How to Linearize Einstein's Field Equations

In order to linearize Einstein's field equations it is useful to know why you would want to do so. Well, it is easier to solve a system of linear equations than nonlinear equations. So, if you are doing a calculation where gravity is weak then you may use the linearized field equations.

Put another way, spacetime is "nearly flat" when gravity is weak. And since the absence of gravity leaves spacetime flat, a weak gravitational field is one in which spacetime is "nearly flat" [1](p. 200). So we may approximate spacetime for weak gravity by starting with the flat (or Minkowski or Lorentz or Galilean) metric components then adding a small perturbation as follows.

$$g_{ab} = h_{ab} + h_{ab} \tag{1}$$

are the components of the metric tensor

where (2)

$$|h_{ab}| << 1$$

are the components of the perturbation

and (3)

$$h_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

are the metric components of flat spacetime

So we may say that the components of the metric tensor, $g_{\alpha\beta}$, represent a "nearly flat" (or "nearly Minkowski" or "nearly Lorentz" or "nearly Galilean") spacetime.

(Note that since $g_{\alpha\beta}$ is symmetric and $\eta_{\alpha\beta}$ is symmetric then $h_{\alpha\beta}$ is symmetric. Therefore $g_{\alpha\beta} = g_{\alpha\beta}$, $\eta_{\alpha\beta} = \eta_{\beta\alpha}$, and $h_{\alpha\beta} = h_{\beta\alpha}$. We will use this concept later when manipulating tensor equations.)

Next, let's take a look at Einstein's field equations (also known as the field equations of General Relativity). The general form may be written as

$$R_{mn} - \frac{1}{2} g_{mn} R = \frac{8pG}{c^4} T_{mn} \tag{4}$$

For convenience, we will substitute the geometrized values for the gravitational constant G and the speed of light c (G=1 and c =1). (The conversion factor from SI units to the geometrized units is $1 = G/c^2 = 7.425 \times 10^{-28}$ m kg^{-1}.) So the geometrized form of the field equations is

$$R_{mn} - \frac{1}{2} g_{mn} R = 8pT_{mn} \tag{5}$$

$R_{\alpha\beta}$ is the *Ricci curvature tensor*, R is the *scalar curvature* and $T_{\mu\nu}$ is the *energy-momentum-stress tensor* (also known as the *stress tensor*).

The Ricci curvature tensor may be written as follows. (Don't worry that the symbols used for the indices are different since the indices in this example range from 1-4, or equivalently, from 0-3, no matter what symbols are used.)

$$R_{lm} = \frac{\partial}{\partial x^m}\left[{}_{s}{}^{s}{}_{l}\right] - \frac{\partial}{\partial x^s}\left[{}_{l}{}^{s}{}_{m}\right] + \left[{}_{l}{}^{t}{}_{s}\right]\left[{}_{m}{}^{s}{}_{t}\right] - \left[{}_{l}{}^{t}{}_{m}\right]\left[{}_{t}{}^{s}{}_{s}\right] \tag{6}$$

$$\text{where} \quad \left[{}_{k}{}^{l}{}_{i}\right] = \Gamma^{l}_{k\,i} = \frac{1}{2} g^{lj}\left(\frac{\partial}{\partial x^k} g_{ij} + \frac{\partial}{\partial x^i} g_{jk} - \frac{\partial}{\partial x^j} g_{ki}\right) \tag{7}$$

is the *Christoffel symbol of the second kind*.

Equation (6) is often written in the following form:

$$R_{lm} = \partial_m \Gamma^{s}_{s\,l} - \partial_s \Gamma^{s}_{l\,m} + \Gamma^{t}_{l\,s}\,\Gamma^{s}_{m\,t} - \Gamma^{t}_{l\,m}\,\Gamma^{s}_{t\,s} \tag{8}$$

$$\text{where} \quad \partial_n \equiv \frac{\partial}{\partial x^n} \tag{9}$$

Or, equivalently, the Ricci curvature tensor, (6), may be written as

$$R_{lm} = \Gamma^{s}_{s\,l,m} - \Gamma^{s}_{l\,m,s} + \Gamma^{t}_{l\,s}\,\Gamma^{s}_{m\,t} - \Gamma^{t}_{l\,m}\,\Gamma^{s}_{t\,s} \tag{10}$$

$$\text{where} \quad f_{,n} \equiv \frac{\partial}{\partial x^n} f \tag{11}$$

uses the comma to represent a derivative.

Throughout the rest of this document I will use the "comma derivative" notation, (11), to represent derivatives and the Γ^{l}_{ki}.notation, (7), to represent a Christoffel symbol of the second kind.

With this notation (7) is now written as

$$\Gamma^{l}_{k\,i} = \frac{1}{2} g^{lj} \left(g_{ij,k} + g_{jk,i} - g_{ki,j} \right) \tag{12}$$

The scalar curvature R in the l.h.s. (left hand side) of (5) is given by

$$R = g^{ab} R_{ab} = R^{a}_{a} \tag{13}$$

Before we linearize Einstein's equations (5) using our weak field approximation (1), we must first lay out the rules to follow:

(a) Everywhere there is a $g_{\mu\nu}$ in the field equations (5) substitute ($\eta_{\mu\nu} + h_{\mu\nu}$), equation (1).
(b) Ignore all combinations of products between $h_{\mu\nu}$ and or its derivatives $h_{\mu\nu,\lambda}$ That is, keep only the terms that are linear in $h_{\mu\nu}$.
(c) Raise and lower indices using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ rather than $g^{\mu\nu}$ and $g_{\mu\nu}$ (in keeping with rule (b)).

Other relationships that will be helpful:

$$g^{ab} = h^{ab} - h^{ab} \quad \text{(See Appendix A for proof)} \tag{14}$$

$$h_{ab,n} = 0 \quad \text{since } \eta_{\alpha\beta} \text{ are constant} \tag{15}$$

"Gymnastics" for raising and lowering indices (For a longer list see page 85 of Misner et al (1973) also know as "MTW", see reference [2]):

$$\eta^{\nu\beta}\, h_{\mu\beta,\lambda} = h_{\mu}{}^{\nu}{}_{,\lambda} \tag{16}$$

$$\eta^{\nu\lambda}\, h_{\mu\beta,\lambda} = h_{\mu\beta,\lambda}\, \eta^{\nu\lambda} = h_{\mu\beta}{}^{,\nu} \text{ (See Appendix B)} \tag{17}$$

Use (1) and (14) to linearize (12). That is, substitute ($\eta_{\mu\nu} + h_{\mu\nu}$) for $g_{\mu\nu}$ and ($\eta^{\mu\nu} - h^{\mu\nu}$) for $g^{\mu\nu}$ in the Christoffel symbol of the second kind, $\Gamma^{l}_{ki} = (1/2)g^{lj}(g_{ij,k} + g_{jk,i} - g_{ki,j})$.

$$\Gamma^{l}_{ki} = (1/2)\,(\eta^{lj} - h^{lj})\,[(\eta_{ij} + h_{ij})_{,k} + (\eta_{jk} + h_{jk})_{,i} - (\eta_{ki} + h_{ki})_{,j}]$$

$$= (1/2)\,(\eta^{lj} - h^{lj})\,[(\eta_{ij,k} + h_{ij,k}) + (\eta_{jk,i} + h_{jk,i}) - (\eta_{ki,j} + h_{ki,j})]$$

