Normal approximation for statistics of Gibbsian input in geometric probability

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Abstract

This paper concerns the asymptotic behavior of a random variable W_λ resulting from the summation of the functionals of a Gibbsian spatial point process over windows $Q_\lambda \uparrow \mathbb{R}^d$. We establish conditions ensuring that W_λ has volume order fluctuations, that is they coincide with the fluctuations of functionals of Poisson spatial point processes. We combine this result with Stein's method to deduce rates of normal approximation for W_λ, as $\lambda \to \infty$. Our general results establish variance asymptotics and central limit theorems for statistics of random geometric and related Euclidean graphs on Gibbsian input. We also establish similar limit theory for claim sizes of insurance models with Gibbsian input, the number of maximal points of a Gibbsian sample, and the size of spatial birth-growth models with Gibbsian input.

Key words and phrases. Gibbs point process, Berry–Esseen bound, Stein's method, random Euclidean graphs, maximal points, spatial birth-growth models.

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1 Introduction and main results

Functionals of large geometric structures on finite input $\mathcal{X} \subset \mathbb{R}^d$ often consist of sums of spatially dependent terms admitting the representation

$$\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}), \tag{1.1}$$

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where the $\mathbb{R}^+$-valued *score function* ξ, defined on pairs $(x, \mathcal{X})$, represents the *interaction* of x with respect to $\mathcal{X}$. The sums (1.1) typically describe a global feature of an underlying geometric property in terms of a sum of local contributions $\xi(x, \mathcal{X})$.

A large and diverse number of functionals and statistics in stochastic geometry, applied geometric probability, and spatial statistics may be cast in the form (1.1) for appropriately chosen ξ. The behavior of these statistics on random input $\mathcal{X}$ on windows $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d \uparrow \mathbb{R}^d$ can be deduced from general limit theorems [5, 6, 30, 31, 34, 35] for (1.1) provided $\mathcal{X}$ is either a Poisson or binomial point process. This has led to solutions of problems in random sequential packing [33], random graphs [30, 31, 32, 34, 40], percolation models [23], analysis of data on manifolds [36], and convex hulls of i.i.d. samples [8, 9, 10], among others.

When $\mathcal{X}$ is neither Poisson nor binomial input, the statistics (1.1) are less well understood. When $\mathcal{X}$ belongs to a restricted family of Gibbs point processes, then [38] shows that in the low temperature regime the behavior of the statistics (1.1) closely parallels the behavior when the input is Poisson or binomial. Our main purpose here is to build on this first step in ways which will be described shortly.

For all $\lambda \in [1, \infty)$ consider the functionals

$$W_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}), \tag{1.2}$$

where $\mathcal{P}_\lambda^{\beta\Psi}$ is the restriction of a Gibbs point process $\mathcal{P}^{\beta\Psi}$ on $\mathbb{R}^d$ to Q_λ. The process $\mathcal{P}^{\beta\Psi}$ has a Hamiltonian (also known as energy functional) Ψ, it is locally absolutely continuous with respect to a reference homogeneous Poisson point process $\tilde{\mathcal{P}}_\tau$ of intensity (activity) τ, and β is the inverse temperature. In general, even for the simplest of score functions ξ, as $\lambda \to \infty$, the Gibbsian functional W_λ may neither enjoy asymptotic normality nor will W_λ have volume order fluctuations, i.e., $\mathrm{Var} W_\lambda$ may not be of order $\mathrm{Vol}(Q_\lambda)$; see [24]. On the other hand, as shown in [38], if both the Gibbsian input and the score function have rapidly decaying spatial dependencies, then one could expect that W_λ behaves like a sum of i.i.d. random variables.

We have three goals. The first is to show that given a Hamiltonian Ψ, there is a range of inverse temperature and activity parameters β and τ such that for any locally determined score function, the Gibbsian functional W_λ has volume order fluctuations. In other words, the fluctuations for W_λ coincide with those when $\mathcal{P}_\lambda^{\beta\Psi}$ is replaced by Poisson or binomial input. This extends [38], which shows normal approximation for W_λ, but only under an assumption of volume order fluctuations. However [38] stops short of spelling out sufficient conditions guaranteeing volume order fluctuations. This paper redresses this. Our second goal is to prove a rate of convergence to the normal for $(W_\lambda - \mathbb{E}\, W_\lambda)/\sqrt{\mathrm{Var} W_\lambda}$ for general score functions ξ, including those which are non-translation invariant. The setting of non-translation invariant score functions yields normal approximation of maximal points of Gibbsian samples, which lies outside the scope of [38]. Formal statements of these results are given in Theorems 1.1-1.3. Our third goal is to show that the standard rates of normal convergence for some classical geometric statistics on Poison input [5, 22, 30, 34, 35] extend to the setting of Gibbsian input. This includes showing rates of normal convergence for (i) statistics of random geometric and Euclidean graphs on Gibbsian input, (ii) the number of claims in an insurance model with claim locations and times given by Gibbsian input, (iii) the number of maximal points in a Gibbs sample, as well as (iv) functionals of spatial birth-growth models with Gibbsian input.

1.1 Notation and terminology

(i) **Gibbs point processes.** We consider the class $\mathbf{\Psi}^*$ of Hamiltonians Ψ defined on finite point sets $\mathcal{X} \subset \mathbb{R}^d$ which are (a) *monotonic*: $\Psi(\mathcal{X}) \leq \Psi(\mathcal{X}')$ for all $\mathcal{X} \subset \mathcal{X}'$; (b) *translation invariant*: $\Psi(\mathcal{X} + x) = \Psi(\mathcal{X})$ for all $x \in \mathbb{R}^d$; (c) *rotation invariant*: $\Psi(\mathcal{X}) = \Psi(\mathcal{X}')$ for all rotations $\mathcal{X}'$ of $\mathcal{X}$; (d) *non-degenerate*: $\Psi(\{x\}) < \infty$ for all $x \in \mathbb{R}^d \cup \{\emptyset\}$; (e) nonnegative. Monotonicity (a) ensures that the interaction is *repulsive*; see e.g. Ch. 6 of [27]. The non-negativity condition (e) is, without loss, equivalent to the assumption $\Psi(\emptyset) > -\infty$, which must hold in every Gibbsian setup. The class $\mathbf{\Psi}^*$ contains Hamiltonians defined in terms of pair potentials without negative part, area interaction Hamiltonians, hard core Hamiltonians and Hamiltonians generating a truncated Poisson point process. We refer to the appendix for more details on the Hamiltonians in the class $\mathbf{\Psi}^*$.

Define for $\Psi \in \mathbf{\Psi}^*$ and finite $\mathcal{X} \subset \mathbb{R}^d$ the local energy function

$$\Delta^{\Psi}(\mathbf{0}, \mathcal{X}) := \Psi(\mathcal{X} \cup \{\mathbf{0}\}) - \Psi(\mathcal{X}),\ \mathbf{0} \notin \mathcal{X}.$$

Here $\mathbf{0}$ denotes a point at the origin of $\mathbb{R}^d$. Proposition 2.1 (i) of [38] shows that for $\mathcal{X} \subset \mathbb{R}^d$ locally finite,

$$\Delta^{\Psi}(\mathbf{0}, \mathcal{X}) := \lim_{r \to \infty} \Delta^{\Psi}(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})) \tag{1.3}$$

is well-defined, where $B_r(x) := \{y : \ |x - y| \leq r\}$ is the Euclidean ball with center x and radius r. Note that monotonicity of Ψ implies non-negativity of Δ^{Ψ}. Ψ has *finite or bounded range* if there is $r^{\Psi} \in (0, \infty)$ such that for all finite $\mathcal{X} \subset \mathbb{R}^d$ we have $\Delta^{\Psi}(\mathbf{0}, \mathcal{X}) = \Delta^{\Psi}(\mathbf{0}, \mathcal{X} \cap B_{r^{\Psi}}(\mathbf{0}))$. With the exception of the Hamiltonian defined by the pair potential, all Hamiltonians in $\mathbf{\Psi}^*$ have finite range (Lemma 3.1 of [38]). For such Ψ we put

$$m_0^{\Psi} := \inf_{\mathcal{X} \text{ locally finite}} \Delta^{\Psi}(\mathbf{0}, \mathcal{X})$$

and

$$\mathcal{R}^\Psi := \{(\tau, \beta) \in (\mathbb{R}^+)^2 : \ \tau v_d \exp(-\beta m_0^\Psi)(r^\Psi + 1)^d < 1\}, \tag{1.4}$$

where $v_d := \pi^{d/2}[\Gamma(1 + d/2)]^{-1}$ is the volume of the unit ball in $\mathbb{R}^d$. The parameter τ is called the *activity* or *fugacity* and $1/\beta$ the *temperature*. When Ψ is a Hamiltonian with a pair potential, then the factor $(r^\Psi + 1)^d$ in (1.4) is replaced by the moment of an exponentially decaying random variable as in (3.7) of [38].

Quantifying spatial dependencies of Gibbs point processes is difficult in general. However spatial dependencies readily become transparent when a Gibbs point process is viewed as an algorithmic construct. As shown in [38], this is feasible whenever Ψ belongs to the class $\mathbf{\Psi}^*$. We review the algorithmic construction of Gibbs point processes developed in [38], and inspired by [18]. Let $(\varrho(t))_{t\in\mathbb{R}}$ be a stationary homogeneous free birth and death process on $\mathbb{R}^d$ with these dynamics:

- A new point $x \in \mathbb{R}^d$ is born in ϱ_t during the time interval $[t - dt, t]$ with probability $\tau dx dt$,
- An existing point $x \in \varrho_t$ dies during the time interval $[t - dt, t]$ with probability dt, that is the lifetimes of points of the process are independent standard exponential.

The unique stationary and reversible measure for this process is the law of the Poisson point process $\tilde{\mathcal{P}}_\tau$.

Following [38], for each $\Psi \in \mathbf{\Psi}^*$, we use a dependent thinning procedure on $(\varrho(t))_{t\in\mathbb{R}}$ to algorithmically construct a suitably rarified Gibbs point process $\mathcal{P}^{\beta\Psi}$ on $\mathbb{R}^d$, one whose law is locally absolutely continuous with respect to the reference point process $\tilde{\mathcal{P}}_\tau$. Section 3 recalls some of the salient properties of $\mathcal{P}^{\beta\Psi}$.

