Mathematics 205: Linear Methods

Professor Linghai Zhang. Office: Christmas-Saucon Hall 236.
Office Hours: Monday, Wednesday, Friday, 2:10PM-3:00PM.
Telephone: 610-758-4116. Email: liz5@lehigh.edu
Textbook: Differential Equations and Linear Algebra.
Fourth Edition. Authors: Stephen W. Goode and Scott A. Annin.
Prerequisite: Math 22.
Examination One: 100 points. Monday, July 17, 2017.
Examination Two: 100 points. Monday, July 31, 2017.
Final Examination: 200 points. August 2017.
Quizzes: 100 points.
Total score: 500 points.
Grading policy: A: 471-500,
A-: 451-470,
B+: 435-450,
B: 417-434,
B-: 401-416,
C+: 385-400,
C: 367-384,
C-: 351-366,
D: 301-350,
F: 0-300. Homework is assigned every week and it will be collected in the following week (Monday, if the professor teaches on Monday, Wednesday and Friday; Tuesday, if the professor teaches on Tuesday and Thursday). No late homework is accepted. Homework problems should be turned in before class, not after class. Homework turned in after a class is considered to be late. The lowest two homework will be dropped.

While in Math 205 classroom, no cellphones, laptop computers, calculators or any other electronic devices are allowed to use. No calculators are allowed to use in the midterm exams and final exam. Graphing calculators, or those capable of symbolic manipulations, are absolutely not allowed to use. Attendance is strongly required on every meeting day. If a student is absent from a midterm exam or the final exam without any reason, then that student will receive a zero in the midterm exam or the final exam. If a student is absent for three or more times without any reason, then the professor may have a Section three report for that student.

Students are expected to spend at least six hours every week to get familiar with the materials and to do the homework problems. The homework is an important part of the course: no one can learn mathematics passively and as learning the material is your responsibility, you should try to solve all problems assigned as soon as possible. I strongly recommend doing the homework problems right after the material has been covered in class - preferably on the same day. You will benefit far more from the lectures if you familiarize yourself with the material to be covered before class and come to prepared to ask questions. It is important that students keep up with both the homework and the lectures as the material rapidly builds upon itself. If this becomes a problem, please ask the professor for help as soon as possible.

Accommodations for students with disabilities: If somebody has disability for which that student may be requesting accommodations, please contact both your mathematics professor and the Office of Academic Supporting Services, University Center 212 (Telephone: 610-758-4152) as early as possible. You must have documentation from the Academic Supporting Services Office before accommodations can be granted.

Lehigh University endorses "The Principles of Our Equitable Community" (http://www4.lehigh.edu/diversity/principles). We expect each member of this class to acknowledge and practice these Principles. Respect for each other and for differing viewpoints is a vital component of the learning environment inside and outside the classroom.

Schedule for Mathematics 205 in 2016:

Week 1 (August 29 - September 4, 2016) Chapter 1: First Order Differential Equations (separable differential equations, first order linear differential equations) 1.1, 1.2, 1.3, 1.4, 1.5

Week 2 (September 5 - September 11, 2016) 1.6, 1.7 Chapter 2: Matrices and Systems of Linear Equations (matrix operations, elementary row operations, reduced row echelon form of matrix, the method of Gauss elimination, inverse matrix,, solutions of system of equations) 2.1, 2.2, 2.3, 2.4

Week 3 (September 12 - September 18, 2016) 2.5, 2.6

Week 4 (September 19 - September 25, 2016) Chapter 3: Determinants (definition of determinants, properties of determinants, adjoin matrix, Crammer's rule) 3.1, 3.2, 3.3

Week 5 (September 26 - October 2, 2016) 3.4 Chapter 4: Vector Spaces (definition of vector spaces, subspaces, spanning set, linear dependence and linear independence, bases and dimension, null space, row space, column space) 4.1, 4.2

Week 6 (October 3 - October 9, 2016) 4.3, 4.4

Week 7 (October 10 - October 16, 2016) 4.5, 4.6, 4.8

Week 8 (October 17 - October 23, 2016), 4.9, Chapter 6: Linear Transformations (definition of linear transformations, properties of linear transformation, one-to-one transformations, onto transformations) 6.1, 6.2, 6.3, 6.4, 6.5

Week 9 (October 24 - October 30, 2016) Chapter 7: Eigenvalues and Eigenvectors (eigenvalues, eigenvectors and eigenspaces, diagonalization) 7.1

Week 10 (October 31 - November 6, 2016) 7.2, 7.3

Week 11 (November 7 - November 13, 2016) Chapter 8: Linear Differential Equations of Order n (second order and higher order differential equations, complementary solutions, particular solutions, general solutions, annihilators, applications of differential equations)

Week 12 (November 14 - November 20, 2016) 8.1, 8.2, 8.3, 8.5

Week 13 (November 21 - November 27, 2016) 8.6, 8.7

Week 14 (November 28 - December 4, 2016) Chapter 9: Systems of Differential Equations (first order system of differential equations, the method of variation of parameters, the method of undetermined coefficients), 9.1, 9.2, 9.3, 9.4

Week 15 (December 5 - December 11, 2016) 9.6, Review for the Final Exam

Week 16 (December - December , 2016) Final Exam

Week 17 (December - December , 2016)

Week 18 Christmas

Mathematics 205 Practice problems

1.4: 8, 10, 12. 1.5: 9, 10, 14. 1.6: 4, 6, 13, 23. 1.7 1, 4.

2.1: 21-26, 2.2: 12, 15. 2.3: 9-11. 2.4: 22, 24, 26. 2.5: 25, 26, 40. 2.6: 9, 15, 18, 20, 25.

3.1: 20, 30, 40. 3.2: 5, 9, 18, 22, 25. 3.3: 15, 20, 23, 30. 3.4: 6, 10, 21, 27.

4.1: 9, 10. 4.2: 11, 13, 20, 21, 28, 29. 4.3: 11, 14, 20, 22. 4.4: 10, 13, 16, 20, 33. 4.5: 8, 12, 14, 18, 26, 35, 43, 44. 4.6: 22, 23, 30, 36, 40, 44, 46, 48, 54. 4.8: 9, 14. 4.9: 8, 10, 12.