By (15), since $\eta_{\alpha\beta}$ are constant, $\eta_{\alpha\beta,\nu}$ are zero (That is, the derivative of a constant is zero.).

$$\Gamma^{l}_{ki} = (1/2)\,(\eta^{lj} + h^{lj})\,[(0 + h_{ij,k}) + (0 + h_{jk,i}) - (0 + h_{ki,j})]$$

$$= (1/2)\,(\eta^{lj} + h^{lj})\,[h_{ij,k} + h_{jk,i} - h_{ki,j}]$$

$= (1/2)\,(\eta^{lj})\,[h_{ij,k} + h_{jk,i} - h_{ki,j}] - (1/2)\,(h^{lj})\,[h_{ij,k} + h_{jk,i} - h_{ki,j}]$

$= (1/2)\,[\eta^{lj}\, h_{ij,k} + \eta^{lj}\, h_{jk,i} - \eta^{lj}\, h_{ki,j}] - (1/2)\,[h^{lj}\, h_{ij,k} + h^{lj}\, h_{jk,i} - h^{lj}\, h_{ki,j}]$

$= (1/2)\,[\eta^{lj}\, h_{ij,k} + \eta^{lj}\, h_{jk,i} - \eta^{lj}\, h_{ki,j}] - (1/2)\,\mathcal{O}[2]$

By rule (b) $\mathcal{O}[2] = 0$, since we are only keeping the terms that are linear (first order) in $h_{\mu\nu}$.

$\Gamma^{l}{}_{ki} = (1/2)\,[\eta^{lj}\, h_{ij,k} + \eta^{lj}\, h_{jk,i} - \eta^{lj}\, h_{ki,j}]$

Using the gymnastics (16) and (17) we now have a linearized expression for the Christoffel symbol of the second kind:

$$\Gamma^{l}{}_{ki} = (1/2)\,[h_{i}{}^{l}{}_{,k} + h^{l}{}_{k,i} - h_{ki}{}^{,l}] \qquad (18)$$

We can now use (18) to linearize the expression for the Ricci curvature tensor $R_{\lambda\nu} = \Gamma^{\sigma}{}_{\sigma\lambda,\mu} - \Gamma^{\sigma}{}_{\lambda\mu,\sigma} + \Gamma^{\tau}{}_{\lambda\sigma}\,\Gamma^{\sigma}{}_{\mu\tau} - \Gamma^{\tau}{}_{\lambda\mu}\,\Gamma^{\sigma}{}_{\tau\sigma}$. Substituting (18) into (10) we have:

$R_{\lambda\nu} = (1/2)\,[h_{\lambda}{}^{\sigma}{}_{,\sigma} + h^{\sigma}{}_{\sigma,\lambda} - h_{\sigma\lambda}{}^{,\sigma}]_{,\mu} - (1/2)\,[h_{\mu}{}^{\sigma}{}_{,\lambda} + h^{\sigma}{}_{\lambda,\mu} - h_{\lambda\mu}{}^{,\sigma}]_{,\sigma}$

$+ (1/4)\,[h_{\sigma}{}^{\tau}{}_{,\lambda} + h^{\tau}{}_{\lambda,\sigma} - h_{\lambda\sigma,}{}^{\tau}]\,[h_{\tau}{}^{\sigma}{}_{,\mu} + h^{\sigma}{}_{\mu,\tau} - h_{\mu\tau}{}^{,\sigma}]$

$- (1/4)\,[h_{\mu}{}^{\tau}{}_{,\lambda} + h^{\tau}{}_{\lambda k,\mu} - h_{\lambda\mu}{}^{,\tau}]\,[h_{\sigma}{}^{\sigma}{}_{,\tau} + h^{\sigma}{}_{\tau,\sigma} - h_{\tau\sigma}{}^{,\sigma}]$

Since the products in the last two terms above result in second order terms we may write

$R_{\lambda\nu} = (1/2)\,[h_{\lambda}{}^{\sigma}{}_{,\sigma} + h^{\sigma}{}_{\sigma,\lambda} - h_{\sigma\lambda}{}^{,\sigma}]_{,\mu} - (1/2)\,[h_{\mu}{}^{\sigma}{}_{,\lambda} + h^{\sigma}{}_{\lambda,\mu} - h_{\lambda\mu}{}^{,\sigma}]_{,\sigma}$

$+ (1/4)\,\mathcal{O}[2] - (1/4)\,\mathcal{O}[2]$

where $\mathcal{O}[2] = 0$ by rule (b).

Rewriting, while ignoring the second order terms, we have

$R_{\lambda\nu} = (1/2)\,[h_{\lambda}{}^{\sigma}{}_{,\sigma} + h^{\sigma}{}_{\sigma,\lambda} - h_{\sigma\lambda}{}^{,\sigma}]_{,\mu} - (1/2)\,[h_{\mu}{}^{\sigma}{}_{,\lambda} + h^{\sigma}{}_{\lambda,\mu} - h_{\lambda\mu}{}^{,\sigma}]_{,\sigma}$

$= (1/2)\,[h_{\lambda}{}^{\sigma}{}_{,\sigma\mu} + h^{\sigma}{}_{\sigma,\lambda\mu} - h_{\sigma\lambda}{}^{,\sigma}{}_{\mu}] - (1/2)\,[h_{\mu}{}^{\sigma}{}_{,\lambda\sigma} + h^{\sigma}{}_{\lambda,\mu\sigma} - h_{\lambda\mu}{}^{,\sigma}{}_{\sigma}]$

The linearized Ricci curvature tensor may now be represented by

$$R_{\lambda\nu} = (1/2)\,[h_{\lambda}{}^{\sigma}{}_{,\sigma\mu} + h^{\sigma}{}_{\sigma,\lambda\mu} - h_{\sigma\lambda}{}^{,\sigma}{}_{\mu} - h_{\mu}{}^{\sigma}{}_{,\lambda\sigma} - h^{\sigma}{}_{\lambda,\mu\sigma} + h_{\lambda\mu}{}^{,\sigma}{}_{\sigma}] \qquad (19)$$

Note here that it is completely permissible to change the symbols of the indices as long as we are consistent. So, before proceeding let's change the symbols for the indices for two reasons: (1) to show that you can, and (2) so that the symbols will match those used by popular text books on the subject (See references [2] and [3]).