For arbitrary $(\tau, \beta)\in (\mathbb{R}^+)^2$ and arbitrary Ψ, the asymptotic behavior of W_λ at (1.2) may involve non-standard scaling and non-standard limits. However, if $\mathcal{P}^{\beta\Psi}$ is *admissible in the sense that* $(\tau, \beta) \in \mathcal{R}^\Psi$ *and* $\Psi \in \mathbf{\Psi}^*$, *then we shall show that* W_λ *behaves like a classical sum of i.i.d. random variables.* Without further mention, we shall always assume that $\mathcal{P}^{\beta\Psi}$ is admissible. We remark that the class of admissible Gibbsian point processes $\mathcal{P}^{\beta\Psi}, \Psi \in \mathbf{\Psi}^*$, is restricted to those having low activity as a function of temperature. As such, this paper, together with [38], represent but a first step in extending the limit theory of geometric functionals of i.i.d. point sets to more general Gibbsian point sets. It remains unclear, at least to us, whether the limit theory of this paper may be extended to Gibbsian input significantly more general than the admissible input considered here.

Recall that $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d$ and put $Q_\infty := \mathbb{R}^d$. Given $\lambda \in [1, \infty]$, $\Psi \in \mathbf{\Psi}^*$, and $(\tau, \beta) \in \mathcal{R}^\Psi$, we let

$$\mathcal{P}_\lambda^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap Q_\lambda. \tag{1.5}$$

By convention we have $\mathcal{P}_\infty^{\beta\Psi} := \mathcal{P}^{\beta\Psi}$.

(ii) **Poisson-like point processes.** A point process Ξ on $\mathbb{R}^d$ is *stochastically dominated* by the reference process $\tilde{\mathcal{P}}_\tau$ if for all Borel sets $B \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ we have $\mathbb{P}[\mathrm{card}(\Xi \cap B) \geq n] \leq \mathbb{P}[\mathrm{card}(\tilde{\mathcal{P}}_\tau \cap B) \geq n]$. As in [38], we say that Ξ is *Poisson-like* if (a) Ξ is stochastically dominated by $\tilde{\mathcal{P}}_\tau$ and (b) there exists $c \in (0, \infty)$ and $r_1 \in (0, \infty)$ such that for all $r \in (r_1, \infty)$, $x \in \mathbb{R}^d$, and any point set $\mathcal{E}_r(x)$ in $B_r^c(x)$, the conditional probability of $B_r(x)$ not being hit by Ξ, given that $\Xi \cap B_r(x)^c$ coincides with $\mathcal{E}_r(x)$, satisfies

$$\mathbb{P}[\Xi \cap B_r(x) = \emptyset \ | \{\Xi \cap B_r(x)^c = \mathcal{E}_r(x)\}] \leq \exp(-cr^d). \tag{1.6}$$

Poisson-like processes have void probabilities analogous to those of homogeneous Poisson processes, justifying the choice of terminology. Lemma 3.3 of [38] shows that admissible Gibbs processes $\mathcal{P}^{\beta\Psi}$ are Poisson-like.

(iii) **Translation invariance**. ξ is *translation invariant* if for all $x \in \mathbb{R}^d$ and locally finite $\mathcal{X} \subset \mathbb{R}^d$ we have $\xi(x, \mathcal{X}) = \xi(x + y, \mathcal{X} + y)$ for all $y \in \mathbb{R}^d$.

(iv) **Moment conditions.** Let $\|X\|_q$ denote the q norm of the random variable X. Say that ξ satisfies the q-moment condition if

$$w_q := \sup_{\lambda \in [1,\infty]} \sup_{x \in Q_\lambda} \|\xi(x, \mathcal{P}_\lambda^{\beta\Psi} \cup \{x\})\|_q < \infty. \tag{1.7}$$

(v) **Stabilization.** Given a locally finite point set $\mathcal{X}$, write $\mathcal{X}^z$ for $\mathcal{X} \cup \{z\}$ if $z \in \mathbb{R}^d$ and $\mathcal{X}^z = \mathcal{X}$ if $z = \emptyset$. The following definition of stabilization, used extensively in [38], is similar to that in [4, 30, 31, 34, 35] except now we consider Gibbsian input, instead of Poisson or binomial input.

Definition 1.1 *ξ is a stabilizing functional with respect to the Poisson-like process Ξ if for all $x \in \mathbb{R}^d$, all $z \in \mathbb{R}^d \cup \{\emptyset\}$, and almost all realizations $\mathcal{X}$ of Ξ there exists $R := R^\xi(x, \mathcal{X}^z) \in (0, \infty)$ (a 'radius of stabilization') such that*

$$\xi(x, \mathcal{X}^z \cap B_R(x)) = \xi(x, (\mathcal{X}^z \cap B_R(x)) \cup \mathcal{Y}) \tag{1.8}$$

for all locally finite point sets $\mathcal{Y} \subseteq \mathbb{R}^d \setminus B_R(x)$.

Stabilization of ξ on Ξ implies that $\xi(x, \mathcal{X}^z)$ is wholly determined by the point configuration $\mathcal{X}^z \cap B_{R^\xi}(x)$. It also yields $\xi(x, \mathcal{X}^z \cap B_r(x)) = \xi(x, \mathcal{X}^z \cap B_{R^\xi}(x))$ for $r \in [R^\xi, \infty)$. Stabilizing functionals can thus be a.s. extended to the entire process Ξ^z, that is to say for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d \cup \{\emptyset\}$ we have

$$\xi(x, \Xi^z) := \lim_{r \to \infty} \xi(x, \Xi^z \cap B_r(x)) \quad \text{a.s.} \tag{1.9}$$

Given $s > 0$ and any simple point process Ξ, including Poisson-like processes, define the conditional tail probability

$$t(\Xi, s) := \sup_{x \in \mathbb{R}^d} \sup_{z \in \mathbb{R}^d \cup \{\emptyset\}} \mathbb{P}[R^\xi(x, \Xi^z) > s | \Xi\{x\} = 1].$$

The conditional distribution of Ξ given that $\Xi\{x\} = 1$ is the Palm distribution of Ξ at x [21, Chapter 10] and the conditional probability can be intuitively interpreted as

$$\mathbb{P}[R^\xi(x, \Xi^z) > s | \Xi\{x\} = 1] = \lim_{\epsilon \downarrow 0} \mathbb{P}[\sup_{y \in B_\epsilon(x) \cap \Xi} R^\xi(y, \Xi^z) > s | \Xi(B_\epsilon(x)) = 1].$$

We say that ξ is *stabilizing in the wide sense* [38] if for every Poisson-like process Ξ we have $t(\Xi, s) \to 0$ as $s \to \infty$. Further, ξ is *exponentially stabilizing in the wide sense* if for every Poisson-like process Ξ we have

$$\limsup_{s \to \infty} s^{-1} \ln t(\Xi, s) < 0. \tag{1.10}$$

Exponential stabilization of ξ with respect to the augmented point set Ξ^z ensures that covariances of scores at points x and y, as given at (1.16), decays exponentially fast with $|x - y|$, implying that W_λ has at most volume order fluctuations, as seen in the proof of Lemma 4.6 below. Notice that for λ large we have $R^\xi(x, \Xi^z \cap Q_\lambda) \leq R^\xi(x, \Xi^z)$ and thus (1.10) holds with $t(\Xi, s)$ replaced by

$$\limsup_{\lambda \to \infty} \sup_{x \in Q_\lambda} \sup_{z \in \mathbb{R}^d \cup \{\emptyset\}} \mathbb{P}[R^\xi(x, \Xi^z \cap Q_\lambda) > s | \Xi\{x\} = 1]. \tag{1.11}$$

The above definitions and terminology are part of the literature and are included for the convenience of the reader. We now give two new definitions which, to our knowledge, are new. As we shall see, they are central to showing volume order fluctuations for W_λ. For a set $E \subset \mathbb{R}^d$, let $\mathrm{Vol}_d(E)$ denote the d-dimensional volume of E. For $u \in (0, \infty)$, we let $Q_u \subset \mathbb{R}^d$ be the cube centered at the origin having $\mathrm{Vol}_d(Q_u) = u$.

(vi) **Non-degeneracy with respect to $\mathcal{P}^{\beta\Psi}$.** Say that ξ satisfies non-degeneracy with respect to $\mathcal{P}^{\beta\Psi}$ if there exists $r \in (0, \infty)$ and $b_0 := b_0(r) \in (0, \infty)$ such that given $\mathcal{P}^{\beta\Psi} \cap Q_r^c$, the sum $\sum_{x \in \mathcal{P}^{\beta\Psi} \cap Q_t} \xi(x, \mathcal{P}^{\beta\Psi})$ has expected variability bounded below by b_0, uniformly in $t \in [r, \infty)$. In other words, we have

$$\inf_{t \in [r, \infty)} \mathbb{E} \operatorname{Var}[\sum_{x \in \mathcal{P}^{\beta\Psi} \cap Q_t} \xi(x, \mathcal{P}^{\beta\Psi}) | \; \mathcal{P}^{\beta\Psi} \cap Q_r^c] \geq b_0. \tag{1.12}$$

As shown in Section 2, functionals of interest often satisfy (1.12). In fact, apart from scores ξ which are identically zero on $\mathcal{P}^{\beta\Psi}$, it is difficult to find examples where (1.12) necessarily fails to hold. There is nothing special about using cubes Q_r in (1.12) and, as can be seen from the proofs, Q_r could be replaced by any compact convex subset of $\mathbb{R}^d$.

If f and g are two functions satisfying $\liminf_{\lambda\to\infty} f(\lambda)/g(\lambda) > 0$ then we write $f(\lambda) = \Omega(g(\lambda))$. If, in addition we have $f(\lambda) = O(g(\lambda))$ then we write $f(\lambda) = \Theta(g(\lambda))$.

