac{g_0'(Q^*) - g_0'(Q_0)}{Q^* - Q_0} \geq g_0''(Q^*) \tag{18}$$

by the concavity of g_0' (see, e.g., Bazaraa, Sherali and Shetty 1993). But $g_0'(Q_0) = 0$ since Q_0 minimizes g_0, so we have

$$\frac{g_0'(Q^*)}{Q^* - Q_0} \geq g_0''(Q^*). \tag{19}$$

Equations (18) and (19) both hold whether $Q_0 \leq Q^*$ or $Q_0 > Q^*$. If $Q_0 \leq Q^*$, we have

$$0 \leq \frac{Q^* - Q_0}{Q^*} \leq \frac{g_0'(Q^*)}{Q^* g_0''(Q^*)}, \tag{20}$$

while if $Q_0 > Q^*$, we have

$$0 \geq \frac{Q^* - Q_0}{Q^*} \geq \frac{g_0'(Q^*)}{Q^* g_0''(Q^*)} \tag{21}$$

(in this case $g_0'(Q^*) < 0$). Combining (20) and (21),

$$\frac{|Q^* - Q_0|}{Q^*} \leq \frac{|g_0'(Q^*)|}{Q^* g_0''(Q^*)}, \tag{22}$$

as desired. □

Proof of Theorem 7.

First note that since $r = 1$,

$$\frac{\beta - \beta_0(Q^* - \theta)}{\beta} = e^{-(\lambda+\mu)(Q^*-\theta)/D} = \frac{\beta - \beta_0(Q^*)}{\beta} \cdot \frac{\beta - \beta_0(-\theta)}{\beta},$$

so

$$\beta_0(Q^* - \theta) = \beta_0(Q^*) + \beta_0(-\theta) - \frac{\beta_0(Q^*)\beta_0(-\theta)}{\beta}. \tag{23}$$

By Proposition 5, $Q_0 \leq Q^*$.

$$\begin{aligned} g_0(Q_0) &= \frac{h\mu Q_0^2/2 + KD\mu + D^2 p\beta_0(Q_0)}{Q_0\mu + \beta_0(Q_0)D} \\ &\geq \frac{h\mu(Q^* - \theta)^2/2 + KD\mu + D^2 p\beta_0(Q^* - \theta)}{Q^*\mu + \beta_0(Q^*)D} \end{aligned} \tag{24}$$

The inequality follows from Theorem 6(a) (note that the absolute values can be removed since $Q_0 \leq Q^*$ and $g_0'(Q^*) > 0$) and the fact that $\beta_0(Q)$ is increasing in Q. Then

$$\begin{aligned} \frac{g_0(Q^*) - g_0(Q_0)}{g_0(Q_0)} &\leq \frac{h\mu(Q^*)^2/2 + KD\mu + D^2 p\beta_0(Q^*)}{h\mu(Q^* - \theta)^2/2 + KD\mu + D^2 p\beta_0(Q^* - \theta)} - 1 \\ &= \frac{h\mu\theta(2Q^* - \theta)/2 - D^2 p\beta_0(-\theta)\left[1 - \frac{\beta_0(Q^*)}{\beta}\right]}{h\mu(Q^* - \theta)^2/2 + KD\mu + D^2 p\beta_0(Q^* - \theta)} \end{aligned}$$

using (23) and (24). □

Proof of Proposition 8.

(a) For all $Q > 0$,

$$g(Q) - g_E(Q) = \frac{h\mu Q^2/2 + KD\mu + D^2 p\beta}{Q\mu + \beta D} - \left[\frac{KD}{Q} + \frac{hQ}{2}\right]$$
$$= \frac{D^2 p\beta - (KD^2\beta/Q) - (hQ\beta D)/2}{Q\mu + \beta D}$$

Thus $g(Q) > g_E(Q)$ iff

$$\frac{KD}{Q} + \frac{hQ}{2} < Dp,$$

i.e., iff $g_E(Q) < Dp$.

(b) By part (a), it suffices to prove $g_E(Q^*) < Dp$.

$$g_E(Q^*) < Dp \iff \frac{h}{2}(Q^*)^2 - DpQ^* + KD < 0$$
$$\iff \frac{Dp - \sqrt{(Dp)^2 - 2KDh}}{h} < Q^* < \frac{Dp + \sqrt{(Dp)^2 - 2KDh}}{h}$$
$$\iff Dp - \sqrt{(Dp)^2 - 2KDh} < Q^* h = g(Q^*) < Dp - \sqrt{(Dp)^2 - 2KDh}$$

The equality follows from Theorem 10. The second inequality follows from Lemma 21. To prove the first inequality, note that for any a, b, $c > 0$ such that $a < b$ and $c \le a^2$, $b - \sqrt{b^2 - c} < a - \sqrt{a^2 - c}$ by the concavity of the square-root function. Since $\sqrt{2KDh} < Dp$ by assumption, we have

$$Dp - \sqrt{(Dp)^2 - 2KDh} < \sqrt{2KDh} - \sqrt{2KDh - 2KDh} = \sqrt{2KDh} < g(Q^*)$$

by Lemma 21, confirming the first inequality. □

Proof of Proposition 9.