In changing the indices from λ to μ, μ to ν, and σ to α (19) becomes

$$R_{\mu\nu} = (1/2)\,[h_\mu{}^\alpha{}_{,\alpha\nu} + h^\alpha{}_{\alpha,\mu\nu} - h_{\alpha\mu}{}^{,\alpha}{}_\nu - h_\nu{}^\alpha{}_{,\mu\alpha} - h^\alpha{}_{\mu,\nu\alpha} + h_{\mu\nu}{}^{,\alpha}{}_\alpha\,] \tag{20}$$

We could leave the Ricci curvature tensor in this form, but instead, let's continue to rearrange the it so that we get a form that matches equation (5.3) on page 163 of Foster [3].
Using (16) and (17), the gymnastics for raising and lowering indices, we may write the following expressions:

$$h_{\mu\nu}{}^{,\alpha}{}_\alpha = \eta^{\alpha\beta}\, h_{\mu\nu,\beta\alpha} = h_{\mu\nu,\beta\alpha}\, \eta^{\alpha\beta} = h_{\mu\nu,}{}^\alpha{}_\alpha = h_{\mu\nu,\alpha}{}^\alpha \tag{21}$$

$$h_{\alpha\mu}{}^{,\alpha}{}_\nu = \eta^{\alpha\beta}\, h_{\alpha\mu,\beta\nu} = h_{\alpha\mu,\beta\nu}\, \eta^{\alpha\beta} = h_{\alpha\mu,}{}^\alpha{}_\nu \tag{22}$$

$$h^\alpha{}_{\alpha,\mu\nu} = \eta^{\alpha\beta}\, h_{\alpha\beta,\mu\nu} = h_{,\mu\nu} \tag{23}$$

$$h_\mu{}^\alpha{}_{,\alpha\nu} = \eta^{\alpha\alpha}\, h_{\mu\alpha,\alpha\nu} = \eta^{\alpha\alpha}\, h_{\alpha\mu,\alpha\nu} = h_{\alpha\mu,\alpha\nu}\, \eta^{\alpha\alpha} = h_{\alpha\mu,}{}^\alpha{}_\nu \tag{24}$$

Substituting (21) through (24) into (20) we have

$$R_{\mu\nu} = (1/2)\,[h_{\alpha\mu,}{}^\alpha{}_\nu + h_{,\mu\nu} - h_{\alpha\mu,}{}^\alpha{}_\nu - h_\nu{}^\alpha{}_{,\mu\alpha} - h^\alpha{}_{\mu,\nu\alpha} + h_{\mu\nu,\alpha}{}^\alpha\,]$$

The first and third term above cancel one another to give

$$R_{\mu\nu} = (1/2)\,[h_{,\mu\nu} - h_\nu{}^\alpha{}_{,\mu\alpha} - h^\alpha{}_{\mu,\nu\alpha} + h_{\mu\nu,\alpha}{}^\alpha\,]$$

As noted before, since $h_{\alpha\beta}$ is symmetric then $h_{\alpha\beta} = h_{\beta\alpha}$. Therefore $h^\alpha{}_\beta = h_\beta{}^\alpha$ and we may rewrite the third term on the r.h.s. (right hand side) above. The Ricci curvature tensor then becomes

$$R_{\mu\nu} = (1/2)\,[h_{,\mu\nu} - h_\nu{}^\alpha{}_{,\mu\alpha} - h_\mu{}^\alpha{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^\alpha\,] \tag{25}$$

(This form of the Ricci curvature tensor now matches equation (5.3) on page 163 of Foster [3].)

We will now linearize the expression for the scalar curvature, R, given by (13):

$$R = g^{\mu\nu} R_{\mu\nu} = R_\mu{}^\mu$$

Substituting $(\eta^{\mu\nu} - h^{\mu\nu})$ for $g^{\mu\nu}$ and (25) for $R_{\mu\nu}$ in R above we have

$$R = (\eta^{\mu\nu} - h^{\mu\nu})\,(1/2)\,[h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha}]$$

$$= (1/2)\,[\eta^{\mu\nu} h_{,\mu\nu} - \eta^{\mu\nu} h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \eta^{\mu\nu} h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + \eta^{\mu\nu} h_{\mu\nu,\alpha}{}^{\alpha}]$$

$$- (1/2)\,[h^{\mu\nu} h_{,\mu\nu} - h^{\mu\nu} h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h^{\mu\nu} h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h^{\mu\nu} h_{\mu\nu,\alpha}{}^{\alpha}]$$

$$= (1/2)\,[\eta^{\mu\nu} h_{,\mu\nu} - \eta^{\mu\nu} h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \eta^{\mu\nu} h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + \eta^{\mu\nu} h_{\mu\nu,\alpha}{}^{\alpha}] - (1/2)\,\mathcal{O}[2]$$

By rule (b) we have

$$R = (1/2)\,[\eta^{\mu\nu} h_{,\mu\nu} - \eta^{\mu\nu} h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \eta^{\mu\nu} h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + \eta^{\mu\nu} h_{\mu\nu,\alpha}{}^{\alpha}] \tag{26}$$

Again, using the gymnastics for raising and lowering indices, (16) and (17), we may write the following expressions:

$$\eta^{\mu\nu} h_{,\mu\nu} = h_{,\mu\nu}\,\eta^{\mu\nu} = h_{,}{}^{\nu}{}_{\nu} = h_{,\alpha\nu}\,\eta^{\alpha\nu} = h_{,\alpha}{}^{\alpha} \tag{27}$$

$$\eta^{\mu\nu} h_{\nu}{}^{\alpha}{}_{,\mu\alpha} = h^{\mu\alpha}{}_{,\mu\alpha} = \eta^{\beta\mu} h_{\beta}{}^{\alpha}{}_{,\mu\alpha} = h_{\beta}{}^{\alpha}{}_{,\mu\alpha}\,\eta^{\beta\mu} = h_{\beta}{}^{\alpha,\beta}{}_{\alpha} = \eta_{\beta\beta}\,\eta^{\beta\beta} h_{\beta}{}^{\alpha}{}_{,\mu\alpha}$$

$$= h^{\beta\alpha}{}_{,\beta\alpha} = h^{\alpha\beta}{}_{,\beta\alpha} = h^{\alpha\beta}{}_{,\alpha\beta} \tag{28}$$

$$\eta^{\mu\nu} h_{\mu}{}^{\alpha}{}_{,\nu\alpha} = h^{\nu\alpha}{}_{,\nu\alpha} = \eta^{\nu\beta}\,\eta_{\nu\beta}\, h^{\nu\alpha}{}_{,\nu\alpha} = \eta_{\nu\beta}\, h^{\nu\alpha}{}_{,\nu\alpha}\,\eta^{\nu\beta} = h_{\beta}{}^{\alpha,\beta}{}_{\alpha}$$

$$= \eta_{\beta\beta}\,\eta^{\beta\beta} h_{\beta}{}^{\alpha,\beta}{}_{\alpha} = h^{\beta\alpha}{}_{,\beta\alpha} = h^{\alpha\beta}{}_{,\beta\alpha} = h^{\alpha\beta}{}_{,\alpha\beta} \tag{29}$$

$$\eta^{\mu\nu} h_{\mu\nu,\alpha}{}^{\alpha} = h_{\mu}{}^{\mu}{}_{,\alpha}{}^{\alpha} = \eta^{\mu\alpha} h_{\mu\alpha,\alpha}{}^{\alpha} = \eta^{\mu\alpha}\,\eta_{\mu\alpha}\, h^{\alpha}{}_{\alpha,\alpha}{}^{\alpha} = (1)\, h^{\alpha}{}_{\alpha,\alpha}{}^{\alpha}$$

$$= h^{\alpha}{}_{\alpha,\alpha}{}^{\alpha} = h_{,\alpha}{}^{\alpha} \tag{30}$$

Note that

1. the order of differentiation does not matter so that $h^{\sigma\nu}{}_{,\beta\alpha} = h^{\sigma\nu}{}_{,\alpha\beta}$.
2. for a symmetric matrix $\eta_{\alpha\beta}$, $\eta^{\alpha\beta} = 1/\eta_{\alpha\beta}$ so that $\eta_{\alpha\beta}\,\eta^{\alpha\beta} = 1$ (Unless you are summing over α and β in which case $\eta_{\alpha\beta}\,\eta^{\alpha\beta} = 4$).