From the standpoint of applications, it is useful to have a version of (1.12) for score functions which are not translation invariant and for input

$$\tilde{\mathcal{P}}_\lambda^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap \tilde{S}_\lambda, \tag{1.13}$$

where $\tilde{S}_\lambda \subset \mathbb{R}^d$ satisfies $\mathrm{Vol}_d(\tilde{S}_\lambda) = \Omega(1)$. In all that follows, $\tilde{Q}_u \subset \mathbb{R}^d$ denotes a cube with $\mathrm{Vol}_d(\tilde{Q}_u) = u$, but not necessarily centered at the origin.

(vii) **Non-degeneracy with respect to $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$.** Say that ξ satisfies non-degeneracy with respect to $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ if there is $r \in (0,\infty)$ and $b_0 := b_0(r) \in (0,\infty)$, such that for λ large there is $\tilde{Q}_r \subset \tilde{S}_\lambda$ satisfying

$$\mathbb{E}\,\mathrm{Var}[\sum_{x\in\tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi}) |\ \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{Q}_r^c] \geq b_0. \tag{1.14}$$

Given $\rho \in (r,\infty)$, let $\mathcal{C}(\rho, r, \tilde{S}_\lambda)$ be a maximal collection of d-dimensional volume r cubes $\tilde{Q}_{i,r}, 1 \leq i \leq n(\rho, r, \tilde{S}_\lambda)$, which are separated by 4ρ and which satisfy (1.14).

For all x and y in $\mathbb{R}^d$, as in [38], we put

$$c^\xi(x) := \mathbb{E}\,\xi(x, \mathcal{P}^{\beta\Psi})\exp(-\beta\Delta(x, \mathcal{P}^{\beta\Psi})), \tag{1.15}$$

and

$$c^\xi(x,y) := c^\xi(x)c^\xi(y) - \mathbb{E}\,\xi(x, \mathcal{P}^{\beta\Psi}\cup\{y\})\xi(y, \mathcal{P}^{\beta\Psi}\cup\{x\})\cdot\exp(-\beta\Delta(\{x,y\}, \mathcal{P}^{\beta\Psi})). \tag{1.16}$$

Put

$$\sigma^2(\xi,\tau) := c^{\xi^2}(\mathbf{0}) - \tau\int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y)dy. \tag{1.17}$$

1.2 Main results

The following are our main results. Applications follow in Section 2. Our first result gives conditions under which the Gibbsian functional W_λ defined at (1.2) has volume order fluctuations.

Theorem 1.1 *Assume that ξ is translation invariant, exponentially stabilizing in the wide sense* (1.10)*, and satisfies the q-moment condition* (1.7) *for some $q \in (2,\infty)$. Then*

$$\lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Var}W_\lambda = \tau\sigma^2(\xi,\tau) \in [0,\infty). \tag{1.18}$$

If, in addition, ξ satisfies non-degeneracy (1.12)*, then $\sigma^2(\xi,\tau) > 0$.*

Recall that the Kolmogorov distance between the distributions of random variables X_1 and X_2 is defined as

$$d_K(X_1, X_2) := \sup_{t\in\mathbb{R}} |\mathbb{P}[X_1 \le t] - \mathbb{P}[X_2 \le t]|.$$

Theorem 1.2 *Assume that ξ is exponentially stabilizing in the wide sense* (1.10) *and satisfies the q-moment condition* (1.7) *for some $q \in (2,\infty)$. For all $p \in (2,q)$, put $p_3 := p_3(p) := \min\{p, 3\}$. Then*

$$d_K\left(\frac{W_\lambda - \mathbb{E}\,W_\lambda}{\sqrt{\mathrm{Var} W_\lambda}}, N(0,1)\right) = O((\ln\lambda)^{d(p_3-1)}\lambda(\mathrm{Var} W_\lambda)^{-p_3/2}). \tag{1.19}$$

Furthermore, if ξ is translation invariant, satisfies non-degeneracy (1.12) *and the q-moment condition* (1.7) *for some $q \in (3,\infty)$, then*

$$d_K\left(\frac{W_\lambda - \mathbb{E}\,W_\lambda}{\sqrt{\mathrm{Var} W_\lambda}}, N(0,1)\right) = O((\ln\lambda)^{2d}\lambda^{-1/2}) \tag{1.20}$$

and therefore as $\lambda \to \infty$

$$\lambda^{-1/2}(W_\lambda - \mathbb{E}\,W_\lambda) \xrightarrow{\mathcal{D}} N(0, \tau\sigma^2(\xi,\tau)).$$

Remarks. (i) (Theorem 1.1.) The proof of volume order variance asymptotics is indirect. We first show that $\mathrm{Var} W_\lambda$ is of volume order up to a logarithmic term (Lemma 4.3). Putting $\hat{W}_\lambda := \sum_{x\in\mathcal{P}^{\beta\Psi}_\lambda} \xi(x, \mathcal{P}^{\beta\Psi}\backslash\{x\})$ we then show in Lemma 4.6 the dichotomy that either $\mathrm{Var}\hat{W}_\lambda = \Omega(\lambda)$ or $\mathrm{Var}\hat{W}_\lambda = O(\lambda^{(d-1)/d})$. Closeness of $\mathrm{Var} W_\lambda$ and $\mathrm{Var}\hat{W}_\lambda$, as shown in Lemma 4.5, completes the argument, whose full details are in Section 3. Under condition (1.12) we obtain volume order variance asymptotics when $\mathcal{P}^{\beta\Psi}$ is replaced by a homogeneous Poisson point process, which is of independent interest. Verifying condition (1.12) for Gibbsian input is comparable to verifying the non-degeneracy conditions of Theorem 2.1 of [32] or Theorem 1.2 of [16]

(ii) (Theorem 1.2.) Theorem 2.3 of [38] shows the rate of convergence $O((\ln\lambda)^{3d}\lambda^{-1/2})$ in (1.20). However this result assumes that $\mathrm{Var} W_\lambda = \Theta(\lambda)$, which may not always hold, particularly when the scaling is not volume order. Theorem 1.2 contains no such assumption and also provides a slight improvement in the rate. Theorem 1.2 extends Corollary 3.1 of [5] to Gibbsian input. We do not take up the question of laws of large numbers for W_λ as this is addressed in [38].

(iii) (Point processes with marks.) Let $(\mathcal{E}, \mathcal{F}_{\mathcal{E}}, \mu_{\mathcal{E}})$ be a probability space (the mark space) and consider the marked reference Poisson point process $\{(x,a); x \in \tilde{\mathcal{P}}_\tau, a \in \mathcal{E}\}$ in the space $\mathbb{R}^d \times \mathcal{E}$ such that given a realisation $\{x_i\}$ of $\tilde{\mathcal{P}}_\tau$, their marks $\{a_i\}$ are independent and follow the distribution $\mu_{\mathcal{E}}$ [14, Definition 6.4.III.]. Then the proofs of Theorems 1.1 and 1.2 go through in this setting, where it is understood that in the algorithmic construction the process $\mathcal{P}_\lambda^{\beta\Psi}$ inherits the marks from $\tilde{\mathcal{P}}_\tau$ and the interactions in $\mathcal{P}_\lambda^{\beta\Psi}$ depend only on the spatial positions and not on the marks. Here the cubes Q_r in condition (1.12) are replaced with cylinders $C_r := Q_r \times \mathcal{E}$. This generalization is used in Section 2.5 to deduce central limit theorems for spatial birth-growth models with Gibbsian input.

Next we consider the analog of W_λ on input $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ defined at (1.13), namely

$$\tilde{W}_\lambda := \sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \setminus \{x\}).$$

Say that ξ satisfies the q-moment condition with respect to $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ if

$$\sup_{\lambda \in [1,\infty)} \sup_{x \in \tilde{S}_\lambda} \|\xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cup \{x\})\|_q < \infty. \tag{1.21}$$

The following result does not assume that ξ is translation invariant. Recall the definition of $n(\rho, r, \tilde{S}_\lambda)$ defined after (1.14).

Theorem 1.3 *Assume that ξ is exponentially stabilizing in the wide sense* (1.10) *and satisfies the q-moment condition* (1.21) *for some $q \in (2,\infty)$. For all $p \in (2,q)$, put $p_3 := p_3(p) := \min\{p,3\}$. Then*

$$d_K\left(\frac{\tilde{W}_\lambda - \mathbb{E}\,\tilde{W}_\lambda}{\sqrt{\mathrm{Var}\tilde{W}_\lambda}}, N(0,1)\right) = O\left((\ln\lambda)^{d(p_3-1)}\mathrm{Vol}_d(\tilde{S}_\lambda)(\mathrm{Var}\tilde{W}_\lambda)^{-p_3/2}\right). \tag{1.22}$$

Furthermore, if ξ satisfies non-degeneracy (1.14) *and $\rho \in (c\ln\lambda, \infty)$, c large, then*

$$\mathrm{Var}\tilde{W}_\lambda \geq c^{-1} b_0 n(\rho, r, \tilde{S}_\lambda). \tag{1.23}$$

If $q \in (3,\infty)$ we thus have

$$d_K\left(\frac{\tilde{W}_\lambda - \mathbb{E}\,\tilde{W}_\lambda}{\sqrt{\mathrm{Var}\tilde{W}_\lambda}}, N(0,1)\right) = O\left((\ln\lambda)^{2d}\mathrm{Vol}_d(\tilde{S}_\lambda) n(\rho, r, \tilde{S}_\lambda)^{-3/2}\right). \tag{1.24}$$

Remark. The bound (1.23) shows volume order growth for $\mathrm{Var}\tilde{W}_\lambda$, but only up to the logarithmic factor $(\ln\lambda)^d$. When ξ is translation invariant we are able to remove this factor, as described in Remark (i) following Theorem 1.2. However for non-translation invariant ξ, we are unable to remove the logarithmic factor. Consequently, the bound (1.20) is smaller than the bound (1.24) by a factor $(\ln\lambda)^{3d/2}$.

2 Applications

We deduce variance asymptotics and central limit theorems for six well-studied functionals in geometric probability. Save for some special cases as noted below, the limit theory for these functionals has, up to now, been largely confined to Poisson or binomial input. Our examples are not exhaustive. For example, there is scope for treating the limit theory of coverage processes on Gibbsian input, and, more generally, the limit theory of functionals of germ-grain models, with germs given by the realization of $\mathcal{P}^{\beta\Psi}$. One could also treat the limit theory of functionals arising in percolation and nucleation models having Gibbsian input, extending [23] and [20], respectively.