(a) By Lemma 20,

$$(Q^*)^2 = \frac{2KD}{h} + \frac{2D\beta}{h\mu}(Dp - hQ^*).$$

By Lemma 21, $Dp - hQ^* > 0$, so $(Q^*)^2 > 2KD/h$, i.e., $Q^* > \sqrt{2KD/h}$.

(b) As $p \to \infty$, $Q^* \to \infty$ but Q_E stays constant.

(c),(d) Follow from Theorem 10 and the analogous result for the EOQ. □

Proof of Theorem 11.

(a) By (6) and Theorem 10,

$$g(Q^*) = \frac{\sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2 p\beta)} - \beta Dh}{\mu}. \tag{25}$$

First consider h. For convenience, define $a \equiv \beta D$ and $b \equiv 2\mu(KD\mu + D^2 p\beta)$. Define $\phi(h) \equiv \sqrt{a^2 + b/h}$, and let $g^*(h)$ denote the optimal cost, $g(Q^*)$, taken as a function of h. Then $g^*(h) =$

$h(\phi(h) - a)/\mu$. Note that

$$\begin{aligned}
\phi'(h) &= \frac{-b}{2\phi(h)h^2} < 0 \\
\phi''(h) &= \frac{2b\phi(h) + bh\phi'(h)}{2\phi(h)^2h^3} \\
&= \frac{2b\phi(h) - b^2h/2\phi(h)h^2}{2\phi(h)^2h^3} \\
&= \frac{4b\phi(h)^2h - b^2}{4\phi(h)^3h^4} \\
&= \frac{4a^2bh + 3b^2}{4\phi(h)^3h^4} > 0
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dg^*}{dh} &= \frac{h\phi'(h) + \phi(h) - a}{\mu} \\
\frac{d^2g^*}{dh^2} &= \frac{h\phi''(h) + 2\phi'(h)}{\mu} \\
&= \frac{4a^2bh^2 + 3b^2h - 4b\phi(h)^2h^2}{4\mu\phi(h)^3h^4} \\
&= \frac{-b^2}{4\phi(h)^3h^3} < 0
\end{aligned}$$

Therefore $g^*(h)$ is strictly concave in h. Furthermore, $g^*(h)$ is increasing because it is strictly concave, defined on an infinite domain, and bounded below by 0.

Monotonicity and concavity with respect to p and K are evident from (25) since each appears only under the square root, which has a positive coefficient. The proof of monotonicity and concavity with respect to D is similar to that for h.

(b) Monotonicity and concavity with respect to D, p, and K are immediate from part (a) and Theorem 10. Now, let $Q^*(h)$ denote the optimal order quantity, Q^*, taken as a function of h. Then $Q^*(h) = (\phi(h) - a)/\mu$, $dQ^*/dh = \phi'(h)/\mu < 0$, and $d^2Q^*/dh^2 = \phi''(h)/\mu > 0$. Therefore Q^* is decreasing and strictly convex in h. □

Proof of Theorem 13. By Theorem 10,

$$\begin{aligned}
\frac{g(Q)}{g(Q^*)} &= \frac{h\mu Q^2/2 + KD\mu + D^2p\beta}{Q\mu + \beta D}\cdot\frac{1}{hQ^*}\\
&= \frac{KD\mu + D^2p\beta}{hQ^*(Q\mu+\beta D)} + \frac{\mu Q^2}{2Q^*(Q\mu+\beta D)}\\
&= \left(\frac{KD\mu + D^2p\beta}{hQ^*Q\mu} + \frac{\mu Q^2}{2Q^*Q\mu}\right)\frac{Q\mu}{Q\mu+\beta D}\\
&= \left(\frac{\frac{h\mu}{2}(Q^*)^2 + \beta DhQ^*}{hQQ^*\mu} + \frac{Q}{2Q^*}\right)\frac{Q\mu}{Q\mu+\beta D} \text{ (using Lemma 20)}\\
&= \left(\frac{Q^*}{2Q} + \frac{\beta D}{Q\mu} + \frac{Q}{2Q^*}\right)\frac{Q\mu}{Q\mu+\beta D}\\
&= \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right)\frac{Q\mu}{Q\mu+\beta D} + \frac{\beta D}{Q\mu+\beta D}\\
&= \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right) - \left[\frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right) - 1\right]\frac{\beta D}{Q\mu+\beta D}\\
&= \epsilon\left(\frac{Q^*}{Q}\right) - \left[\epsilon\left(\frac{Q^*}{Q}\right) - 1\right]\frac{\beta D}{Q\mu+\beta D}
\end{aligned}$$

as desired. □

Proof of Lemma 15. We first prove the equality, then the inequality.