Substituting (27)-(30) into (26) we have

$$R = (1/2)\,[h_{,\alpha}{}^{\alpha} - h^{\alpha\beta}{}_{,\alpha\beta} - h^{\alpha\beta}{}_{,\alpha\beta} + h_{,\alpha}{}^{\alpha}]$$

After combining like terms we have

$$R = (1/2)\,[2\, h_{,\alpha}{}^{\alpha} - 2\, h^{\alpha\beta}{}_{,\alpha\beta}]$$

which reduces to

$$R = [h_{,\alpha}{}^{\alpha} - h^{\alpha\beta}{}_{,\alpha\beta}\ ] \qquad (31)$$

(This expression for the scalar curvature matches the form of equation (5.4) on page 163 of Foster [3].)

Now let’s substitute our expressions for the Ricci curvature tensor (25), the metric tensor (1), and the scalar curvature (31) into Einstein’s field equations (5):

First recall the geometrized form of Einstein’s field equations (5):

$$R_{mn} - \frac{1}{2} g_{mn} R = 8pT_{mn}$$

Upon substitution we have

$$\underbrace{\{(1/2)\,[h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha}\ ]\}}_{R_{\mu\nu}} - (1/2)\,\underbrace{(\eta_{\mu\nu} + h_{\mu\nu}\ )}_{g_{\mu\nu}}\,\underbrace{[h_{,\alpha}{}^{\alpha} - h^{\alpha\beta}{}_{,\alpha\beta}\ ]}_{R} = 8\,\pi\ T_{\mu\nu}$$

Multiplying through by 2 we have

$$[h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha}\ ] - (\eta_{\mu\nu} + h_{\mu\nu}\ )\ [h_{,\alpha}{}^{\alpha} - h^{\alpha\beta}{}_{,\alpha\beta}\ ] = 16\,\pi\ T_{\mu\nu}$$

$$[h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha}\ ] - (\eta_{\mu\nu})\ [h_{,\alpha}{}^{\alpha} - h^{\alpha\beta}{}_{,\alpha\beta}\ ] - (h_{\mu\nu}\ )\ [h_{,\alpha}{}^{\alpha} - h^{\alpha\beta}{}_{,\alpha\beta}\ ] = 16\,\pi\ T_{\mu\nu}$$

Collecting second order terms we have

$$[h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha}\ ] - (\eta_{\mu\nu})\ [h_{,\alpha}{}^{\alpha} - h^{\alpha\beta}{}_{,\alpha\beta}\ ] - \mathcal{O}[2] = 16\,\pi\ T_{\mu\nu}$$

By rule (b) $\mathcal{O}[2]$ can be ignored. Now we have

$$[h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha}\ ] - (\eta_{\mu\nu})\ [h_{,\alpha}{}^{\alpha} - h^{\alpha\beta}{}_{,\alpha\beta}\ ] = 16\,\pi\ T_{\mu\nu}$$

$$[h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha}\ ] - \eta_{\mu\nu}\ h_{,\alpha}{}^{\alpha} + \eta_{\mu\nu}\ h^{\alpha\beta}{}_{,\alpha\beta} = 16\,\pi\ T_{\mu\nu}$$

And Einstein’s linearized field equations are

$$h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\ h_{,\alpha}{}^{\alpha} + \eta_{\mu\nu}\ h^{\alpha\beta}{}_{,\alpha\beta} = 16\,\pi\ T_{\mu\nu} \qquad (32)$$

We will now introduce a change of variables to simplify the form of (32).

Let

$$\overline{h}_{mn} \equiv h_{mn} - \frac{1}{2} h \, h_{mn} \tag{33}$$

I will deviate from tradition (because I don't have a convenient font) and use an "$\hbar$" instead of an "$\overline{h}$":

So, let

$$\hbar_{\mu\nu} = h_{\mu\nu} - (1/2)\, h\, \eta_{\mu\nu} \tag{34}$$

$$\text{where } \hbar_{\mu\nu} = \overline{h}_{mn}$$

Now the task is to find h, $h_{\mu\nu}$, $h_\nu{}^\alpha$, $h_\mu{}^\alpha$, and $h^{\alpha\beta}$, in terms of the new variable, $\hbar$. Then we will be able express (32) in terms of the new variable.

So, starting with (34) operate with $\eta^{\mu\nu}$

$\eta^{\mu\nu}\, \hbar_{\mu\nu} = \eta^{\mu\nu}\, [h_{\mu\nu} - (1/2)\, h\, \eta_{\mu\nu}\,]$

$\hbar_\mu{}^\mu = \eta^{\mu\nu}\, h_{\mu\nu} - (1/2)\, \eta^{\mu\nu}\, \eta_{\mu\nu}\, h$

$\hbar_\mu{}^\mu = h_\mu{}^\mu - (1/2)\, \eta^{\mu\nu}\, \eta_{\mu\nu}\, h$

$\hbar = h - (1/2)\, \eta^{\mu\nu}\, \eta_{\mu\nu}\, h$

$\hbar = h - (1/2)\, [\eta^{\mu\nu}\, \eta_{\mu\nu}\,]\, h$

$$\begin{aligned}\hbar = h - (1/2)\, [&\eta^{11}\eta_{11} + \eta^{12}\eta_{12} + \eta^{13}\eta_{13} + \eta^{14}\eta_{14} \\ &+ \eta^{21}\eta_{21}\, \eta^{22}\eta_{22} + \eta^{23}\eta_{23} + \eta^{24}\eta_{24} \\ &+ \eta^{31}\eta_{31} + \eta^{32}\eta_{32} + \eta^{33}\eta_{33} + \eta^{34}\eta_{34} \\ &+ \eta^{41}\eta_{41} + \eta^{42}\eta_{42} + \eta^{43}\eta_{43} + \eta^{44}\eta_{44}]\, h\end{aligned}$$

$$\begin{aligned}= h - (1/2)\, [&(1)(1) + 0 + 0 + 0 \\ &+ 0\, (-1)(-1) + 0 + 0 \\ &+ 0 + 0 + (-1)(-1) + 0 \\ &+ 0 + 0 + 0 + (-1)(-1)]\, h\end{aligned}$$

$= h - (1/2)\, [1 + 1 + 1 + 1]\, h$

$= h - (1/2)\, [4]\, h$

$= h - 2\, h$

$= - h$

And now we have an expression for h in terms of $\hbar$:

$$h = - \hbar \qquad (35)$$

Starting with (34) solve for $h_{\mu\nu}$

$\hbar_{\mu\nu} = h_{\mu\nu} - (1/2)\, h\, \eta_{\mu\nu}$

$- h_{\mu\nu} = - \hbar_{\mu\nu} - (1/2)\, h\, \eta_{\mu\nu}$

$h_{\mu\nu} = \hbar_{\mu\nu} + (1/2)\, h\, \eta_{\mu\nu}$

Substituting (35) for h

$h_{\mu\nu} = \hbar_{\mu\nu} + (1/2)\,(- \hbar\,)\, \eta_{\mu\nu}$

Now we have $h_{\mu\nu}$ in terms of $\hbar$:

$$h_{\mu\nu} = \hbar_{\mu\nu} - (1/2)\, \hbar\, \eta_{\mu\nu} \qquad (36)$$

Again let's return to (34),

$\hbar_{\mu\nu} = h_{\mu\nu} - (1/2)\, h\, \eta_{\mu\nu}$

Operate on both sides of (34) with $\eta^{\alpha\mu}$.