We first state three lemmas needed in showing the non-degeneracy condition (1.12). The proofs are deferred to section 6.

Lemma 2.1 *For every Borel set $B \subset \mathbb{R}^d$ with $\mathrm{Vol}_d(B) < \infty$, and for any $A \in \mathcal{G}_{B^c} := \sigma(\mathcal{P}^{\beta\Psi} \cap B^c)$, we have*

$$\mathbb{P}\left[\mathcal{P}^{\beta\Psi} \cap B = \emptyset \middle| A\right] \geq e^{-\tau \mathrm{Vol}_d(B)}.$$

Lemma 2.2 *Let $m \geq 1$ be a finite integer, $F_1, \ldots, F_m$ be bounded disjoint Borel subsets of $\mathbb{R}^d$ and $F = \cup_{i=1}^m F_i$. For integers $0 \leq l_i \leq k_i$, $1 \leq i \leq m$, we have*

$$\begin{aligned} &\mathbb{P}\left[\bigcap_{i=1}^m \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = l_i\right\} \middle| \sigma(\mathcal{P}^{\beta\Psi} \cap F^c)\right] \\ &\geq e^{-\tau \mathrm{Vol}_d(F)} \mathbb{P}\left[\bigcap_{i=1}^m \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = k_i\right\} \middle| \sigma(\mathcal{P}^{\beta\Psi} \cap F^c)\right] \quad a.s. \end{aligned} \tag{2.1}$$

Lemma 2.3 *For a random variable Y and Borel sets A_1 and A_2 of $\mathbb{R}$, let $d(A_1, A_2) := \inf\{|x_1 - x_2| : x_1 \in A_1, x_2 \in A_2\}$. If $\mathbb{P}[Y \in A_i] \geq p_i$, $i = 1, 2$, then*

$$\mathrm{Var}[Y] \geq \frac{1}{4} d(A_1, A_2)^2 (p_1 \wedge p_2).$$

2.1 Clique counts in random geometric graphs

Let $\mathcal{X} \subset \mathbb{R}^d$ be locally finite and put $s \in (0, \infty)$. The geometric graph on $\mathcal{X}$, here denoted $GG_s(\mathcal{X})$, is obtained by connecting points $x, y \in \mathcal{X}$ with an edge whenever $|x - y| \leq s$. If there is a subset $S := S(s, k)$ of $\mathcal{X}$ of size $k+1$ with all points of S within a distance s of each other, then the k simplex formed by S has edges in $GG_s(\mathcal{X})$. The Vietoris-Rips complex $\mathcal{R}^s(\mathcal{X})$, or Rips complex, is the simplicial complex arising as the union of of all k-simplices $S(s, k) \subset GG_s(\mathcal{X})$. The Vietoris-Rips complex and the closely related Cech complex (which has a simplex for every finite subset of balls in $GG_s(\mathcal{X})$ with non-empty intersection) are used to model the topology of ad hoc sensor and wireless networks and they are also useful in the statistical analysis of high-dimensional data sets. Note that $\mathcal{C}_k^s(\mathcal{X})$ is the number of cliques of order $k+1$ in $GG_s(\mathcal{X})$. For $\mathcal{X}$ random, the number $\mathcal{C}_k^s(\mathcal{X})$ of k-simplices in $GG_s(\mathcal{X})$ is of theoretical and applied interest (see e.g. [29]). The limit theory for $\mathcal{C}_k^s(\mathcal{X})$ is well understood when $\mathcal{X}$ is Poisson or binomial input on $\mathbb{R}^d$ [29] or on a manifold [36]. We are unaware of limit theory for $\mathcal{C}_k^s(\cdot)$ on Gibbsian input. For all $k = 1, 2, ...$ and all $s \in (0, \infty)$ let $\xi_k(x, \mathcal{X}) := \xi_k^{(s)}(x, \mathcal{X})$ be $(k+1)^{-1}$ times the number of k-simplices in $\mathcal{R}^s(\mathcal{X})$ containing the vertex x.

Theorem 2.1 *For all $k = 1, 2, ...$ and all $s > 0$ satisfying $\mathbb{P}[\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) \geq k+1] > 0$, we have*

$$\lim_{\lambda \to \infty} \lambda^{-1} \mathrm{Var}[\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi})] = \tau\sigma^2(\xi_k, \tau) > 0,$$

and

$$d_K\left(\frac{\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi}) - \mathbb{E}\,\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi})}{\sqrt{\mathrm{Var}[\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi})]}}, N(0,1)\right) = O((\ln \lambda)^{2d} \lambda^{-1/2}).$$

Remark. If $\mathcal{P}^{\beta\Psi}$ is the hard-core Gibbs point process and s is bounded by the hard-core radius, then $\xi_k \equiv 0$ and non-degeneracy (1.12) as well as the claim in Theorem 2.1 clearly fail. Notice that if s is large enough we always have $\mathbb{P}[\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) \geq k+1] > 0$, since otherwise $\mathcal{P}^{\beta\Psi}$ would be degenerate.

Proof of Theorem 2.1. We have $\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi}) = \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi_k(x, \mathcal{P}_\lambda^{\beta\Psi})$. It suffices to show that ξ_k satisfies the conditions of Theorems 1.1 and 1.2. Given $x \in \mathbb{R}^d$ and $k = 1, 2, ...$ we note that $\xi_k(x, \mathcal{P}_\lambda^{\beta\Psi})$ is generously bounded by $\left(\sum_{X_i \in \mathcal{P}_\lambda^{\beta\Psi}} \mathbf{1}(|x - X_i| \leq s)\right)^k$, as $\mathcal{P}_\lambda^{\beta\Psi}$ is stochastically dominated by $\tilde{\mathcal{P}}_\tau$, this is in turn bounded by the kth power of a Poisson random variable with parameter $\tau \mathrm{Vol}_d(B_s(x))$. Since all moments of Poisson random variables are finite, it follows that ξ_k satisfies the moment condition (1.7) for all $q \in (1, \infty)$. Clearly ξ_k is translation invariant and exponentially stabilizing with stabilization radius equal to s. It remains to show that ξ_k satisfies non-degeneracy (1.12). With s fixed, put $r := (3s)^d$. Let E' be the event that $\mathcal{P}_\lambda^{\beta\Psi}$ puts no points in $Q_r \setminus Q_{s^d}$, E_1 be the event that $\mathcal{P}_\lambda^{\beta\Psi}$ puts $k+1$ points in Q_{s^d} and E_2 be the event that $\mathcal{P}_\lambda^{\beta\Psi}$ puts no points in Q_{s^d}. On the event $E' \cap E_1$ (resp. $E' \cap E_2$) we have $V_r := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap Q_r} \xi_k^{(s)}(x, \mathcal{P}_\lambda^{\beta\Psi}) = 1$ (resp. 0). We apply the conditional analog of Lemma 2.3 with A_i as the range of $V_r(E' \cap E_i)$, $i = 1, 2$, to obtain

$$\mathrm{Var}[V_r |\, \mathcal{P}^{\beta\Psi} \cap Q_r^c] \geq \frac{1}{4}(\mathbb{P}[E' \cap E_1 |\, \mathcal{P}^{\beta\Psi} \cap Q_r^c] \wedge \mathbb{P}[E' \cap E_2 |\, \mathcal{P}^{\beta\Psi} \cap Q_r^c]). \tag{2.2}$$

Since $\mathbb{P}[\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) \geq k+1] > 0$, there exists a $k_1 \geq k+1$ such that $\mathbb{P}[\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) = k_1] > 0$. Using Lemma 2.2 with $m = 2$, k_1 as above, $l_1 = k+1$ for $E' \cap E_1$ and

$l_1 = 0$ for $E' \cap E_2$, $F_1 = Q_{s^d}$, $k_2 = l_2 = 0$, $F_2 = Q_r \setminus Q_{s^d}$, we have

$$\mathbb{P}[E' \cap E_i | \mathcal{P}^{\beta\Psi} \cap Q_r^c] \geq e^{-\tau r} \mathbb{P}[E' \cap \{\text{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) = k_1\} | \mathcal{P}^{\beta\Psi} \cap Q_r^c], \ i = 1, \ 2,$$

which implies

$$\begin{aligned}
&\mathbb{E}\,\text{Var}[V_r | \mathcal{P}^{\beta\Psi} \cap Q_r^c] \\
&\geq \frac{1}{4} e^{-\tau r} \mathbb{E}\,\mathbb{P}[E' \cap \{\text{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) = k_1\} | \mathcal{P}^{\beta\Psi} \cap Q_r^c] \\
&= \frac{1}{4} e^{-\tau r} \mathbb{P}[E' \cap \{\text{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) = k_1\}] \\
&= \frac{1}{4} e^{-\tau r} \mathbb{P}[E' | \{\text{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) = k_1\}] \mathbb{P}[\{\text{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) = k_1\}] \\
&> \frac{1}{4} e^{-2\tau r} \mathbb{P}[\{\text{card}(\mathcal{P}^{\beta\Psi} \cap Q_{s^d}) = k_1\}] =: b_0,
\end{aligned}$$

where the last inequality is due to Lemma 2.1. This ensures the non-degeneracy (1.12). □

2.2 Functionals of Euclidean graphs

Many functionals of Euclidean graphs on Gibbsian input satisfy (1.18) and (1.19), as shown in [38]. However [38] left open the question of showing variance lower bounds, which is essential to showing that (1.19) is meaningful. We now redress this and assert that the functionals in [38] satisfy non-degeneracy (1.12), and thus $\sigma^2(\xi, \tau) > 0$. We illustrate this for select functionals in [38], leaving it to the reader to verify this assertion for the remaining functionals, namely those arising in random sequential adsorption, component counts in random geometric graphs, and Gibbsian loss networks.