$$\begin{aligned}
f\left(\frac{3}{2}\hat{T}\right) - f\left(\frac{3}{4}\hat{T}\right) &= \frac{\frac{h\mu D}{2}\left(\frac{3}{2}\hat{T}\right)^2 + K\mu + Dp\beta}{\frac{3}{2}\hat{T}\mu + \beta} - \frac{\frac{h\mu D}{2}\left(\frac{3}{4}\hat{T}\right)^2 + K\mu + Dp\beta}{\frac{3}{4}\hat{T}\mu + \beta}\\
&= \frac{3\hat{T}\mu[9h\mu D\hat{T}^2 + 18h\beta D\hat{T} - 16(K\mu + Dp\beta)]}{8(3\hat{T}\mu + 2\beta)(3\hat{T}\mu + 4\beta)}
\end{aligned} \tag{26}$$

Since

$$\begin{aligned}
\hat{T}^2 &= \frac{2(\beta h)^2 + \frac{16}{9}h\mu\left(\frac{K\mu}{D} + p\beta\right) - 2\beta h\sqrt{(\beta h)^2 + \frac{16}{9}h\mu\left(\frac{K\mu}{D} + p\beta\right)}}{(h\mu)^2}\\
&= \frac{2}{h\mu}\left[\frac{8}{9}\left(\frac{K\mu}{D} + p\beta\right) - \beta h\hat{T}\right],
\end{aligned}$$

the numerator of (26) equals

$$3\hat{T}\mu\left[9h\mu D\cdot\frac{2}{h\mu}\left[\frac{8}{9}\left(\frac{K\mu}{D} + p\beta\right) - \beta h\hat{T}\right] + 18h\beta D\hat{T} - 16(K\mu + Dp\beta)\right] = 0.$$

This proves that $f\left(\frac{3}{4}\hat{T}\right) = f\left(\frac{3}{2}\hat{T}\right)$. We next prove that $f\left(\frac{3}{2}\hat{T}\right) \le \frac{3\sqrt{2}}{4}f(T^*)$. By Theorem 13,

$$\begin{aligned}
\frac{f\left(\frac{3}{2}\hat{T}\right)}{f(T^*)} = \frac{g\left(\frac{3}{2}D\hat{T}\right)}{g(Q^*)} &\le \frac{1}{2}\left(\frac{Q^*}{\frac{3}{2}D\hat{T}} + \frac{\frac{3}{2}D\hat{T}}{Q^*}\right)\\
&= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{\sqrt{1+2\alpha}-1}{\sqrt{1+\frac{16}{9}\alpha}-1} + \frac{3}{2}\cdot\frac{\sqrt{1+\frac{16}{9}\alpha}-1}{\sqrt{1+2\alpha}-1}\right)\\
&= \frac{1}{2}\left(\frac{2\sqrt{1+2\alpha}-2}{\sqrt{9+16\alpha}-3} + \frac{\sqrt{9+16\alpha}-3}{2\sqrt{1+2\alpha}-2}\right) \equiv \psi(\alpha),
\end{aligned}$$

where $\alpha = \frac{h\mu(KD\mu+D^2p\beta)}{(\beta Dh)^2} \geq 0$. We will show that $\psi(\alpha)$ is increasing in α, thus it attains its maximum value in the limit as $\alpha \to \infty$.

$$\frac{d\psi}{d\alpha} = \frac{\left[1 - 3\sqrt{9+16\alpha} + 8\sqrt{1+2\alpha}\right]\left[4\left(\sqrt{1+2\alpha}-1\right)^2 - \left(\sqrt{9+16\alpha}-3\right)^2\right]}{4\sqrt{1+2\alpha}\sqrt{9+16\alpha}\left(\sqrt{1+2\alpha}-1\right)^2\left(\sqrt{9+16\alpha}-3\right)^2}.$$

The denominator is clearly positive, and both terms in the numerator are negative:

$$\begin{aligned} 1 - 3\sqrt{9+16\alpha} + 8\sqrt{1+2\alpha} &< 0 \\ \iff 1 + 16\sqrt{1+2\alpha} + 64 + 128\alpha &< 81 + 144\alpha \\ \iff \sqrt{1+2\alpha} &< 1+\alpha, \end{aligned}$$

which holds since $1+\alpha = \sqrt{1+2\alpha+\alpha^2}$. Similarly,

$$\begin{aligned} 4\left(\sqrt{1+2\alpha}-1\right)^2 - \left(\sqrt{9+16\alpha}-3\right)^2 &< 0 \\ \iff \sqrt{4+8\alpha}+1 &< \sqrt{9+16\alpha} \\ \iff 4+8\alpha+2\sqrt{4+8\alpha}+1 &< 9+16\alpha \\ \iff 2\sqrt{4+8\alpha} &< 4+8\alpha, \end{aligned}$$

which holds since $2 < \sqrt{4+8\alpha}$. Thus, $\frac{d\psi}{d\alpha} > 0$, so

$$\begin{aligned} \max_{\alpha} \psi(\alpha) &= \lim_{\alpha\to\infty} \psi(\alpha) \\ &= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{\sqrt{2}}{\sqrt{\frac{16}{9}}} + \frac{3}{2}\cdot\frac{\sqrt{\frac{16}{9}}}{\sqrt{2}}\right) \\ &= \frac{3\sqrt{2}}{4} \approx 1.06, \end{aligned}$$

proving the lemma. □

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