$\eta^{\alpha\mu}\, \hbar_{\mu\nu} = \eta^{\alpha\mu}\, h_{\mu\nu} - (1/2)\, h\, \eta^{\alpha\mu}\, \eta_{\mu\nu}$

$\hbar^{\alpha}{}_{\nu} = h^{\alpha}{}_{\nu} - (1/2)\, h\, (\eta^{\alpha\mu}\, \eta_{\mu\nu}\,)$

$\hbar^{\alpha}{}_{\nu} = h^{\alpha}{}_{\nu} - (1/2)\, h\, (\eta^{\mu\alpha}\, \eta_{\mu\nu}\,)$

$\hbar^{\alpha}{}_{\nu} = h^{\alpha}{}_{\nu} - (1/2)\, h\, (\eta^{1\alpha}\, \eta_{1\nu} + \eta^{2\alpha}\, \eta_{2\nu} + \eta^{3\alpha}\, \eta_{3\nu} + \eta^{4\alpha}\, \eta_{4\nu})$

$\hbar^{\alpha}{}_{\nu} = h^{\alpha}{}_{\nu} - (1/2)\, h\, (\delta^{\alpha}{}_{\nu})$

$\hbar^{\alpha}{}_{\nu} = h^{\alpha}{}_{\nu} - (1/2)\, \delta^{\alpha}{}_{\nu}\, h$

Solving for $h^{\alpha}{}_{\nu}$ we have

$h^{\alpha}{}_{\nu} = \hbar^{\alpha}{}_{\nu} + (1/2)\, \delta^{\alpha}{}_{\nu}\, h$

Substituting (35) we have

$h^{\alpha}{}_{\nu} = \hbar^{\alpha}{}_{\nu} + (1/2)\, \delta^{\alpha}{}_{\nu}\, (- \hbar)$

And

$$h^{\alpha}{}_{\nu} = \hbar^{\alpha}{}_{\nu} - (1/2)\, \delta^{\alpha}{}_{\nu}\, \hbar \qquad (37)$$

$$= h_{\nu}{}^{\alpha} = \hbar_{\nu}{}^{\alpha} - (1/2)\, \delta_{\nu}{}^{\alpha}\, \hbar$$

Likewise

$$h^{\alpha}{}_{\mu} = \hbar^{\alpha}{}_{\mu} - (1/2)\, \delta^{\alpha}{}_{\mu}\, \hbar \qquad (38)$$

$$= h_{\mu}{}^{\alpha} = \hbar_{\mu}{}^{\alpha} - (1/2)\, \delta_{\mu}{}^{\alpha}\, \hbar$$

Starting with (36) operate on both sides with $\eta^{\nu\nu}$

$h_{\mu\nu} = \hbar_{\mu\nu} - (1/2)\, \hbar\, \eta_{\mu\nu}$

$\eta^{\nu\nu}\, h_{\mu\nu} = \eta^{\nu\nu}\, \hbar_{\mu\nu} - (1/2)\, \hbar\, \eta^{\nu\nu}\, \eta_{\mu\nu}$

$h_{\mu}{}^{\nu} = \hbar_{\mu}{}^{\nu} - (1/2)\, \hbar\, \eta_{\mu}{}^{\nu}$

Now operate on both sides with $\eta^{\mu\mu}$

$h^{\mu\nu} = \hbar^{\mu\nu} - (1/2)\, \hbar\, \eta^{\mu\nu}$

And now we have $h^{\mu\nu}$ in terms of $\hbar$

$$h^{\mu\nu} = \hbar^{\mu\nu} - (1/2)\, \eta^{\mu\nu}\, \hbar \qquad (39)$$

Likewise,

$$h^{\alpha\beta} = \hbar^{\alpha\beta} - (1/2)\, \eta^{\alpha\beta}\, \hbar \qquad (40)$$

Now we can express Einstein's linearized equations (32) in terms of $\hbar$ by substituting equations (35) through (39)

Starting with (32) we have

$$h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\, h_{,\alpha}{}^{\alpha} + \eta_{\mu\nu}\, h^{\alpha\beta}{}_{,\alpha\beta} = 16\,\pi\; T_{\mu\nu}$$

or

$$16\,\pi\; T_{\mu\nu} = h_{,\mu\nu} - h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\, h_{,\alpha}{}^{\alpha} + \eta_{\mu\nu}\, h^{\alpha\beta}{}_{,\alpha\beta}$$

Upon substitution we have

$$\begin{aligned}
16\,\pi\; T_{\mu\nu} = {} & [-\hbar]_{,\mu\nu} \\
& - [\hbar_{\nu}{}^{\alpha} - (1/2)\,\delta_{\nu}{}^{\alpha}\,\hbar]_{,\mu\alpha} \\
& - [\hbar_{\mu}{}^{\alpha} - (1/2)\,\delta_{\mu}{}^{\alpha}\,\hbar]_{,\nu\alpha} \\
& + [\hbar_{\mu\nu} - (1/2)\,\hbar\,\eta_{\mu\nu}]_{,\alpha}{}^{\alpha} \\
& - \eta_{\mu\nu}\,[-\hbar]_{,\alpha}{}^{\alpha} \\
& + \eta_{\mu\nu}\,[\hbar^{\alpha\beta} - (1/2)\,\eta^{\alpha\beta}\,\hbar]_{,\alpha\beta} \\
= {} & -\hbar_{,\mu\nu} \\
& - [\hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} - (1/2)\,\delta_{\nu}{}^{\alpha}\,\hbar_{,\mu\alpha}] \\
& - [\hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha} - (1/2)\,\delta_{\mu}{}^{\alpha}\,\hbar_{,\nu\alpha}] \\
& + [\hbar_{\mu\nu,\alpha}{}^{\alpha} - (1/2)\,\hbar_{,\alpha}{}^{\alpha}\,\eta_{\mu\nu}] \\
& - [-\eta_{\mu\nu}\,\hbar_{,\alpha}{}^{\alpha}] \\
& + [\eta_{\mu\nu}\,\hbar^{\alpha\beta}{}_{,\alpha\beta} - (1/2)\,\eta_{\mu\nu}\,\eta^{\alpha\beta}\,\hbar_{,\alpha\beta}]
\end{aligned}$$