(i) k-nearest neighbors graph. The k-nearest neighbors (undirected) graph on the vertex set $\mathcal{X}$, denoted $NG(\mathcal{X})$, is the graph obtained by including $\{x, y\}$ as an edge whenever y is one of the k points nearest to x and/or x is one of the k points nearest to y. The k-nearest neighbors (directed) graph on $\mathcal{X}$, denoted $NG'(\mathcal{X})$, is obtained by placing a directed edge between each point and its k-nearest neighbors. In case $\mathcal{X} = \{x\}$ is a singleton, x has no nearest neighbor and the *nearest neighbor* distance for x is set by convention to 0.

Total edge length of k-nearest neighbors graph. Given $x \in \mathbb{R}^d$ and a locally finite point set $\mathcal{X} \subset \mathbb{R}^d$, the *nearest neighbors length functional* $\xi_{NG}(x, \mathcal{X})$ is one half the sum of the edge lengths of edges in $NG(\mathcal{X} \cup \{x\})$ which are incident to x. The total edge length of $NG(\mathcal{P}^{\beta\Psi} \cap Q_\lambda)$ is given by

$$W_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi_{NG}(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}).$$

Theorem 5.2 in [38] shows that W_λ satisfies the rate of convergence to the normal at (1.19). This follows since ξ_{NG} is translation invariant, exponentially stabilizing in the wide sense, and satisfies the moment condition (1.7) for all $q \in (2, \infty)$. However that theorem leaves open the question of variance lower bounds for $\mathrm{Var} W_\lambda$ and thus the rate of convergence is possibly useless. The next result resolves this question and also gives a slightly better bound than that in [38].

Theorem 2.2 *We have* $\lim_{\lambda \to \infty} \lambda^{-1} \mathrm{Var} W_\lambda = \tau \sigma^2(\xi_{NG}, \tau) > 0$ *and*

$$d_K\left(\frac{W_\lambda - \mathbb{E}\, W_\lambda}{\sqrt{\mathrm{Var} W_\lambda}}, N(0,1) \right) = O((\ln \lambda)^{2d} \lambda^{-1/2}).$$

Proof. We show that non-degeneracy (1.12) holds and then apply Theorem 1.1 to deduce the variance asymptotics and we apply (1.20) to deduce the rate of normal approximation. We do this by modifying the proof of Lemma 6.3 of [32]. This goes as follows. Let $C_0 := \kappa^d Q_1$, where Q_1 is the unit cube centered at the origin, and where κ is a constant. The annulus $Q_{(4\kappa)^d} \setminus C_0$ will be called the moat; notice that $Q_{(4\kappa)^d}$ has edge length 4κ. Partition the annulus $Q_{(6\kappa)^d} \setminus Q_{(4\kappa)^d}$ into a finite collection $\mathcal{U}$ of cubes of edge length κ. Now define the following events. Let E' be the event that there are no points of $\mathcal{P}_\lambda^{\beta\Psi}$ in the moat, E_2 be the event that there are exactly $k+1$ points of $\mathcal{P}_\lambda^{\beta\Psi}$ in each of the subcubes in $\mathcal{U}$. Let E_1 be the intersection of E_2 and the event that there is 1 point in C_0; let E_0 be the intersection of E_2 and the event that there are no points in C_0.

Put $F_1 = C_0$, $F_2 = Q_{(4\kappa)^d} \setminus C_0$, $F_3, \ldots, F_m$ enumerating all cubes in $\mathcal{U}$. Since $\mathcal{P}^{\beta\Psi}$ is non-degenerate, when κ is large enough, we have

$$\mathbb{P}\left[\bigcap_{i=1}^m \left\{ \mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) \geq k+1 \right\} \right] > 0,$$

which ensures that there exist $k_i \geq k+1$, $1 \leq i \leq m$, such that

$$\mathbb{P}\left[\bigcap_{i=1}^m \left\{ \mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) = k_i \right\} \right] > 0. \tag{2.3}$$

Define $Q_r := Q_{(6\kappa)^d}$, i.e., put $r = (6\kappa)^d$. Given any configuration $\mathcal{P}^{\beta\Psi} \cap Q_r^c$, then conditional on the event that $E_0 \cap E'$ occurs, the sum

$$V_r := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap Q_r} \xi_{NG}(x, \mathcal{P}_\lambda^{\beta\Psi} \backslash \{x\})$$

is strictly less than the same sum V_r, conditional on the event $E_1 \cap E'$, by at least $1.5k\kappa$. This is because on the event $E_1 \cap E'$ there are k additional edges crossing the moat, each of length at least 1.5κ.

We apply the conditional analog of Lemma 2.3 with A_i as the range of $V_r(E' \cap E_{i-1})$, $i = 1, 2$, to obtain

$$\operatorname{Var}[V_r|\, \mathcal{P}^{\beta\Psi} \cap Q_r^c] \geq \frac{1}{4}(1.5k\kappa)^2 \mathbb{P}[E' \cap E_0|\, \mathcal{P}^{\beta\Psi} \cap Q_r^c) \wedge \mathbb{P}(E' \cap E_1|\, \mathcal{P}^{\beta\Psi} \cap Q_r^c]. \quad (2.4)$$

However, using Lemma 2.2 with above m and F_i's, k_i's, $l_1 = 1$ for $E' \cap E_1$ and $l_1 = 0$ for $E' \cap E_0$, $l_2 = 0$, $l_3 = \cdots = l_m = k + 1$, we have

$$\mathbb{P}[E' \cap E_i|\, \mathcal{P}^{\beta\Psi} \cap Q_r^c] \geq e^{-\tau r} \mathbb{P}\left[E' \cap \bigcap_{i=1}^m \left\{ \operatorname{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) = k_i \right\} \middle|\, \mathcal{P}^{\beta\Psi} \cap Q_r^c \right],\ i = 0,\ 1,$$

which, together with (2.4), gives

$$\begin{aligned}
&\mathbb{E}\operatorname{Var}[V_r|\, \mathcal{P}^{\beta\Psi} \cap Q_r^c] \\
&\geq \frac{1}{4}(1.5k\kappa)^2 e^{-\tau r} \mathbb{E}\,\mathbb{P}\left[E' \cap \bigcap_{i=1}^m \left\{ \operatorname{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) = k_i \right\} \middle|\, \mathcal{P}^{\beta\Psi} \cap Q_r^c \right] \\
&= \frac{1}{4}(1.5k\kappa)^2 e^{-\tau r} \mathbb{P}\left[E' \cap \bigcap_{i=1}^m \left\{ \operatorname{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) = k_i \right\} \right] \\
&= \frac{1}{4}(1.5k\kappa)^2 e^{-\tau r} \mathbb{P}\left[E' \middle| \bigcap_{i=1}^m \left\{ \operatorname{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) = k_i \right\} \right] \mathbb{P}\left[\bigcap_{i=1}^m \left\{ \operatorname{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) = k_i \right\} \right] \\
&> \frac{1}{4}(1.5k\kappa)^2 e^{-2\tau r} \mathbb{P}\left[\bigcap_{i=1}^m \left\{ \operatorname{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) = k_i \right\} \right] =: b_0,
\end{aligned}$$

where the last inequality is from Lemma 2.1. This gives the non-degeneracy (1.12). □

(ii) Gibbs-Voronoi tessellations. Given $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, the set of points in $\mathbb{R}^d$ closer to x than to any other point of $\mathcal{X}$ is the interior of a possibly unbounded convex polyhedral cell $C(x, \mathcal{X})$. The Voronoi tessellation induced by $\mathcal{X}$ is the collection of cells $C(x, \mathcal{X}), x \in \mathcal{X}$. When $\mathcal{X}$ is the realization of the Poisson point set $\mathcal{P}_\tau$, this generates the Poisson-Voronoi tessellation of $\mathbb{R}^d$. Here, given the Gibbs point process $\mathcal{P}^{\beta\Psi}$, consider the Voronoi tessellation of this process, sometimes called the Ord process [27].

Total edge length of Gibbs-Voronoi tessellations. Given $\mathcal{X} \subset \mathbb{R}^2$, let $\xi_{\mathrm{Vor}}(x, \mathcal{X})$ denote one half the total edge length of the *finite* length edges in the cell $C(x, \mathcal{X} \cup \{x\})$ (thus we do not take infinite edges into account). The total edge length of the Voronoi graph on $\mathcal{P}^{\beta\Psi}$ is given by

$$W_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi_{\mathrm{Vor}}(x, \mathcal{P}_\lambda^{\beta\Psi} \backslash \{x\}).$$

It may be shown [38] that ξ_{Vor} is exponentially stabilizing in the wide sense (1.10), that it satisfies the moment condition (1.7) for $q \in (2, \infty)$, and, as in Theorem 5.4 of [38] that W_λ satisfies the rate of convergence to the normal as in (1.19).

However that theorem leaves open the question of variance lower bounds for $\mathrm{Var} W_\lambda$ and thus the rate of convergence is possibly useless. The next result resolves this question and gives a better rate than that in [38].

Theorem 2.3 *We have* $\lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Var} W_\lambda = \tau\sigma^2(\xi_{\mathrm{Vor}}, \tau) > 0$ *and*

$$d_K\left(\frac{W_\lambda - \mathbb{E}\, W_\lambda}{\sqrt{\mathrm{Var} W_\lambda}}, N(0,1)\right) = O((\ln\lambda)^{2d}\lambda^{-1/2}).$$

Proof. We need only show that non-degeneracy (1.12) is satisfied and then apply Theorem 1.1 and (1.20). We do this by modifying the proof of Lemma 8.2 of [32]. This goes as follows.

Consider the construction used in the proof of Theorem 2.2. Let E' be the event that there are no points of $\mathcal{P}_\lambda^{\beta\Psi}$ in the moat, E_2 be the event that there is exactly one point of $\mathcal{P}_\lambda^{\beta\Psi}$ in each of the subcubes in $\mathcal{U}$. Fix ε small ($< 1/100$). Choose points $x_1, x_2, x_3 \in \mathbb{R}^2$ forming an equilateral triangle of side-length 0.5κ, centered at the origin. Let E_0 be the intersection of E_2 and the event that there is exactly one point of $\mathcal{P}_\lambda^{\beta\Psi}$ in each of $B_{\varepsilon\kappa}(x_i)$, and the event that there is no other point in $C_0 \setminus (\cup_{i=1}^3 B_{\varepsilon\kappa}(x_i))$, except for a point z in the ball $B_{\varepsilon\kappa\delta}(0)$, where $\delta \in (0,1)$ will be chosen shortly. Let E_1 be the intersection of E_2, the event that there is exactly one point of $\mathcal{P}_\lambda^{\beta\Psi}$ in each of $B_{\varepsilon\kappa\delta}(\delta x_i)$, and the event that there is no other point in $C_0 \setminus (\cup_{i=1}^3 B_{\delta\varepsilon\kappa}(\delta x_i))$, except for the point z in the ball $B_{\varepsilon\kappa\delta}(0)$.