6.1: 4, 8, 12, 18. 6.3: 4, 12, 14, 16, 18. 6.4: 17, 20, 22. 6.5: 4.
7.1: 20, 28, 32. 7.2: 14, 16, 21. 7.3: 8, 12, 16.
8.1: 23, 29, 38. 8.2: 9, 19, 21, 36. 8.3: 11, 12, 25, 29, 35. 8.5: 1, 4, 8, 12. 8.6: 6, 8. 8.7: 1, 3, 12, 18. 9.1: 8, 10, 14, 19. 9.2: 1, 6, 9. 9.3: 3, 6, 7. 9.4: 9, 10, 13, 14. 9.6: 5, 6, 8, 9. Note: Sections 4.7 and 5.5 may be covered, but we will keep it light.

Mathematics 205 - Differential Equations and Linear Algebra

Chapter One: First Order Differential Equations

Section 1.1 Differential Equations Everywhere

Section 1.2 Basic Ideas and Terminology

Section 1.3 The Geotry of First Order Differential Equations

Section 1.4 Separable Differential Equations

Solve the following separable differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\exp(3x)\cos(4x)}{\exp(5y)\cos(12y)}.$$

Solution:

$$\begin{aligned} & \frac{1}{169}\exp(5y)[5\cos(12y) + 12\sin(12y)] \\ = \ & \frac{1}{25}\exp(3x)[3\cos(4x) + 4\sin(4x)] + C. \end{aligned}$$

Section 1.5 Some Simple Population Models

Section 1.6 First order lLinear dDifferential eEquations

Let $p = p(x)$ and $q = q(x)$ be given real functions. Consider the following differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + p(x)u = q(x).$$

The integrating factor is defined by

$$\mu(x) = \exp\left[\int p(x)\mathrm{d}x\right].$$

Multiplying the given differential equation by the integrating factor $\mu(x)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\exp\left[\int p(x)\mathrm{d}x\right]u(x)\right\} = q(x)\exp\left[\int p(x)\mathrm{d}x\right].$$

Solving this equation, we obtain the general solution

$$\exp\left[\int p(x)\mathrm{d}x\right]u(x) = \int\left\{q(x)\exp\left[\int p(x)\mathrm{d}x\right]\right\}\mathrm{d}x.$$

Chapter Two: Matrices and Systems of Linear Equations

Section 2.1: Matrices: Definitions and Notations

An $m \times n$ matrix is represented by $A = (a_{ij})_{m\times n}$, or more explicitly by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

There are m rows and n columns in the matrix.

The matrix

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{pmatrix}$$

is called the transposed matrix of A.

If $m = n$, then the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

is called a square matrix.

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

is called the main diagonal of the matrix A.

The matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

is a diagonal matrix.

The matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called an identity matrix.

The matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{pmatrix}$$

is called a symmetric matrix. Note that $A^T = A$.

The matrix

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n} \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{1n} & -a_{2n} & -a_{3n} & \cdots & 0 \end{pmatrix}$$

is called a skew-symmetric matrix. Note that $A^T = -A$.

The matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

is called an upper triangular matrix.

The matrix

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

is called a lower triangular matrix.

Section 2.2 Matrix algebra

Definition Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be matrices. Let α be a real constant. Define

$$A + B = (a_{ij} + b_{ij})_{m\times n}, A - B = (a_{ij} - b_{ij})_{m\times n}, \alpha A = (\alpha a_{ij})_{m\times n}.$$

More explicitly

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} & \cdots & a_{2n} + b_{2n} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} & \cdots & a_{3n} + b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & a_{m3} + b_{m3} & \cdots & a_{mn} + b_{mn} \end{pmatrix},$$

and

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} & \cdots & \alpha a_{2n} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} & \cdots & \alpha a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha a_{m1} & \alpha a_{m2} & \alpha a_{m3} & \cdots & \alpha a_{mn} \end{pmatrix}.$$

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$ be matrices. Define

$$AB = (c_{ij})_{m\times p}, \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

The properties

$$
\begin{aligned}
(AB)C &= A(BC), \\
(A+B)C &= AC+BC, \\
A(B+C) &= AB+AC, \\
(A^T)^T &= A, \\
(A+B)^T &= A^T+B^T, \\
(AB)^T &= B^TA^T.
\end{aligned}
$$

If $A=(a_{ij})$ and $B=(b_{ij})$ are $n\times n$ square matrices, show that in general

$$(A+B)^2 \neq A^2+2AB+B^2.$$

Show, however, that the equality always holds if $AB=BA$.
Solution: Let

$$A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $AB\neq BA$, so

$$(A+B)^2 \neq A^2+2AB+B^2.$$

Section 2.3 Terminology for Systems of Linear Equations

Consider the following $m\times n$ system of linear equations

$$
\begin{aligned}
a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots+a_{1n}x_n &= b_1, \\
a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots+a_{2n}x_n &= b_2, \\
a_{31}x_1+a_{32}x_2+a_{33}x_3+\cdots+a_{3n}x_n &= b_3, \\
\cdots\cdots\cdots\cdots &\quad \cdots \\
a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\cdots+a_{mn}x_n &= b_m,
\end{aligned}
$$

where a_{ij} and b_i are given real constants. The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

is called the coefficient matrix of the system $A\mathbf{x} = \mathbf{b}$. The matrix

$$A^{\#} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{pmatrix}$$

is called the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$.

Definition. If there exists a solution, then the system $A\mathbf{x} = \mathbf{b}$ is called consistent. If there exists no solution, then the system $A\mathbf{x} = \mathbf{b}$ is call inconsistent.

Section 2.4 Elementary Row Operations and Row Echelon Matrices

Definition. The elementary row operations include:

(1) Interchange two rows.

(2) Multiply a row by a nonzero constant.

(3) Multiply a row by a real constant and add the result to another row.

Definition. An $m \times n$ matrix $A = (a_{ij})$ is called a row-echelon matrix if it satisfies the following three conditions.

(1) If there are any rows consisting entirely of zeros, they are grouped together at the bottom of the matrix.

(2) The first nonzero element in any nonzero row is a leading 1.