$$\begin{aligned}
= {} & -\hbar_{,\mu\nu} \\
& - \hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} + (1/2)\,\delta_{\nu}{}^{\alpha}\,\hbar_{,\mu\alpha} \\
& - \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha} + (1/2)\,\delta_{\mu}{}^{\alpha}\,\hbar_{,\nu\alpha} \\
& + \hbar_{\mu\nu,\alpha}{}^{\alpha} - (1/2)\,\hbar_{,\alpha}{}^{\alpha}\,\eta_{\mu\nu} \\
& + \eta_{\mu\nu}\,\hbar_{,\alpha}{}^{\alpha} \\
& + \eta_{\mu\nu}\,\hbar^{\alpha\beta}{}_{,\alpha\beta} - (1/2)\,\eta_{\mu\nu}\,\eta^{\alpha\beta}\,\hbar_{,\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
= {} & -\hbar_{,\mu\nu} \\
& - \hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} + (1/2)\,\hbar_{,\mu\nu} \\
& - \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha} + (1/2)\,\hbar_{,\nu\mu} \\
& + \hbar_{\mu\nu,\alpha}{}^{\alpha} - (1/2)\,\hbar_{,\alpha}{}^{\alpha}\,\eta_{\mu\nu} \\
& + \eta_{\mu\nu}\,\hbar_{,\alpha}{}^{\alpha} \\
& + \eta_{\mu\nu}\,\hbar^{\alpha\beta}{}_{,\alpha\beta} - (1/2)\,\eta_{\mu\nu}\,\eta^{\alpha\beta}\,\hbar_{,\alpha\beta}
\end{aligned}$$

After rearranging,

$$\begin{aligned}
16\,\pi\; T_{\mu\nu} = {} & + \hbar_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\,\hbar^{\alpha\beta}{}_{,\alpha\beta} - \hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha} \\
& - \hbar_{,\mu\nu} + (1/2)\,\hbar_{,\mu\nu} + (1/2)\,\hbar_{,\nu\mu} \\
& - (1/2)\,\hbar_{,\alpha}{}^{\alpha}\,\eta_{\mu\nu} + \eta_{\mu\nu}\,\hbar_{,\alpha}{}^{\alpha} - (1/2)\,\eta_{\mu\nu}\,\eta^{\alpha\beta}\,\hbar_{,\alpha\beta}
\end{aligned}$$

$$= \hbar_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\,\hbar^{\alpha\beta}{}_{,\alpha\beta} - \hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha}$$

$$- \cancel{\hbar}_{,\mu\nu} + \cancel{(1/2)\,\hbar}_{,\mu\nu} + \cancel{(1/2)\,\hbar}_{,\nu\mu} \qquad \text{(These 3 terms cancel)}$$
$$- (1/2)\, \hbar_{,\alpha}{}^{\alpha}\, \eta_{\mu\nu} + \eta_{\mu\nu}\, \hbar_{,\alpha}{}^{\alpha} - (1/2)\, \eta_{\mu\nu}\, \eta^{\alpha\beta}\, \hbar_{,\alpha\beta}$$

Raise the index on the last term and we have

$$16\,\pi\, T_{\mu\nu} = \hbar_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\, \hbar^{\alpha\beta}{}_{,\alpha\beta} - \hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha}$$
$$- (1/2)\, \hbar_{,\alpha}{}^{\alpha}\, \eta_{\mu\nu} + \eta_{\mu\nu}\, \hbar_{,\alpha}{}^{\alpha} - (1/2)\, \eta_{\mu\nu}\, \hbar_{,\alpha}{}^{\alpha}$$

$$= \hbar_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\, \hbar^{\alpha\beta}{}_{,\alpha\beta} - \hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha}$$
$$- \cancel{(1/2)\,\hbar}_{,\alpha}{}^{\alpha}\, \eta_{\mu\nu} + \cancel{\hbar}_{,\alpha}{}^{\alpha}\, \eta_{\mu\nu} - \cancel{(1/2)\,\hbar}_{,\alpha}{}^{\alpha}\, \eta_{\mu\nu} \qquad \text{(These 3 terms cancel)}$$

And we are left with

$$\hbar_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\, \hbar^{\alpha\beta}{}_{,\alpha\beta} - \hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} - \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha} = 16\,\pi\, T_{\mu\nu} \qquad (41)$$

where

$$\hbar_{\mu\nu} = h_{\mu\nu} - (1/2)\, h\, \eta_{\mu\nu}$$

and

$$h_{mn} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Equation (41) matches the form of equation (5.5) on page 163 of Foster [3].

Now to check that (41) is equivalent to equation (18.7) on page 437 of MTW [2] we must first multiply the l.h.s. of (41) by a -1 since MTW uses the other signature convention for $\eta_{\alpha\beta}$. See (42) below

$$h_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (42)$$

MTW's signature convention [2]

Multiplying the l.h.s. of (41) by -1 we have

$$- \hbar_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu} \hbar^{\alpha\beta}{}_{,\alpha\beta} + \hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} + \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha} = 16\,\pi\, T_{\mu\nu} \qquad (43)$$

Next, using the gymnastics for raising and lowering indices, we may write the following expressions:

$$\hbar^{\alpha\beta}{}_{,\alpha\beta} = \eta_{\beta\beta}\, \eta^{\beta\beta}\, \hbar^{\alpha\beta}{}_{,\alpha\beta} = \eta_{\beta\beta}\, \hbar^{\alpha\beta}{}_{,\alpha\beta}\, \eta^{\beta\beta} = \hbar^{\alpha}{}_{\beta,\alpha}{}^{\beta} = \eta_{\alpha\alpha}\, \eta^{\alpha\alpha}\, \hbar^{\alpha}{}_{\beta,\alpha}{}^{\beta}$$
$$= \eta_{\alpha\alpha}\, \hbar^{\alpha}{}_{\beta,\alpha}{}^{\beta}\, \eta^{\alpha\alpha} = \hbar_{\alpha\beta,}{}^{\alpha\beta} \qquad (44)$$

$$\hbar_{\nu}{}^{\alpha}{}_{,\mu\alpha} = \hbar_{\nu}{}^{\alpha}{}_{,\alpha\mu} = \eta_{\alpha\alpha}\, \eta^{\alpha\alpha}\, \hbar_{\nu}{}^{\alpha}{}_{,\alpha\mu} = \eta_{\alpha\alpha}\, \hbar_{\nu}{}^{\alpha}{}_{,\alpha\mu}\, \eta^{\alpha\alpha}$$
$$= \hbar_{\nu\alpha,}{}^{\alpha}{}_{\mu} = \hbar_{\nu\alpha,}{}^{\alpha}{}_{\mu} \qquad (45)$$

$$\hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha} = \eta_{\alpha\alpha}\, \eta^{\alpha\alpha}\, \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha} = \eta_{\alpha\alpha}\, \hbar_{\mu}{}^{\alpha}{}_{,\nu\alpha}\, \eta^{\alpha\alpha} = \hbar_{\mu\alpha,\nu}{}^{\alpha}$$
$$= \hbar_{\mu\alpha,}{}^{\alpha}{}_{\nu} = \hbar_{\mu\alpha,}{}^{\alpha}{}_{\nu} \qquad (46)$$

Substituting (44) through (46) into (43) we have an equivalent version of the field equations.

$$- \hbar_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\, \hbar_{\alpha\beta,}{}^{\alpha\beta} + \hbar_{\nu\alpha,}{}^{\alpha}{}_{\mu} + \hbar_{\mu\alpha,}{}^{\alpha}{}_{\nu} = 16\,\pi\, T_{\mu\nu} \qquad (47)$$

where

$$\hbar_{\mu\nu} = h_{\mu\nu} - (1/2)\, h\, \eta_{\mu\nu}$$

and

$$h_{mn} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This form of Einstein's linearized field equations matches the form of equation (18.7) on page 437 in MTW [2].