On the event $E_0 \cap E'$, the presence of z near the origin leads to three edges, namely the edges of a (nearly equilateral) triangular cell T around the origin. It removes the parts of the three edges of the Voronoi graph (on all points except z) which intersect T. The difference between the sum of the lengths of the added edges and the sum of the lengths of the three removed edges exceeds $\alpha\kappa$ for some fixed positive number α (the reason is this: given an equilateral triangle T, and a point P inside it, the sum of the lengths of the three edges joining P to the vertices of T is strictly less than the perimeter of T since the length of each of the three edges is less than the common length of the side of T. If T is nearly equilateral (our case) this is still true).

On the other hand, on the event $E_1 \cap E'$, the presence of z cannot increase the total edge length by more than the total edge length of triangular cell around the origin, and this increase is bounded by a constant δ' multiple of $\delta\kappa$, which is less than $\alpha\kappa$ if δ is small enough. Thus if δ is small enough, the events $E_0 \cap E'$ and $E_1 \cap E'$ give rise to values of $V_r := \sum_{x\in\mathcal{P}_\lambda^{\beta\Psi}\cap Q_r} \xi_{\mathrm{Vor}}(x, \mathcal{P}_\lambda^{\beta\Psi}\setminus\{x\})$ which differ by at least $(\alpha - \delta'\delta)\kappa$.

Using the conditional analog of Lemma 2.3 with A_i as the range of $V_r(E' \cap E_{i-1})$, $i = 1, 2$, gives

$$\mathrm{Var}[V_r |\, \mathcal{P}^{\beta\Psi} \cap Q_r^c] \geq \frac{1}{4}(\alpha - \delta'\delta)^2\kappa^2(\mathbb{P}\left[E' \cap E_0 |\, \mathcal{P}^{\beta\Psi} \cap Q_r^c\right] \wedge \mathbb{P}\left[E' \cap E_1 |\, \mathcal{P}^{\beta\Psi} \cap Q_r^c\right]). \tag{2.5}$$

Put $F_1 = B_{\varepsilon\kappa\delta}(0)$, $l_1 = 1$, $F_2 = B_{\varepsilon\kappa}(x_1)$ (resp. $B_{\varepsilon\kappa\delta}(\delta x_1)$), $l_2 = 1$, $F_3 = B_{\varepsilon\kappa}(x_2)$ (resp. $B_{\varepsilon\kappa\delta}(\delta x_2)$), $l_3 = 1$, $F_4 = B_{\varepsilon\kappa}(x_3)$ (resp. $B_{\varepsilon\kappa\delta}(\delta x_3)$), $l_4 = 1$, $F_5 = C_0 \setminus (\cup_{i=1}^4 F_i)$, $l_5 = 0$, $F_6 = Q_{(4\kappa)^d} \setminus C_0$, $l_6 = 0$, $F_7, \ldots, F_m$ enumerating all cubes in $\mathcal{U}$, and $l_7 = \cdots = l_m = 1$. It's obvious that when $\kappa\delta$ is sufficiently large,

$$\mathbb{P}\left[\bigcap_{i=1}^{m} \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) \geq 1\right\}\right] > 0,$$

which ensures that there exist $k_i \geq 1$, $1 \leq i \leq m$, such that

$$\mathbb{P}\left[\bigcap_{i=1}^{m} \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = k_i\right\}\right] > 0. \tag{2.6}$$

Hence it follows from Lemma 2.2 with above F_i, k_i and l_i that

$$\mathbb{P}\left[E' \cap E_i \middle| \mathcal{P}^{\beta\Psi} \cap Q_r^c\right] \geq e^{-\tau r}\mathbb{P}\left[E' \cap \bigcap_{i=1}^{m} \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = k_i\right\} \middle| \mathcal{P}^{\beta\Psi} \cap Q_r^c\right],\ i = 0, 1,$$

which, together with (2.5), ensures

$$\begin{aligned}
&\mathbb{E}\,\mathrm{Var}[V_r |\, \mathcal{P}^{\beta\Psi} \cap Q_r^c] \\
&\geq \frac{1}{4}(\alpha - \delta'\delta)^2\kappa^2 e^{-\tau r}\mathbb{E}\,\mathbb{P}\left[E' \cap \bigcap_{i=1}^{m} \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = k_i\right\} \middle| \mathcal{P}^{\beta\Psi} \cap Q_r^c\right] \\
&= \frac{1}{4}(\alpha - \delta'\delta)^2\kappa^2 e^{-\tau r}\mathbb{P}\left[E' \cap \bigcap_{i=1}^{m} \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = k_i\right\}\right] \\
&= \frac{1}{4}(\alpha - \delta'\delta)^2\kappa^2 e^{-\tau r}\mathbb{P}\left[E' \middle| \bigcap_{i=1}^{m} \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = k_i\right\}\right] \mathbb{P}\left[\bigcap_{i=1}^{m} \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = k_i\right\}\right] \\
&\geq \frac{1}{4}(\alpha - \delta'\delta)^2\kappa^2 e^{-2\tau r}\mathbb{P}\left[\bigcap_{i=1}^{m} \left\{\mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) = k_i\right\}\right] =: b_0,
\end{aligned}$$

where the last inequality is from Lemma 2.1. Combining with (2.6) yields the non-degeneracy (1.12). □

2.3 Insurance models

The modeling of insurance claims has been of considerable interest in the literature. The thrust of the modeling is to set up a claim process $\{N_t,\ t \geq 0\}$ to record the number and time of claims and a sequence of random variables $\{X_i,\ i \geq 1\}$ representing the claim sizes. The aggregate claim size by time t can then be represented as $S_t = \sum_{i=1}^{N_t} X_i$. Most of the literature assumes that $\{X_i,\ i \geq 1\}$ are independent and identically distributed random variables, and are independent of the claim process $\{N_t,\ t \geq 0\}$

[17]. When $\{N_t,\ t \geq 0\}$ is a Poisson process, the process $\{S_t,\ t \geq 0\}$ becomes a compound Poisson process and is also known as the Cramér–Lundberg model ([17], p. 22). Significant effort has been devoted to generalize the model so that it represents real situations more closely, e.g., making the claim process a more general counting process such as a renewal process, a negative binomial process, or a stationary point process [37]. To address the interdependence of claim sizes, [5] introduces a strictly stationary process $\{Y_t,\ t \geq 0\}$ representing a random environment of the claims and a simple point process H on $[0,T] \times \mathbb{N}$ recording the times and sizes of clusters of claims. The total claim amount X_a for $a = (t,n)$ is assumed to be the sum of n independent and identically distributed random variables with distribution determined by the value of Y_t. Assuming that $\{Y_t\}$ is independent of H and both $\{Y_t\}$ and H are locally dependent with a 'uniform dependence radius h_0' such that for all $0 < t_1 < t_2 < \infty$, $Y|_{[t_1,t_2]}$ is independent of $Y|_{\mathbb{R}^+ \setminus (t_1-h_0,t_2+h_0)}$ and $H|_{[t_1,t_2]\times\mathbb{N}}$ is independent of $H|_{(\mathbb{R}^+ \setminus (t_1-h_0,t_2+h_0))\times\mathbb{N}}$, [5] proves that the aggregate claim size $W_T := \int_{a=(t,n):\ t \leq T} X_a H(da)$, when standardized, can be approximated in distribution by the standard normal with an approximation error of order $O(T^{-1/2})$.

In disastrous events, insurance claims may involve dependence amongst the time, size and environment of the claims. In applications, local dependence with a uniform dependence radius may be violated. In this subsection, we aim to address these issues. To this end, let the time and spatial location of claims of insurances be represented by a Gibbs point process in $\mathbb{R}^+ \times \mathbb{R}^{d-1}$. In practice, we have $d \in \{3,4\}$ and the space is typically restricted to a compact convex set $\mathbb{D} \subset \mathbb{R}^{d-1}$ with $\mathrm{Vol}_{d-1}(\mathbb{D}) > 0$. Consequently, we consider the restriction of $\mathcal{P}^{\beta\Psi}$ to $[0,T] \times \mathbb{D}$, namely we set $\tilde{\mathcal{P}}_T^{\beta\Psi} := \mathcal{P}^{\beta\Psi}|_{[0,T]\times\mathbb{D}}$. Let $\xi((t,\mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi})$ be the value of the claim at $(t,\mathbf{s})$ with $t \in \mathbb{R}^+$ and $\mathbf{s} \in \mathbb{R}^d$. The aggregate claim size in the time interval $[0,T]$ is $\tilde{W}_T := \int_{[0,T]\times\mathbb{D}} \xi((t,\mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi}) \tilde{\mathcal{P}}_T^{\beta\Psi}(dt, d\mathbf{s})$. The proof of the next result makes use of Lemma 4.2 and is thus deferred to Section 4.