(3) The leading 1 of any row below the first row is to the right of the leading 1 of the row above it.

Definition. An $m \times n$ matrix $A = (a_{ij})$ is called a reduced row-echelon matrix if it is a row-echelon matrix and it satisfies the additional condition:

(4) Any column that contains a leading 1 has zeros everywhere else.

Definition. The number of nonzero rows in the reduced row echelon matrix is called the rank of the matrix.

Examples:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Find the reduced row echelon form and the rank of all matrix

$$\begin{pmatrix} 3 & 2 & -5 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 3 & -1 & 4 & 2 \\ 1 & -1 & 2 & 3 \\ 7 & -1 & 8 & 0 \end{pmatrix}$$

Section 2.5 Gaussian Elimination

....

....

....

....

Theorem. Let $A = (a_{ij})_{m\times n}$. Consider the $m \times n$ system of linear equations $A\mathbf{x} = \mathbf{b}$.
Let $A^{\#}$ represent the augmented matrix of the system.
If rank(A) =rank($A^{\#}$) $= n$, then the system has a unique solution.
If rank(A) <rank($A^{\#}$), then the system has no solution.
If rank(A) =rank($A^{\#}$) $= r < n$, then the system has infinitely many solutions, with $n - r$ free variables.

Section 2.6 The Inverse Matrix of a Square Matrix

Definition Let $A = (a_{ij})_{n\times n}$. If there exists another matrix $B = (b_{ij})_{n\times n}$, such that

$$AB = BA = I,$$

then we say that the inverse matrix of A exists and $A^{-1} = B$.

Some properties of inverse matrix:

$$(A^{-1})^{-1} = A,$$
$$(AB)^{-1} = B^{-1}A^{-1},$$
$$(A^T)^{-1} = (A^{-1})^T.$$

If there exists a matrix B, such that $AB = I$ or $BA = I$, then the inverse matrix A^{-1} exists and $B = A^{-1}$.

The inverse matrix A^{-1} exists if and only if the rank of A is equal to n, if and only if there exists a unique solution to $A\mathbf{x} = \mathbf{b}$, for any vector b.

There exists a unique solution $x = A^{-1}b$ to the system $A\mathbf{x} = \mathbf{b}$, if the inverse matrix A^{-1} exists.

Example. Let A be a real square matrix with $A^{10} = I$. Find A^{-1}. Solution:

$$A^{-1} = A^9.$$

Example. Let A be a matrix, satisfying the equation

$$\alpha A^2 - \beta A = I,$$

where $\alpha \neq 0$, β is a constant and I is the identity matrix. Find the inverse matrix A^{-1} of A. Solution:

$$A^{-1} = \alpha A - \beta I.$$

Example 1.

Example. Find the inverse matrix

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{pmatrix}^{-1} = \frac{1}{14}\begin{pmatrix} 11 & -16 & 1 \\ -6 & 10 & 2 \\ 3 & 2 & -1 \end{pmatrix}.$$

Solution: Performing elementary row operations to the augmented matrix (A, I), we have **Example.** Use elementary row operations to find the inverse matrix of

$$\begin{pmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Example 3. (I) Let $A^3 = 0$. Find $(I - 2A)^{-1} = I + 2A + 4A^2$. (II) Given that $A^2 = 5A + 2I$. Find the inverse matrix of A. Solution:

$$\begin{aligned} \lambda I - A &= \begin{pmatrix} \lambda - 1 & 3 \\ 2 & \lambda - 4 \end{pmatrix}, \\ \det(\lambda I - A) &= \lambda^2 - 5\lambda - 2, \\ A^{-1} &= \frac{1}{2}(A - 5I). \end{aligned}$$

Solve the system of equations

$$\begin{aligned} x + y + 3z &= 2, \\ y + 2z &= 1, \\ 3x + 5y - z &= 4. \end{aligned}$$

Example. Let α, β, γ, δ be real numbers, such that $\alpha\delta - \beta\gamma \neq 0$. Then the inverse matrix of

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is

$$A^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Example. Let $x \in \mathbb{R}$ and define the matrix

$$A = \frac{1}{1 + 2x^2} \begin{pmatrix} 1 & -2x & 2x^2 \\ 2x & 1 - 2x^2 & -2x \\ 2x^2 & 2x & 1 \end{pmatrix}.$$

Show that the inverse matrix $A^{-1} = A^T$.

Section 2.7 Elementary Matrices and the LU Factorization

The matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

is real symmetric.

Section 2.8 The Inverse Matrix Theorem I

Section 2.9 Chapter review

Chapter 3 Determinants

Section 3.1 The definition of the determinant

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

$$\begin{aligned}
&\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\
= \;& a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} \\
- \;& a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{21}a_{12}a_{33}.
\end{aligned}$$

Let $A = (a_{ij})_{n\times n}$. Let C_{ij} represent the cofactor of a_{ij}.

Section 3.3 Cofactor expansions

Let $A = (a_{ij})_{n\times n}$. The element a_{ij} is in row i and in column j. Let us cross out row i and column j in A.

$$\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn}
\end{pmatrix}$$

The cofactor C_{ij} of a_{ij} is the determinant of the reduced matrix times $(-1)^{i+j}$.

Definition. Define the determinant by using row expansion

$$\det A = \sum_{k=1}^{n} a_{ik}C_{ik} = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in},$$

where $i = 1, 2, 3, \cdots, n$.

Define the determinant by using column expansion

$$\det A = \sum_{k=1}^{n} a_{kj}C_{kj} = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj},$$

where $j = 1, 2, 3, \cdots, n$.

Section 3.2 The Properties of Determinants

Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ be square matrices.

If every element in one row of A is zero, then $\det A = 0$.

If two rows of A are identical, then $\det A = 0$.

If one row is equal to another row multiplied by a constant α, then $\det A = 0$.