Lastly, we will check that (47), and thus (41), is equivalent to equation (8.32) on page 204 of Schutz [1]. Since Schutz uses the same signature for $\eta_{\mu\nu}$ as MTW [2], start with (47):

$- \hbar_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\, \hbar_{\alpha\beta}{}^{,\alpha\beta} + \hbar_{\nu\alpha,}{}^{\alpha}{}_{\mu} + \hbar_{\mu\alpha,}{}^{\alpha}{}_{\nu} = 16\,\pi\; T_{\mu\nu}$

If we let $8\,\pi\; T_{\mu\nu} = G_{\mu\nu}$ then (43) becomes

$- \hbar_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\, \hbar_{\alpha\beta}{}^{,\alpha\beta} + \hbar_{\nu\alpha,}{}^{\alpha}{}_{\mu} + \hbar_{\mu\alpha,}{}^{\alpha}{}_{\nu} = 2\; G_{\mu\nu}$

After rearranging we have

$$- (1/2)\,[\hbar_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\, \hbar_{\alpha\beta}{}^{,\alpha\beta} - \hbar_{\nu\alpha,}{}^{\alpha}{}_{\mu} - \hbar_{\mu\alpha,}{}^{\alpha}{}_{\nu}\,] = G_{\mu\nu} \qquad (48)$$

where $G_{\mu\nu}$ is the *Einstein tensor*

Next we may write the following expressions since, as stated before, the order of differentiation doesn't matter:

$$\hbar_{\nu\alpha,}{}^{\alpha}{}_{\mu} = \hbar_{\nu\alpha,\mu}{}^{\alpha} \qquad (49)$$

$$\hbar_{\mu\alpha,}{}^{\alpha}{}_{\nu} = \hbar_{\mu\alpha,\nu}{}^{\alpha} \qquad (50)$$

Substituting (49) and (50) into (48) we have:

$$- (1/2)\,[\hbar_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\, \hbar_{\alpha\beta}{}^{,\alpha\beta} - \hbar_{\nu\alpha,\mu}{}^{\alpha} - \hbar_{\mu\alpha,\nu}{}^{\alpha}\,] = G_{\mu\nu} \qquad (51)$$

This is the same form of the linearized field equations as equation (8.32) in Schutz [1]. To make the equation look exactly the same, without changing its meaning, let's change the symbols used for the indices from μ to α, ν to β, β to ν, and α to μ. Then (51) becomes

$$- (1/2)\,[\hbar_{\alpha\beta,\mu}{}^{\mu} + \eta_{\alpha\beta}\, \hbar_{\mu\nu}{}^{,\mu\nu} - \hbar_{\beta\mu,\alpha}{}^{\mu} - \hbar_{\alpha\mu,\beta}{}^{\mu}\,] = G_{\alpha\beta} \qquad (51)$$

or equivalently,

$$- (1/2)\,[\hbar_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\, \hbar_{\mu\nu}{}^{,\mu\nu} - \hbar_{\beta\mu,\alpha}{}^{,\mu} - \hbar_{\alpha\mu,\beta}{}^{,\mu} + \mathcal{O}(2)] = G_{\alpha\beta}$$

where

$$\hbar_{\alpha\beta} = h_{\alpha\beta} - (1/2)\, h\, \eta_{\alpha\beta}$$

and

$$h_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The three forms of Einstein's linearized field equations presented above, (41), (47), and (51), are equivalent. Note that we are not limited to these three forms. I am only showing 3 here.

Appendix A

Show that $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$

Start by taking a guess that $g^{\mu\nu}$ is of the form

$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu} \qquad \text{where } | f^{\mu\nu} | << 1 \tag{A-1}$$

Recall that

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $| h_{\mu\nu} | << 1$

And find a way to relate $f^{\mu\nu}$ to $h_{\mu\nu}$. Here's one way:

Multiply $g^{\mu\sigma}$ by $g_{\sigma\nu}$ and we have

$g^{\mu\sigma} g_{\sigma\nu} = (\eta^{\mu\sigma} + f^{\mu\sigma})(\eta_{\sigma\nu} + h_{\sigma\nu})$

$g^{\mu\sigma} g_{\sigma\nu} = \eta^{\mu\sigma} \eta_{\sigma\nu} + \eta^{\mu\sigma} h_{\sigma\nu} + f^{\mu\sigma} \eta_{\sigma\nu} + f^{\mu\sigma} h_{\sigma\nu}$

Note that $g^{\mu\sigma} g_{\sigma\nu} = \delta_\nu{}^\mu$ and $\eta^{\mu\sigma} \eta_{\sigma\nu} = \delta_\nu{}^\mu$ therefore we have

$\delta_\nu{}^\mu = \delta_\nu{}^\mu + \eta^{\mu\sigma} h_{\sigma\nu} + f^{\mu\sigma} \eta_{\sigma\nu} + f^{\mu\sigma} h_{\sigma\nu}$

$\not{\delta}_\nu{}^\mu = \not{\delta}_\nu{}^\mu + \eta^{\mu\sigma} h_{\sigma\nu} + f^{\mu\sigma} \eta_{\sigma\nu} + f^{\mu\sigma} h_{\sigma\nu}$

where the $\delta_\nu{}^\mu$'s cancel to give

$0 = \eta^{\mu\sigma} h_{\sigma\nu} + f^{\mu\sigma} \eta_{\sigma\nu} + f^{\mu\sigma} h_{\sigma\nu}$

Since $f^{\mu\sigma} h_{\sigma\nu}$ is a second order term we may write

$0 = \eta^{\mu\sigma} h_{\sigma\nu} + f^{\mu\sigma} \eta_{\sigma\nu} + \mathcal{O}[2]$

By rule (b) $\mathcal{O}[2] = 0$ so we have

$0 = \eta^{\mu\sigma} h_{\sigma\nu} + f^{\mu\sigma} \eta_{\sigma\nu}$

and

$- \eta^{\mu\sigma} h_{\sigma\nu} = f^{\mu\sigma} \eta_{\sigma\nu}$

$- h^\mu{}_\nu = f^{\mu\sigma} \eta_{\sigma\nu}$

Operate on both sides with $\eta^{\nu\rho}$:

$\eta^{\nu\rho}$ [- $h^{\mu}{}_{\nu}$] = $\eta^{\nu\rho}$ [$f^{\mu\sigma} \eta_{\sigma\nu}$]

- $\eta^{\nu\rho} h^{\mu}{}_{\nu} = \eta^{\nu\rho}$ [$f^{\mu\sigma} \eta_{\sigma\nu}$]

- $h^{\mu\rho} = \eta^{\nu\rho}$ [$f^{\mu\sigma} \eta_{\sigma\nu}$]

Since $\eta^{\nu\rho}$ is symmetric we can change the order in which it operates:

- $h^{\mu\rho} = f^{\mu\sigma}$ [$\eta^{\nu\rho} \eta_{\sigma\nu}$]

where $\eta^{\nu\rho} \eta_{\sigma\nu} = \delta_{\sigma}{}^{\rho}$ so that

- $h^{\mu\rho} = f^{\mu\sigma}$ [$\delta_{\sigma}{}^{\rho}$]

- $h^{\mu\rho} = f^{\mu\sigma} \delta_{\sigma}{}^{\rho}$

- $h^{\mu\rho} = f^{\mu\rho}$

And finally we have a relationship between f and h.