Theorem 2.4 *Assume that ξ is exponentially stabilizing in the wide sense* (1.10)*, translation invariant in the time coordinate t, and satisfies the q-moment condition* (1.21) *for some $q \in (3,\infty)$. If there exists an $\epsilon > 0$ such that for all large T there is an interval $I \subset (\epsilon T, (1-\epsilon)T)$ of length $\Theta(1)$, such that the conditional distribution $\tilde{W}_T | \tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0,T] \setminus I) \times \mathbb{D}\}$ is non-degenerate, then*

$$d_K\left(\frac{\tilde{W}_T - \mathbb{E}\,\tilde{W}_T}{\sqrt{\mathrm{Var}\tilde{W}_T}}, N(0,1)\right) = O((\ln T)^{3.5} T^{-1/2}).$$

Corollary 2.1 *Assume that the distribution of $\xi((t,\mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi})$ is determined by the k-nearest neighbors of $(t,\mathbf{s})$ and satisfies the q-moment condition* (1.21) *for some $q \in (3,\infty)$. If there exists an $\epsilon > 0$ such that for all large T there is an interval $I \subset (\epsilon T, (1-$*

$\epsilon)T)$ of length $\Theta(1)$, such that the conditional distribution $\tilde{W}_T | \tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0,T] \setminus I) \times \mathbb{D}\}$ is non-degenerate, then

$$d_K \left(\frac{\tilde{W}_T - \mathbb{E}\,\tilde{W}_T}{\sqrt{\mathrm{Var}\tilde{W}_T}}, N(0,1) \right) = O((\ln T)^{3.5} T^{-1/2}).$$

Proof. Using the argument of Section 2.2 (i), one can easily verify that ξ satisfies all the conditions of Theorem 2.4, hence the conclusion follows. $\square$

2.4 Maximal points of Gibbsian samples

Let $K := [0,\infty)^d$. Given $\mathcal{X} \subset \mathbb{R}^d$ locally finite, $x \in \mathcal{X}$ is called K-maximal, or simply maximal if $(K \oplus x) \cap \mathcal{X} = \{x\}$. A point $x = (x_1, ..., x_d) \in \mathcal{X}$ is maximal if there is no other point $(z_1, ..., z_d) \in \mathcal{X}$ with $z_i \geq x_i$ for all $1 \leq i \leq d$. The maximal layer $m_K(\mathcal{X})$ is the collection of maximal points in $\mathcal{X}$. Let $M_K(\mathcal{X}) := \mathrm{card}(m_K(\mathcal{X}))$.

Consider the region

$$A := \{(v,w) : \ v \in D, 0 \leq w \leq F(v)\}$$

where $F : D \to \mathbb{R}$ has continuous negative partials $F_i, 1 \leq i \leq d-1$, bounded away from zero and negative infinity, $D \subset [0,1]^{d-1}$, and $|F| \leq 1$. We are interested in showing asymptotic normality for $M_K([\lambda^{-1/d}\mathcal{P}_\lambda^{\beta\Psi} \oplus (1/2, ..., 1/2)] \cap A)$, with $\mathcal{P}_\lambda^{\beta\Psi}$ as in (1.5). Maximal points are invariant with respect to scaling and translations and it suffices to prove a central limit theorem for $M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d} A)$.

The asymptotic behavior and central limit theorem for $M_K(\mathcal{X})$ with $\mathcal{X}$ either Poisson or binomial input has been studied in [2, 3, 4, 5, 15, 41]; the next theorem extends these results to Gibbsian input.

Theorem 2.5 *We have*

$$d_K \left(\frac{M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d} A) - \mathbb{E}\, M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d} A)}{\sqrt{\mathrm{Var} M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d} A)}}, N(0,1) \right) = O\left((\ln \lambda)^{(7d-1)/2} \lambda^{-(d-1)/2d} \right).$$

Proof. We shall show this is a consequence of Theorem 1.3 for an appropriate $\tilde{S}_\lambda$. For any subset $E \subset \mathbb{R}^d$ and $\epsilon > 0$ let $E^\epsilon := \{x \in \mathbb{R}^d : \ d(x,E) < \epsilon\}$, where $d(x,E)$ denotes the Euclidean distance between x and the set E. Put $\partial A := \{(v, F(v)) : \ v \in D\}$, $\tilde{S}_\lambda := (\lambda^{1/d} \partial A)^{c \ln \lambda}$ for a suitable constant c and in accordance with (1.13), we set $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap \tilde{S}_\lambda$. We first show that it is enough to prove Theorem 2.5 with $\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d} A$ replaced by the smaller set $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \lambda^{1/d} A$.

Define

$$\zeta(x,\mathcal{X}) := \zeta(x,\mathcal{X};\lambda^{1/d}A) := \begin{cases} 1 & \text{if } ((K\oplus x)\cap\lambda^{1/d}A)\cap(\mathcal{X}\cup\{x\}) = \{x\}, \\ 0 & \text{otherwise.}\end{cases}$$

Given any $L \in [1,\infty)$ we observe that if c is large then the event

$$G := \{M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A) = M_K(\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\lambda^{1/d}A)\}$$

may be bounded below by

$$\begin{aligned}\mathbb{P}[G] \quad &\geq 1 - \mathbb{E}\sum_{x\in\mathcal{P}^{\beta\Psi}\cap(\lambda^{1/d}A-\tilde{S}_\lambda)} \zeta(x,\mathcal{P}^{\beta\Psi}) \\ &\geq 1-\mathbb{E}\int_{\lambda^{1/d}A-\tilde{S}_\lambda}\mathbb{P}[\mathcal{P}^{\beta\Psi}\cap B_{\alpha(x)c\ln\lambda}(\theta_x)=\emptyset | x\in\mathcal{P}^{\beta\Psi}]\mathcal{P}^{\beta\Psi}(dx) \geq 1-\lambda^{-L}.\end{aligned}$$

Here θ_x is the center of the largest ball contained in $(K\oplus x)\cap\lambda^{1/d}A$ and $\alpha(x)$ is a constant depending on x; by the assumptions on the partials F_i, $\alpha(x)$ is bounded away from zero uniformly in x and λ. The last inequality is due to Poisson domination (1.6) along with the fact that the integration domain has volume at most $c\lambda$. Since $\mathcal{P}_\lambda^{\beta\Psi}$ is stochastically dominated by $\tilde{\mathcal{P}}_\tau$, the third moment of $M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A)$ is bounded by $O(\lambda^3)$. This implies that

$$\begin{aligned} 0 &\leq \mathbb{E}\left[M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A)^2 - M_K(\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\lambda^{1/d}A)^2\right] \\ &\leq \mathbb{E}\left[M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A)^2\mathbf{1}(G^c)\right] \\ &\leq \mathbb{E}\left[M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A)^3\right]^{2/3}\mathbb{E}\left[\mathbf{1}(G^c)\right]^{1/3} \\ &\leq O(\lambda^{2-L/3})\end{aligned}$$

and

$$\begin{aligned} 0 &\leq \mathbb{E}\left[M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A) - M_K(\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\lambda^{1/d}A)\right] \\ &\leq \mathbb{E}\left[M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A)\mathbf{1}(G^c)\right] \\ &\leq \mathbb{E}\left[M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A)^3\right]^{1/3}\mathbb{E}\left[\mathbf{1}(G^c)\right]^{2/3} \\ &\leq O(\lambda^{1-2L/3}).\end{aligned}$$

This guarantees that $\mathrm{Var}M_K(\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A)$ and $\mathrm{Var}M_K(\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\lambda^{1/d}A)$ have the same asymptotic behavior and thus it is enough to prove Theorem 2.5 with $\mathcal{P}^{\beta\Psi}\cap\lambda^{1/d}A$ replaced by $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\lambda^{1/d}A$.

Notice that ζ is not translation invariant and that

$$M_K(\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\lambda^{1/d}A) = \sum_{x\in\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\lambda^{1/d}A}\zeta(x,\tilde{\mathcal{P}}_\lambda^{\beta\Psi}).$$

To prove Theorem 2.5, it suffices to show that ζ satisfies exponential stabilization in the wide sense (1.10), the moment condition (1.21), non-degeneracy (1.14), and then apply Theorem 1.3. This goes as follows.

Exponential stabilization in the wide sense (1.10). Given $x \in \tilde{S}_\lambda \cap \lambda^{1/d}A$, let $D_1(x) := D_1(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi})$ be the distance between x and the nearest point in $(K \oplus x) \cap \lambda^{1/d}A \cap \tilde{\mathcal{P}}_\lambda^{\beta\Psi}$, if there is such a point; otherwise we let $D_1(x)$ be the *maximal* distance between x and $(K \oplus x) \cap \lambda^{1/d}\partial A$, denoted here by $D(x)$. By the smoothness assumptions on ∂A, it follows that $(K \oplus x) \cap \lambda^{1/d}A \cap B_t(x)$ has volume at least $c_1 t^d$ for all $t \in [0, D(x)]$. It follows from the Poisson domination of $\mathcal{P}_\lambda^{\beta\Psi}$ that uniformly in $x \in \tilde{S}_\lambda \cap \lambda^{1/d}A$ and $\lambda \in [1, \infty)$

$$\mathbb{P}[D_1(x) > t] \leq \exp(-c_1 t^d), \; 0 \leq t \leq D(x). \tag{2.7}$$

For $t \in (D(x), \infty)$, this inequality holds trivially and so (2.7) holds for all $t \in (0, \infty)$.

Let $R(x) := R(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi}) := D_1(x)$. We claim that $R := R(x)$ is a radius of stabilization for ζ at x. Indeed, if $D_1(x) \in (0, D(x))$, then x is not maximal, and so

$$\zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap B_R(x)) = 0$$

and inserting points $\mathcal{Y}$ outside $B_R(x)$ does not modify the score ζ. If $D_1(x) \in [D(x), \infty)$ then

$$\zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap B_R(x)) = 1.$$

Keeping the realization $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap B_R(x)$ fixed, we notice that inserting points $\mathcal{Y}$ outside $B_R(x)$ does not modify the score ζ, since maximality of x is preserved. Thus $R(x)$ is a radius of stabilization for ζ at x and it decays exponentially fast, as demonstrated above.

Moment condition (1.21). This condition is clearly satisfied since ζ is bounded by one.

Non-degeneracy (1.14). We now show that ζ satisfies non-degeneracy (1.14) for a large number of cubes of volume at least $c_2 r$. We do this for $d = 2$, but the proof extends to higher dimensions.

Fix $r \in [1, \infty)$ with a value to be decided later. Let $\tilde{Q}_r \subset \tilde{S}_\lambda$ be such that $\tilde{Q}_r \cap \lambda^{1/d}\partial A \neq \emptyset$. We also assume that $\lambda^{1/d}A$ contains only the lower left corner of $\tilde{Q}_r$, but that $\mathrm{Vol}(\tilde{Q}_r \cap \lambda^{1/d}A) \geq c_3 r$.

