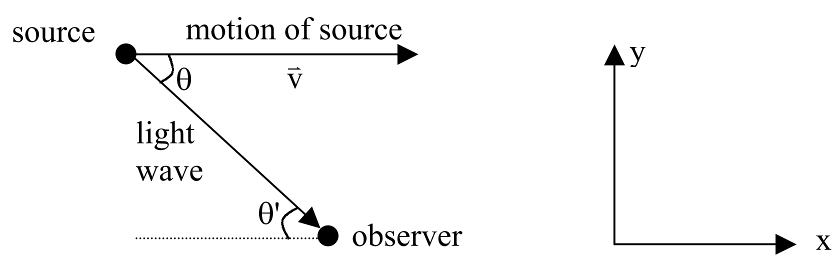
Relativistic Doppler Shift Formula

In class, I wrote the relativistic Doppler shift formula for the source moving in an arbitrary direction relative to the observer as

$$\nu = \nu_0 \frac{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{1/2}}{\left(1-\frac{\mathrm{v}\cos\theta}{c}\right)} \tag{1}$$

The text gives a different formula (2.36) that appears to be inconsistent with the formula (1). However, the difference comes about due to the different definitions of the angles θ and θ'.



θ is the angle between the light wave and the x axis in the frame of the source, while θ' is the angle in the observer's rest frame. These are not the same because the observer sees distances along x affected by length contraction, but those along y are not affected.

The book states that these angles are related by Eq. (2.34)

$$\nu\cos\theta = \nu_0 \frac{\left(\cos\theta' + \frac{\mathrm{v}}{c}\right)}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{1/2}} \qquad \text{or} \qquad \cos\theta = \frac{\nu_0}{\nu} \frac{\left(\cos\theta' + \frac{\mathrm{v}}{c}\right)}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{1/2}}$$

Substituting into Eq. (1) above

$$\frac{\nu}{\nu_0} = \frac{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{1/2}}{\left(1-\frac{\mathrm{v}}{c}\frac{\nu_0}{\nu}\frac{\left(\cos\theta' + \frac{\mathrm{v}}{c}\right)}{\left(1-\frac{\mathrm{v}^2}{c^2}\right)^{1/2}}\right)} = \frac{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)}{\left(1-\frac{\mathrm{v}^2}{c^2}\right)^{1/2} - \frac{\nu_0}{\nu}\left(\frac{\mathrm{v}}{c}\cos\theta' + \frac{\mathrm{v}^2}{c^2}\right)}$$

We can solve this equation for $\frac{\nu}{\nu_0}$:

$$\frac{\nu}{\nu_0}\left(1-\frac{\mathrm{v}^2}{c^2}\right)^{1/2}-\left(\frac{\mathrm{v}}{c}\cos\theta'+\frac{\mathrm{v}^2}{c^2}\right)=\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)$$

$$\frac{\nu}{\nu_0}\left(1-\frac{\mathrm{v}^2}{c^2}\right)^{1/2}-\frac{\mathrm{v}}{c}\cos\theta'=1$$

$$\frac{\nu}{\nu_0}=\frac{1+\frac{\mathrm{v}}{c}\cos\theta'}{\left(1-\frac{\mathrm{v}^2}{c^2}\right)^{1/2}}$$

which is textbook Eq. (2.36)