$f^{\mu\rho} = - h^{\mu\rho}$

We may now rename the index ρ to ν and we have

$$f^{\mu\nu} = - h^{\mu\nu} \quad \text{(A-2)}$$

Substituting (A-2) into (A-1) we have

$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ where $| f^{\mu\nu} | << 1$

We may again change the symbols used for the indices to complete this proof: Change μ to α, and ν to β and we have

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \qquad \text{where } | f^{\alpha\beta} | << 1 \quad \text{(A-3)}$$

Appendix B

Explicit representation of the raised and non-raised "comma derivative"

$$h_{mn,a} = \begin{pmatrix} \frac{\partial}{\partial x^a} h_{11} & \frac{\partial}{\partial x^a} h_{12} & \frac{\partial}{\partial x^a} h_{13} & \frac{\partial}{\partial x^a} h_{14} \\ \frac{\partial}{\partial x^a} h_{21} & \frac{\partial}{\partial x^a} h_{22} & \frac{\partial}{\partial x^a} h_{23} & \frac{\partial}{\partial x^a} h_{24} \\ \frac{\partial}{\partial x^a} h_{31} & \frac{\partial}{\partial x^a} h_{32} & \frac{\partial}{\partial x^a} h_{33} & \frac{\partial}{\partial x^a} h_{34} \\ \frac{\partial}{\partial x^a} h_{41} & \frac{\partial}{\partial x^a} h_{42} & \frac{\partial}{\partial x^a} h_{43} & \frac{\partial}{\partial x^a} h_{44} \end{pmatrix}$$

$$h_{mn}{}^{,a} = \begin{pmatrix} h^{ga} \frac{\partial}{\partial x^g} h_{11} & h^{ga} \frac{\partial}{\partial x^g} h_{12} & h^{ga} \frac{\partial}{\partial x^g} h_{13} & h^{ga} \frac{\partial}{\partial x^g} h_{14} \\ h^{ga} \frac{\partial}{\partial x^g} h_{21} & h^{ga} \frac{\partial}{\partial x^g} h_{22} & h^{ga} \frac{\partial}{\partial x^g} h_{23} & h^{ga} \frac{\partial}{\partial x^g} h_{24} \\ h^{ga} \frac{\partial}{\partial x^g} h_{31} & h^{ga} \frac{\partial}{\partial x^g} h_{32} & h^{ga} \frac{\partial}{\partial x^g} h_{33} & h^{ga} \frac{\partial}{\partial x^g} h_{34} \\ h^{ga} \frac{\partial}{\partial x^g} h_{41} & h^{ga} \frac{\partial}{\partial x^g} h_{42} & h^{ga} \frac{\partial}{\partial x^g} h_{43} & h^{ga} \frac{\partial}{\partial x^g} h_{44} \end{pmatrix}$$

where

$$h^{ga} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Resulting relationship between the raised and non-raised comma derivative:

$$For\ a = 1 \qquad h_{mn}{}^{,a} = h_{mn,a}$$

$$For\ a = 2,3,4 \quad h_{mn}{}^{,a} = -h_{mn,a}$$

$$h_{mn,ab} = \begin{pmatrix} \frac{\partial^2}{\partial x^a \partial x^b} h_{11} & \frac{\partial^2}{\partial x^a \partial x^b} h_{12} & \frac{\partial^2}{\partial x^a \partial x^b} h_{13} & \frac{\partial^2}{\partial x^a \partial x^b} h_{14} \\ \frac{\partial^2}{\partial x^a \partial x^b} h_{21} & \frac{\partial^2}{\partial x^a \partial x^b} h_{22} & \frac{\partial^2}{\partial x^a \partial x^b} h_{23} & \frac{\partial^2}{\partial x^a \partial x^b} h_{24} \\ \frac{\partial^2}{\partial x^a \partial x^b} h_{31} & \frac{\partial^2}{\partial x^a \partial x^b} h_{32} & \frac{\partial^2}{\partial x^a \partial x^b} h_{33} & \frac{\partial^2}{\partial x^a \partial x^b} h_{34} \\ \frac{\partial^2}{\partial x^a \partial x^b} h_{41} & \frac{\partial^2}{\partial x^a \partial x^b} h_{42} & \frac{\partial^2}{\partial x^a \partial x^b} h_{43} & \frac{\partial^2}{\partial x^a \partial x^b} h_{44} \end{pmatrix}$$

$$h_{mn,a}{}^{b} = \begin{pmatrix} h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{11} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{12} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{13} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{14} \\ h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{21} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{22} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{23} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{24} \\ h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{31} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{32} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{33} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{34} \\ h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{41} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{42} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{43} & h^{gb} \frac{\partial^2}{\partial x^a \partial x^g} h_{44} \end{pmatrix}$$

where

$$h^{gb} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Resulting relationship between the raised and non-raised second comma derivative:

$$For\ b = 1 \qquad h_{mn,a}{}^{b} = h_{mn,ab}$$

$$For\ b = 2,3,4 \quad h_{mn,a}{}^{b} = -h_{mn,ab}$$

$$h_{mn}{}^{,ab} = \begin{pmatrix} h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{11} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{12} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{13} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{14} \\ h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{21} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{22} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{23} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{24} \\ h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{31} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{32} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{33} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{34} \\ h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{41} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{42} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{43} & h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s}h_{44} \end{pmatrix}$$

Where

$$h^{ga}h^{sb}\frac{\partial^2}{\partial x^g \partial x^s} = \left(h^{1a}\frac{\partial}{\partial x^1} + h^{2a}\frac{\partial}{\partial x^2} + h^{3a}\frac{\partial}{\partial x^3} + h^{4a}\frac{\partial}{\partial x^4}\right) \times \left(h^{1b}\frac{\partial}{\partial x^1} + h^{2b}\frac{\partial}{\partial x^2} + h^{3b}\frac{\partial}{\partial x^3} + h^{4b}\frac{\partial}{\partial x^4}\right)$$

Resulting factors that relate $h_{\mu\nu}{}^{,\alpha\beta}$ to $h_{\mu\nu,\alpha\beta}$:

	α=1	α=2	α=3	α=4
β=1	+	-	-	-
β=2	-	+	+	+
β=3	-	+	+	+
β=4	-	+	+	+

Examples:

For $\alpha,\beta=1,3$ $\quad h_{\mu\nu}{}^{,\alpha\beta} = -h_{\mu\nu,\alpha\beta}$

For $\alpha,\beta=4,4$ $\quad h_{\mu\nu}{}^{,\alpha\beta} = h_{\mu\nu,\alpha\beta}$

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