If A is an $n \times n$ upper or lower triangular matrix, then $\det A = a_{11}a_{22}a_{33}\cdots a_{nn}$. More explicitly

$$\det \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \lambda_1\lambda_2\lambda_3\cdots\lambda_n.$$

$$\det \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn},$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{aa} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

If the matrix B is obtained by interchanging two rows of the matrix A, then

$$\det(B) = -\det(A).$$

If the matrix B is obtained by multiplying a row of the matrix A by a scalar α, then

$$\det(B) = \alpha \det(A).$$

If the matrix B is obtained by multiplying a row by a scalar and giving the result to another row of the matrix A, then

$$\det(B) = \det(A).$$

$$\det A^T = \det A.$$

$$\det(AB) = \det(A)\det(B)$$

Let A be an invertible $n \times n$ matrix. Then

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Examples.

Let

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Find the determinant of the matrix $\lambda I - A$.

Solutions:

$$\begin{aligned}
\lambda I - A &= \begin{pmatrix} \lambda - 2 & -2 & -2 \\ -2 & \lambda - 2 & -2 \\ -2 & -2 & \lambda - 2 \end{pmatrix} \\
&\to \begin{pmatrix} \lambda - 2 & -2 & -2 \\ -\lambda & \lambda & 0 \\ -\lambda & 0 & \lambda \end{pmatrix}, \\
\det(\lambda I - A) &= \lambda^2 \det \begin{pmatrix} \lambda - 2 & -2 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
&= \lambda^2 \det \begin{pmatrix} \lambda - 6 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \lambda^2(\lambda - 6).
\end{aligned}$$

Let

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

Find the determinant of the matrix $\lambda I - A$.

$$\begin{aligned}
\lambda I - A &= \begin{pmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{pmatrix} \\
&\to \begin{pmatrix} \lambda - 4 & -2 & -2 \\ 2 - \lambda & \lambda - 2 & 0 \\ 2 - \lambda & 0 & \lambda - 2 \end{pmatrix}, \\
\det(\lambda I - A) &= (\lambda - 2)^2 \det \begin{pmatrix} \lambda - 4 & -2 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
&= (\lambda - 2)^2 \det \begin{pmatrix} \lambda - 8 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = (\lambda - 2)^2 (\lambda - 8).
\end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Find the determinant of $\lambda I - A$. Performing elementary row operations to the matrix $\lambda I - A$, we have

$$\begin{aligned}
\lambda I - A &= \begin{pmatrix} \lambda - 1 & -1 & 1 \\ 1 & \lambda - 3 & 1 \\ 1 & -1 & \lambda - 1 \end{pmatrix} \\
&\to \begin{pmatrix} \lambda - 1 & -1 & 1 \\ 2 - \lambda & \lambda - 2 & 0 \\ 2 - \lambda & 0 & \lambda - 2 \end{pmatrix} \\
\det(\lambda I - A) &= (\lambda - 2)^2 \det \begin{pmatrix} \lambda - 1 & -1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
&= (\lambda - 2)^2 \det \begin{pmatrix} \lambda - 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
&= (\lambda - 1)(\lambda - 2)^2.
\end{aligned}$$

Evaluate the determinants of the matrices Show that

$$\det \begin{pmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{pmatrix} = (x + y + z)^2.$$

Solution:

$$
\begin{aligned}
&\det\begin{pmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{pmatrix} \\
= &\det\begin{pmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -x-y-z & 0 & 0 & 0 \end{pmatrix} \\
= &\ (x+y+z)\det\begin{pmatrix} x & y & z \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \\
= &\ (x+y+z)\det\begin{pmatrix} x & y & x+y+z \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
= &\ (x+y+z)^2.
\end{aligned}
$$

If a row vector of A is the sum of two row vectors, $r_i = r_i' + r_i''$, then

$$\det A = \det[r_1, \cdots, r_i', \cdots, r_n] + \det[r_1, \cdots, r_i'', \cdots, r_n].$$

Let $A = (a_{ij})$ be an $n \times n$ matrix and let C_{ij} be the cofactor corresponding to a_{ij}, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Then

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik} = \sum_{k=1}^{n} a_{kj} C_{kj},$$
$$\sum_{k=1}^{n} a_{ik} C_{jk} = 0, \text{ where rows } i \neq j,$$
$$\sum_{k=1}^{n} a_{ki} C_{kj} = 0, \text{ where columns } i \neq j.$$

Therefore

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix}$$
$$= \begin{pmatrix} \det A & 0 & 0 & \cdots & 0 \\ 0 & \det A & 0 & \cdots & 0 \\ 0 & 0 & \det A & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \det A \end{pmatrix}$$
$$= \det A \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Therefore, if $\det A \neq 0$, then the inverse matrix may be represented as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ 8a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix}.$$

Theorem.

$\det(A) \neq 0$ if and only if the inverse matrix A^{-1} exists.

Example. Let a and b be given real constants. Solve the equation for all solutions

$$\det \begin{pmatrix} 1 & 1 & 1 \\ a & b & x \\ a^2 & b^2 & x^2 \end{pmatrix} = 0.$$

Solutions: Performing elementary row operations and use properties of determinants yield

$$\det \begin{pmatrix} 1 & 1 & 1 \\ a & b & x \\ a^2 & b^2 & x^2 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & x-a \\ 0 & b^2-a^2 & x^2-a^2 \end{pmatrix}$$

$$= \det \begin{pmatrix} b-a & x-a \\ b^2-a^2 & x^2-a^2 \end{pmatrix} = (b-a)(x-a)\det \begin{pmatrix} 1 & 1 \\ b+a & x+a \end{pmatrix}$$

$$= \det(b-a)(x-a)\det \begin{pmatrix} 1 & 1 \\ 0 & x-a \end{pmatrix} = (b-a)(x-a)(x-b).$$

The solutions of the equation are $x = a$ and $x = b$.

