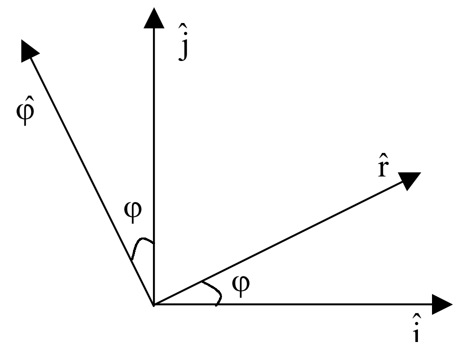
**Relationship between Plane Polar and Cartesian Coordinates**




From geometry we can see that the Cartesian unit vectors $\hat{i}$ and $\hat{j}$ are related to the plane polar unit vectors $\hat{r}$ and $\hat{\varphi}$ by the relations:

$$\hat{r} = \cos\varphi\ \hat{i} + \sin\varphi\ \hat{j}$$

$$\hat{\varphi} = -\sin\varphi\ \hat{i} + \cos\varphi\ \hat{j} \quad .$$

Note also that $$\frac{d\hat{r}}{d\varphi} = -\sin\varphi\ \hat{i} + \cos\varphi\ \hat{j} = \hat{\varphi} \quad .$$

The position vector $\vec{r}_p$ (from the origin to point P) is

$$\vec{r}_p = x\ \hat{i} + y\ \hat{j}$$ in Cartesian coordinates

and $$\vec{r}_p = r\ \hat{r}$$ in plane polar coordinates.

The unit vectors $\hat{i}$ and $\hat{j}$ in Cartesian coordinates are <u>constant.</u> Therefore, the infinitesimal length element in Cartesian coordinates is simply

$$d\vec{\ell} \equiv d\vec{r}_p = dx\ \hat{i} + dy\ \hat{j} \quad .$$

But in plane polar coordinates we have

$$\begin{aligned} d\vec{\ell} \equiv d\vec{r}_p = d(r\ \hat{r}) &= r\ d\hat{r} + \hat{r}\ dr \\ &= r\frac{d\hat{r}}{d\varphi}d\varphi + \hat{r}\ dr \\ &= r\ d\varphi\ \hat{\varphi} + \hat{r}\ dr \end{aligned}$$

where in the last step, we used the result above for $\frac{d\hat{r}}{d\varphi}$.