Let a, b, c be distinct real constants. Use cofactor method to find the inverse matrix of

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}.$$

Solution. First of all, the determinant of the matrix is

$$
\begin{aligned}
& \det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \\
= \ & \det\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \\
= \ & \det\begin{pmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{pmatrix} \\
= \ & (a-b)(c-a)\det\begin{pmatrix} -1 & 1 \\ -a-b & c+a \end{pmatrix} \\
= \ & (a-b)(b-c)(c-a) \neq 0.
\end{aligned}
$$

Second, the transposed matrix of the cofactor matrix is

$$
\begin{pmatrix} -bc(b-c) & (b-c)(b+c) & -(b-c) \\ -ca(c-a) & (c-a)(c+a) & -(c-a) \\ -ab(a-b) & (a-b)(a+b) & -(a-b) \end{pmatrix}
$$

The inverse matrix is given by

$$
\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}^{-1} = \frac{1}{(a-b)(b-c)(c-a)}
$$

$$
\cdot \begin{pmatrix} -bc(b-c) & (b-c)(b+c) & -(b-c) \\ -ca(c-a) & (c-a)(c+a) & -(c-a) \\ -ab(a-b) & (a-b)(a+b) & -(a-b) \end{pmatrix}.
$$

Let α, β, γ be real distinct constants. Solve the following equation for all solutions

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & x \\ \alpha^2 & \beta^2 & \gamma^2 & x^2 \\ \alpha^3 & \beta^3 & \gamma^3 & x^3 \end{pmatrix} = 0.$$

Solutions: Performing elementary row operations and applying properties of determinants, we have

$$\begin{aligned}
&\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & x \\ \alpha^2 & \beta^2 & \gamma^2 & x^2 \\ \alpha^3 & \beta^3 & \gamma^3 & x^3 \end{pmatrix} \\
= \ &\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha - x & \beta - x & \gamma - x & 0 \\ \alpha^2 - x^2 & \beta^2 - x^2 & \gamma^2 - x^2 & 0 \\ \alpha^3 - x^3 & \beta^3 - x^3 & \gamma^3 - x^3 & 0 \end{pmatrix} \\
= \ &-\det\begin{pmatrix} \alpha - x & \beta - x & \gamma - x \\ \alpha^2 - x^2 & \beta^2 - x^2 & \gamma^2 - x^2 \\ \alpha^3 - x^3 & \beta^3 - x^3 & \gamma^3 - x^3 \end{pmatrix} \\
= \ &-(\alpha - x)(\beta - x)(\gamma - x) \\
\cdot \ &\det\begin{pmatrix} 1 & 1 & 1 \\ \alpha + x & \beta + x & \gamma + x \\ \alpha^2 + \alpha x + x^2 & \beta^2 + \beta x + x^2 & \gamma^2 + \gamma x + x^2 \end{pmatrix} \\
= \ &-(\alpha - x)(\beta - x)(\gamma - x) \\
\cdot \ &\det\begin{pmatrix} 1 & 1 & 1 \\ \alpha - \gamma & \beta - \gamma & 0 \\ \alpha^2 - \gamma^2 + (\alpha - \gamma)x & \beta^2 - \gamma^2 + (\beta - \gamma)x & 0 \end{pmatrix} \\
= \ &-(\alpha - x)(\beta - x)(\gamma - x) \\
\cdot \ &\det\begin{pmatrix} \alpha - \gamma & \beta - \gamma \\ \alpha^2 - \gamma^2 + (\alpha - \gamma)x & \beta^2 - \gamma^2 + (\beta - \gamma)x \end{pmatrix} \\
= \ &-(\alpha - x)(\beta - x)(\gamma - x)(\alpha - \gamma)(\beta - \gamma)\det\begin{pmatrix} 1 & 1 \\ \alpha + \gamma + x & \beta + \gamma + x \end{pmatrix} \\
= \ &-(x - \alpha)(x - \beta)(x - \gamma)(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha).
\end{aligned}$$

Therefore, the solutions are $x = \alpha$, $x = \beta$, $x = \gamma$.

Crammer's rule: Let $A = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n]$ be an $n \times n$ matrix, where each $\mathbf{c}_i$ is a column vector of A, let $\mathbf{b}$ be another column vector. Suppose that $\det A \neq 0$. Then the solution of the algebraic system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = A^{-1}\mathbf{b}$, namely

$$x_i = \frac{\det(A_i)}{\det(A)}, \qquad A_i = (\mathbf{a}_1, \cdots, \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, \cdots, \mathbf{a}_n).$$

Solve the system of equations

$$\begin{aligned} ax + by &= r, \\ cx + dy &= s, \end{aligned}$$

where $ad - bc \neq 0$.

The solution is given by

$$x = \frac{\det\begin{pmatrix} r & b \\ s & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \frac{dr - bs}{ad - bc},$$

$$y = \frac{\det\begin{pmatrix} a & r \\ c & s \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \frac{as - rc}{ad - bc}.$$

Let $A = (a_{ij})_{n \times n}$ be a matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. The algebraic system $A\mathbf{x} = \mathbf{b}$ has a unique solution if $\det A \neq 0$.

Section 3.4 Summary of determinants

Section 3.5 Chapter review

Chapter 4. Vector Spaces

Section 4.1 Vectors in $\mathbb{R}^n$

Section 4.2 Definition of a Vector Space

Section 4.3 Subspaces

Let $\mathbf{V} = \mathbb{R}^3$ and let

$$W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 14 \\ 3 \\ 13 \end{pmatrix} \right\}.$$

Is it true that $W = \mathbb{R}^3$? Is it also true that $\begin{pmatrix} 1 \\ -10 \\ -3 \end{pmatrix} \in W$? Show all work to support your answer.

Solutions: Since

$$\begin{pmatrix} 14 \\ 3 \\ 13 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + 3 \cdot \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix},$$

the vector $\begin{pmatrix} 14 \\ 3 \\ 13 \end{pmatrix}$ is spanned by the first two vectors. Therefore, $W \neq \mathbb{R}^3$. Since

$$\begin{pmatrix} 1 \\ -10 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} - 3 \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix},$$

we know that

$$\begin{pmatrix} 1 \\ -10 \\ -3 \end{pmatrix} \in W.$$

Let $\mathbf{V} = \mathbb{R}^3$ and let

$$W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \right\}.$$

Is it true that $W = \mathbb{R}^3$? Show all work to support your answer. Select three vectors from the given five vectors to form a basis of $\mathbb{R}^3$. Verify why this is a basis.

Section 4.4 Spanning Sets

Definition. Let $\mathbf{V}$ be a vector space and let $v_1, v_2, v_3, \cdots\cdots, v_n$ be vectors in $\mathbf{V}$. If every vector v in $\mathbf{V}$ may be written as a linear combination of $v_1, v_2, v_3, \cdots\cdots, v_n$, that is, $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n$, then we say that $\mathbf{V}$ is spanned or generated by $v_1, v_2, v_3, \cdots\cdots, v_n$. We call the set of vectors $\{v_1, v_2, v_3, \cdots\cdots, v_n\}$ a spanning set of the vector space $\mathbf{V}$.

Theorem. Let $v_1, v_2, v_3, \cdots, v_n$ be vectors in $\mathbb{R}^n$. Then $\{v_1, v_2, v_3, \cdots, v_n\}$ spans $\mathbb{R}^n$ if and only if the determinant $\det(v_1, v_2, v_3, \cdots, v_n) \neq 0$, if and only if the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b} \in \mathbb{R}^n$, where $A = (v_1, v_2, v_3, \cdots, v_n)$.

Example. Show that

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\}$$

is a spanning set of the vector space $\mathbb{R}^3$.

Solution: For any real vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, let us see if there exists a solution (α, β, γ) to the system of equations

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

That is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

If the determinant of the coefficient matrix is not zero, then there exists a unique solution to the system. By using properties of determinants, we may calculate the determinant

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} &= \det \begin{pmatrix} 6 & 6 & 6 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \\ = \ 6 \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} &= 6 \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & -1 & 0 \end{pmatrix} \\ = \ 6 \det \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} &= -18 \neq 0. \end{aligned}$$

Hence, there exists a unique solution $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ to the system. Therefore $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a spanning set of $\mathbb{R}^3$.

Theorem. Let v_1, v_2, v_3, $\cdots$, v_n be vectors in a vector space $\mathbf{V}$. Then the subset W spanned by these vectors v_1, v_2, v_3, $\cdots$, v_n is a subspace of $\mathbf{V}$, where

$$W = \left\{ \sum_{k=1}^{n} \alpha_k v_k : \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n \text{ are real numbers} \right\}.$$

Determine whether the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix},$$

span $\mathbb{R}^3$.

If $V = \mathbb{R}^3$ and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Determine the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Does $\mathbf{v}_3$ lie in this subspace?

Find a spanning set for the vector space V consisting of all 3×3 skew-symmetric matrices.

Solution: For all real numbers a_{12}, a_{13}, a_{23}, there holds

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$$

$$= a_{12} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{13} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + a_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Therefore

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

is a spanning set.

Consider the vector space P_2. Let

$$f(x) = x - 4, \qquad g(x) = x^2 - x + 3, \qquad h(x) = 2x^2 - x + 2.$$

Is it true that

$$h \in \text{span}\{f, g\}?$$

Let a and b be real numbers, such that

$$\begin{aligned} af(x) + bg(x) &= a(x-4) + b(x^2 - x + 3) \\ &= bx^2 + (a-b)x + (-4a+3b) = 2x^2 - x + 2. \end{aligned}$$

We find that $a = 1$ and $b = 2$. Therefore, $h \in$span$\{f, g\}$.

Section 4.5 Linear Dependence and Linear Independence

Definition. Let $\mathbf{V}$ be a vector space and let v_1, v_2, v_3, $\cdots$, v_n be vectors in $\mathbf{V}$. If the only solution of the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = \mathbf{0},$$

is the zero solution $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\cdots$, $\alpha_n = 0$, then we say v_1, v_2, v_3, $\cdots$, v_n are linearly independent.

Definition. Let $\mathbf{V}$ be a vector space and let v_1, v_2, v_3, $\cdots$, v_n be vectors in $\mathbf{V}$. If there are scalars α_1, α_2, α_3, $\cdots$, α_n, not all equal to zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = \mathbf{0},$$

then we say $\{v_1, v_2, v_3, \cdots, v_n\}$ are linearly dependent.

Find three nonzero real numbers a, b and c, such that

$$a\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ k \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ k \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix}.$$

Determine all values of the constant k for which the vectors is linearly independent in $\mathbb{R}^4$.

Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}.$$

Determine a linearly independent subset of these vectors that span the same vector subspace of $\mathbb{R}^3$ as $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 7 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 6 \\ 9 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -2 \\ 3 \\ 1 \\ 6 \end{pmatrix}.$$

Determine whether these vectors are linearly independent.

Definition. The set of functions $\{f_1,\ f_2,\ f_3,\ \cdots,\ f_n\}$ is linearly independent on an interval I if and only if the solution of the equation

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) + \cdots + \alpha_n f_n(x) = 0,$$

is $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\cdots$, $\alpha_n = 0$.

Definition. The set of functions $f_1,\ f_2,\ f_3,\ \cdot,\ f_n$ is linearly dependent on an interval I, if and only if there exists at least one non-trivial solution $(\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n) \neq (0, 0, 0, \cdots, 0)$ to the equation

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) + \cdots + \alpha_n f_n(x) = 0.$$

Show that $1, x, x^2, x^3, x^4, x^5, 0$ are linearly dependent. Solution: r are re numbers (o), such that $01 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + 1 \cdot 0 = 0$. Thus they are.

Show that the 1, $\cos^2 x$, $\sin^2 x$, $\cos 2x$ and $\sin 2x$ are linearly dependent by finding five real numbers and α_5 such that $\cdot 1 \cdot \cos^2 \cdot \sin^+ \alpha_4 \cos 25 \cdot \sin 2 = 0$ for all real numbers x. Do not use Wronskian idea. Solution: There are non-zero real numbers, such that

$$1 \cdot 1 + (-1) \cdot (\cos x)^2 + (-1)(\sin x)^2 + 0 \cdot \cos(2x) + 0 \cdot \sin(2x) = 0,$$

for all $x \in \mathbb{R}$. There are also non-zero real numbers, such that

$$0 \cdot 1 + (-1) \cdot (\cos x)^2 + 1 \cdot (\sin x)^2 + 1 \cdot \cos(2x) + 0 \cdot \sin(2x) = 0,$$

for all $x \in \mathbb{R}$. Thus these functions are linearly dependent on $\mathbb{R}$.

Definition. Let f_1, f_2, f_3, $\cdots$, f_n be smooth functions defined on the interval I. The Wronskian of f_1, f_2, f_3, $\cdots$, f_n is the $n \times n$ determinant

$$\det \begin{pmatrix} f_1(x) & f_2(x) & f_3(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & f_3'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) & \cdots & f_n''(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & f_3^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}.$$

Theorem. If the Wronskian of the functions f_1, f_2, f_3, $\cdots$, f_n is not equal to zero, then the functions f_1, f_2, f_3, $\cdots$, f_n are linearly independent. (Note: The converse of the Wronskian test is not necessarily correct.)

Show that $\cos x$ and $\sin x$ are linearly independent. Solution: The Wronskian

$$\det \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = 1.$$

Prove that $1, x, x^2, x^3, x^4$ are linearly independent.
Solution: The Wronskian

$$\det \begin{pmatrix} 1 & x & x^2 & x^3 & x^4 \\ 0 & 1 & 2x & 3x^2 & 4x^3 \\ 0 & 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 0 & 6 & 24x \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} = 288 > 0.$$

Let $r_1 < r_2 < r_3$ be real numbers. Define the following functions

$$f_1(x) = \exp(r_1 x), \qquad f_2(x) = \exp(r_2 x), \qquad f_3(x) = \exp(r_3 x).$$

Show that they are linearly independent.

Solution: Let us calculate the Wronskian $W(f_1, f_2, f_3)(x)$:

$$\begin{aligned}
& \det \begin{pmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{pmatrix} \\
= \ & \det \begin{pmatrix} \exp(r_1 x) & \exp(r_2 x) & \exp(r_3 x) \\ r_1 \exp(r_1 x) & r_2 \exp(r_2 x) & r_3 \exp(r_3 x) \\ r_1^2 \exp(r_1 x) & r_2^2 \exp(r_2 x) & r_3^2 \exp(r_3 x) \end{pmatrix} \\
= \ & \exp[(r_1 + r_2 + r_3)x] \det \begin{pmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{pmatrix} \\
= \ & \exp[(r_1 + r_2 + r_3)x] \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & r_2 - r_1 & r_3 - r_1 \\ 0 & r_2^2 - r_1^2 & r_3^2 - r_1^2 \end{pmatrix} \\
= \ & (r_1 - r_2)(r_3 - r_1) \exp[(r_1 + r_2 + r_3)x] \det \begin{pmatrix} -1 & 1 \\ -r_1 - r_2 & r_3 + r_1 \end{pmatrix} \\
= \ & (r_1 - r_2)(r_2 - r_3)(r_3 - r_1) \exp[(r_1 + r_2 + r_3)x].
\end{aligned}$$

Therefore, the functions $\exp(r_1 x)$, $\exp(r_2 x)$ and $\exp(r_3 x)$ are linearly independent on $\mathbb{R}$.

Prove that $\cos x, \sin x, \cos(2x), \sin(2x)$ are linearly independent.

Solution: The Wronskian

$$
\begin{aligned}
W(x) &= \det\begin{pmatrix} \cos x & \sin x & \cos(2x) & \sin(2x) \\ -\sin x & \cos x & -2\sin(2x) & 2\cos(2x) \\ -\cos x & -\sin x & -4\cos(2x) & -4\sin(2x) \\ \sin x & -\cos x & 8\sin(2x) & -8\cos(2x) \end{pmatrix} \\
&= \det\begin{pmatrix} \cos x & \sin x & \cos(2x) & \sin(2x) \\ -\sin x & \cos x & -2\sin(2x) & 2\cos(2x) \\ 0 & 0 & -3\cos(2x) & -3\sin(2x) \\ 0 & 0 & 6\sin(2x) & -6\cos(2x) \end{pmatrix} \\
&= \det\begin{pmatrix} \cos x & \sin x & 0 & 0 \\ -\sin x & \cos x & 0 & 0 \\ 0 & 0 & -3\cos(2x) & -3\sin(2x) \\ 0 & 0 & 6\sin(2x) & -6\cos(2x) \end{pmatrix} = 18 \neq 0.
\end{aligned}
$$

Therefore, the functions $\cos x$, $\sin x$, $\cos(2x)$ and $\sin(2x)$ are linearly dependent on $\mathbb{R}$.

(A) Let $r_1 < r_2 < r_3$ be real numbers. Show that are linearly independent.

Prove that

$$1, x, \sin x, \cos x$$

are linearly independent.

Solution: If there are numbers α, β, γ, δ such that

$$\alpha + \beta x + \gamma \sin x + \delta \cos x = 0,$$

for all x, then differentiating the equation about x yields

$$\beta + \gamma \cos x - \delta \sin x = 0.$$

Calculating the derivative again gives $\gamma \sin x + \delta \cos x = 0$. By using the Wronskian of $\sin x$ and $\cos x$:

$$W(\sin x, \cos x) = \det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -1,$$

we see $\sin x$ and $\cos x$ are linearly independent, so that $\gamma = 0, \delta = 0$. Hence we have $a + bx = 0$, for all x. It is easy to get $\alpha = 0, \beta = 0$. Therefore $1, x, \sin x, \cos x$ are linearly independent.
Solution: There are real numbers $0, 0, 0, 0, 1$, one of them is nonzero, such that

$$0 \cdot \cos x + 0 \cdot \sin x + 0 \cdot \cos(2x) + 0 \cdot \sin(2x) + 1 \cdot 0 = 0.$$

Thus they are linearly dependent.

Given m vectors $v_1, v_2, v_3, \cdots, v_m$ in $\mathbb{R}^n$, how to determine if they are linearly independent or linear dependent? How to determine if $v_1, v_2, v_3, \cdots, v_m$ span the entire space $\mathbb{R}^n$? Or how to determine if another vector v is in span$\{v_1, v_2, v_3, \cdots, v_m\}$? If the number m of vectors is equal to the dimension n, then compute the determinant of the square matrix $(v_1, v_2, v_3, \cdots, v_n)$. If the determinant is not equal to zero, then $v_1, v_2, v_3, \cdots, v_n$ are linearly independent and they span the whole space $\mathbb{R}^n$. If the determinant is equal to zero, then $v_1, v_2, v_3, \cdots, v_n$ are linearly dependent and they do not span the whole space $\mathbb{R}^n$. If the number of vectors is not equal to the dimension n, then also there is a powerful idea: performing elementary row operations to the matrix $(v_1, v_2, v_3, \cdots, v_m)$ or to the augmented matrix $(v_1, v_2, v_3, \cdots, v_m, b)$, check the reduced row echelon form to make the corresponding decision.

Section 4.6 Bases and Dimension

Definition. A set of vectors $\{v_1, v_2, v_3, \cdots, v_n\}$ in a vector space $\mathbf{V}$ is called a basis if (1) $v_1, v_2, v_3, \cdots, v_n$ are linearly independent and (2) $v_1, v_2, v_3, \cdots, v_n$ span the whole space $\mathbf{V}$.

Theorem. If a finite-dimensional vector space has a basis $\{v_1, v_2, v_3, \cdots, v_n\}$ consisting of n vectors, then any set of more than n vectors is linearly dependent.

Theorem. All bases in a finite-dimensional vector space $\mathbf{V}$ contain the same number of vectors.

Theorem. If a finite-dimensional vector space $\mathbf{V}$ has a basis consisting of n vectors $\{v_1, v_2, v_3, \cdots, v_n\}$, then any spanning set must contain at least n vectors.

Definition. The dimension of a finite-dimensional vecrtor space, written $\dim \mathbf{V}$, is the number of vectors in any basis $\{v_1, v_2, v_3, \cdots, v_n\}$ for $\mathbf{V}$. If $\mathbf{V}$ is the trivial space $\{0\}$, then we define its dimension to be zero.

Theorem. If $\dim \mathbf{V} = n$, then any set of n linearly independent vectors $\{v_1, v_2, v_3, \cdots, v_n\}$ in $\mathbf{V}$ is a basis for $\mathbf{V}$.

Theorem. If $\dim \mathbf{V} = n$, then any set of n vectors $\{v_1, v_2, v_3, \cdots, v_n\}$ in $\mathbf{V}$ that spans $\mathbf{V}$ is a basis for $\mathbf{V}$.$\{v_1, v_2, v_3, \cdots, v_n\}$

Theorem. If $\dim \mathbf{V} = n$ and $\mathbf{W} = \{v_1, v_2, v_3, \cdots, v_n\}$ is a set of vectors in $\mathbf{V}$, then the following statements are equivalent:

(1) $\{v_1, v_2, v_3, \cdots, v_n\}$ is a basis for $\mathbf{V}$.

(2) $\{v_1, v_2, v_3, \cdots, v_n\}$ is linearly independent.

(3) $\{v_1, v_2, v_3, \cdots, v_n\}$ spans $\mathbf{V}$.

Find a basis for the linear vector space

$$\mathbf{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \text{ where } a, b, c, d \text{ are complex numbers} \right\}.$$

Solution: The following set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for the space $\mathbf{V}$.

Theorem. Let $\mathbf{W}$ be a subspace of a finite-dimensional vector space $\mathbf{V}$. If $\dim \mathbf{V} = n$, then $\dim W \leq n$. If $\dim W = \dim V$, then $W = V$.

Theorem. Let $\mathbf{W}$ be a subspace of a finite-dimensional vector space $\mathbf{V}$. Any basis of $\mathbf{W}$ is part of a basis of $\mathbf{V}$.

Is the vector space $\mathbf{V} = C(\mathbb{R}) = \{f\colon\ f$ is a continuous function defined on $\mathbb{R}\}$ finite-dimensional or infinite-dimensional? Show work and give a possible basis.

Solution: It is an infinite-dimensional vector space. For any integer $n \geq 1$, the vectors

$$1, x, x^2, x^3, \cdots, x^n$$

are always linearly independent. A basis is:

$$1, x, x^2, x^3, \cdots, x^n, \cdots,$$

or

$$1, \cos x, \sin x, \cos(2x), \sin(2x), \cos(3x), \sin(3x), \cdots, \cos(nx), \sin(nx), \cdots$$

Let $A = (a_{ij})_{m\times n} = (v_1, v_2, \cdots, v_n)$ be a matrix. Define the null space of A by

$$NS(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$

The null space is a subspace of $\mathbb{R}^n$.

Define the column space by

$$CS(A) = \{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n : \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n \text{ are real con}$$

The column space is a subspace of $\mathbb{R}^m$.

Define the row space by

$$RS(A) = \{\beta_1 r_1 + \beta_2 r_2 + \beta_3 r_3 + \cdots + \beta_m r_m : \beta_1, \beta_2, \beta_3, \cdot, \beta_m \text{ are real consta}$$

The row space is a subspace of $\mathbb{R}^n$.

Theorem. Let $A = (a_{ij})_{m\times n}$ be a matrix. Then

$$\text{rank}(A) + \dim NS(A) = n.$$

Theorem. Let $A = (a_{ij})_{m\times n}$ be a matrix. If $\text{rank}(A) = n$, then $Ax = 0$ only has the trivial solution $x = 0$ and the null space $NS(A) = \{0\}$. If $\text{rank}(A) < n$, then $Ax = 0$ has infinitely many solutions. All of them may be represented as

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{n-r} x_{n-r},$$

where $\{x_1, x_2, \cdots, x_{n-r}\}$ is a linerly independent set of $n - r$ solutions to $Ax = 0$.

Theorem. Let $A = (a_{ij})_{m\times n}$ be a matrix. Consider the system $A\mathbf{x} = \mathbf{b}$. If $\mathbf{b}$ is not in the column space $CS(A)$, then the system has no solution.

If $\mathbf{b}$ is in the column space $CS(A)$, then the system has a unique solution if $\dim(CS(A)) = n$ and the system has infinitely many solutions if $\dim(CS(A)) < n$.

Theorem. Let $A = (a_{ij})_{m\times n}$ be a matrix. If $\text{rank}(A) = r < n$ and $\mathbf{b}$ is in the column space $CS(A)$, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. They may be represented as

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{n-r} x_{n-r} + x_p,$$

Section 4.7 Change of Basis

Section 4.8 Row Space and Column Space

Section 4.9 The Rank Nullity Theorem

Section 4.10 The Invertible Matrix Theorem II

Section 4.11 Inner Product Spaces

Section 4.12 Orthogonal Sets of Vectors and the Gram Schmidt Process

Section 4.13 Chapter Review