Referring to Figure 1, we consider the event E_1 that $\mathrm{card}(\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \mathrm{S}_1) = \mathrm{card}(\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \mathrm{S}_2) = 1$ and $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap S_3 = \emptyset$, where S_1, S_2 and S_3 are the congruent squares in Figure 1. Let E be the event that $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ puts no points in $\tilde{Q}_r \backslash (S_1 \cup S_2 \cup S_3)$. On $E \cap E_1$ we have that

$$V_r := \sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{Q}_r} \zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi})$$

contributes a value of 2 to the total sum $\sum_{x\in\tilde{\mathcal{P}}_\lambda^{\beta\Psi}}\zeta(x,\tilde{\mathcal{P}}_\lambda^{\beta\Psi})$. Let E_2 be the event that $\text{card}(\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap \mathrm{S}_1)=\text{card}(\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap \mathrm{S}_2)=\text{card}(\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap \mathrm{S}_3)=1$. On $E\cap E_2$ we have that V_r contributes a value of 3 to the total sum $\sum_{x\in\tilde{\mathcal{P}}_\lambda^{\beta\Psi}}\zeta(x,\tilde{\mathcal{P}}_\lambda^{\beta\Psi})$.

Using the notation of Lemma 2.2, put $m=4$, $F_i=S_i$, $k_i=1$, $1\le i\le 3$, $F_4=\tilde{Q}_r\setminus(F_1\cup F_2\cup F_3)$. When r is large there is positive probability that each of the six squares in Figure 1 contains one or more points, even if the input arises from the hard-core model. It follows that for large r

$$\mathbb{P}\left[\bigcap_{i=1}^{4}\left\{\text{card}\left(\mathcal{P}^{\beta\Psi}\cap F_i\right)\ge 1\right\}\right]>0.$$

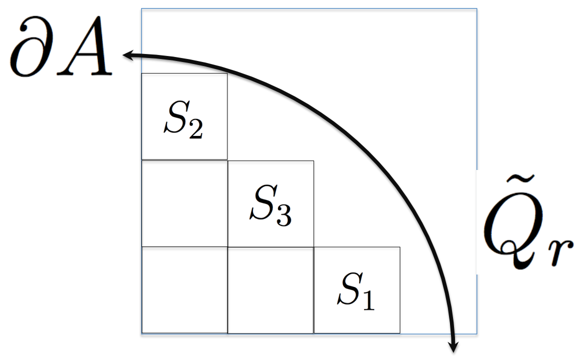

Figure 1: The square $\tilde{Q}_r$ and the subsquares S_1,S_2,S_3

This implies that there exist $k_i\ge 1$, $1\le i\le m$, such that

$$\mathbb{P}\left[\bigcap_{i=1}^{4}\left\{\text{card}\left(\mathcal{P}^{\beta\Psi}\cap F_i\right)=k_i\right\}\right]>0. \tag{2.8}$$

Using the conditional analog of Lemma 2.3 with A_i as the range of $V_r(E\cap E_i)$, $i=1,2$, we obtain

$$\text{Var}[V_r|\ \mathcal{P}^{\beta\Psi}\cap Q_r^c]\ge\frac{1}{4}(\mathbb{P}[E\cap E_1|\ \mathcal{P}^{\beta\Psi}\cap Q_r^c)\wedge\mathbb{P}(E\cap E_2|\ \mathcal{P}^{\beta\Psi}\cap Q_r^c]). \tag{2.9}$$

Now apply Lemma 2.2 with m,F_i, and k_i as above, $l_1=l_2=1$, and $l_4=0$. Put $l_3=0$ for $E\cap E_1$ and $l_3=1$ for $E\cap E_2$. This yields

$$\mathbb{P}[E\cap E_i|\ \mathcal{P}^{\beta\Psi}\cap Q_r^c]\ge e^{-\tau r}\mathbb{P}\left[E\cap\bigcap_{i=1}^{m}\left\{\text{card}\left(\mathcal{P}^{\beta\Psi}\cap F_i\right)=k_i\right\}\middle|\ \mathcal{P}^{\beta\Psi}\cap Q_r^c\right],\ i=1,\ 2.$$

Combining with (2.9) and applying Lemma 2.1 for the second inequality give

$$\begin{aligned}
&\mathbb{E}\,\text{Var}[V_r|\ \mathcal{P}^{\beta\Psi}\cap Q_r^c]\\
&\ge\frac{1}{4}e^{-\tau r}\mathbb{E}\,\mathbb{P}\left[E\cap\bigcap_{i=1}^{4}\left\{\text{card}\left(\mathcal{P}^{\beta\Psi}\cap F_i\right)=k_i\right\}\middle|\ \mathcal{P}^{\beta\Psi}\cap Q_r^c\right]\\
&=\frac{1}{4}e^{-\tau r}\mathbb{P}\left[E\cap\bigcap_{i=1}^{4}\left\{\text{card}\left(\mathcal{P}^{\beta\Psi}\cap F_i\right)=k_i\right\}\right]\\
&=\frac{1}{4}e^{-\tau r}\mathbb{P}\left[E\middle|\bigcap_{i=1}^{4}\left\{\text{card}\left(\mathcal{P}^{\beta\Psi}\cap F_i\right)=k_i\right\}\right]\mathbb{P}\left[\bigcap_{i=1}^{4}\left\{\text{card}\left(\mathcal{P}^{\beta\Psi}\cap F_i\right)=k_i\right\}\right]\\
&\ge\frac{1}{4}e^{-2\tau r}\mathbb{P}\left[\bigcap_{i=1}^{4}\left\{\text{card}\left(\mathcal{P}^{\beta\Psi}\cap F_i\right)=k_i\right\}\right]=:b_0.
\end{aligned} \tag{2.10}$$

This and (2.8) ensure that condition (1.14) holds. Since the surface area of $\lambda^{1/d}\partial A$ is $\Theta(\lambda^{(d-1)/d})$, the number of cubes $\tilde{Q}_r$ having these properties is of order $\Theta((\lambda^{1/d}/\ln\lambda)^{d-1})$, whenever $\rho = \Theta(\ln\lambda)$. Thus we have $n(\rho, r, \tilde{S}_\lambda) = \Theta((\lambda^{1/d}/\ln\lambda)^{d-1})$.

Applying Theorem 1.3 we obtain Theorem 2.5. Noting that $\mathrm{Vol}_d(\tilde{S}_\lambda) = \Theta(\lambda^{(d-1)/d}\ln\lambda)$, the bound (1.24) yields the rate of convergence to the normal

$$= O\left((\ln\lambda)^{2d}\lambda^{(d-1)/d}\ln\lambda\;(\lambda^{(d-1)/d}/(\ln\lambda)^{d-1})^{-3/2}\right) = O\left((\ln\lambda)^{7d/2-1/2}\lambda^{-(d-1)/2d}\right),$$

which was to be shown. □

2.5 Spatial birth-growth models

Consider the following spatial birth-growth model on $\mathbb{R}^d$. Seeds appear at random locations $X_i \in \mathbb{R}^d$ at i.i.d. times $T_i,\ i = 1, 2, ...$ according to a spatial-temporal point process $\mathcal{P} := \{(X_i, T_i) \in \mathbb{R}^d \times [0, \infty)\}$. When a seed is born, it has initial radius zero and then forms a cell within $\mathbb{R}^d$ by growing radially in all directions with a constant speed $v > 0$. Whenever one growing cell touches another, it stops growing in that direction. If a seed appears at X_i and if X_i belongs to any of the cells existing at the time T_i, then the seed is discarded. We assume that the law of $X_i, i \geq 1$, is independent of the law of $T_i, i \geq 1$.

Such growth models have received considerable attention with mathematical contributions given in [11, 12, 13, 20, 28]. First and second order characteristics for Johnson-Mehl growth models on homogeneous Poisson points on $\mathbb{R}^d$ are given in [25, 26]. Using the general Theorem 1.2, we may extend many of these results to growth models with Gibbsian input. We illustrate with the following theorem in which $\hat{\mathcal{P}}$ denotes a marked Gibbs point process such that the Gibbsian positions are endowed with identically distributed time marks which are independent of each other and of the positions.

Given a compact subset K' of $\mathbb{R}^d$, let $N(\mathcal{P}; K')$ be the number of seeds accepted in K'. We shall deduce the following result from Remark (iii) following Theorem 1.2. We let $\hat{\mathcal{P}}_\lambda^{\beta\Psi}$ denote the process of marked points $\{(X_i, T_i):\ X_i \in \mathcal{P}_\lambda^{\beta\Psi},\ T_i \in [0, \infty)\}$. Given a marked point set $\mathcal{X} \subset \mathbb{R}^d \times [0, \infty)$, define the score

$$\nu(x, \mathcal{X}) := \begin{cases} 1 & \text{if the seed at } x \text{ is accepted,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.6 *We have* $\lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Var}N(\hat{\mathcal{P}}_\lambda^{\beta\Psi}; Q_\lambda) = \tau\sigma^2(\nu, \tau) > 0$ *and*

$$d_K\left(\frac{N(\hat{\mathcal{P}}_\lambda^{\beta\Psi}; Q_\lambda) - \mathbb{E}\,N(\hat{\mathcal{P}}_\lambda^{\beta\Psi}; Q_\lambda)}{\sqrt{\mathrm{Var}N(\hat{\mathcal{P}}_\lambda^{\beta\Psi}; Q_\lambda)}}, N(0,1)\right) = O\left((\ln\lambda)^{2d}\lambda^{-1/2}\right).$$

Proof. Notice by the definition of ν we have

$$N(\hat{\mathcal{P}}_\lambda^{\beta\Psi}; Q_\lambda) = \sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap Q_\lambda} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\Psi}).$$

Let K denote the downward right circular cone with apex at the origin of $\mathbb{R}^d$. Then

$$\nu(x, \mathcal{X}) = \begin{cases} 1 & \text{if } (K \oplus x) \cap (\mathcal{X} \cup \{x\}) = x, \\ 0 & \text{otherwise.} \end{cases}$$

We now aim to show that ν satisfies all the conditions of Theorem 1.2. Clearly ν is translation invariant in $\mathbb{R}^d$. The moment condition (1.7) is satisfied, since $|\nu| \leq 1$. We claim that ν satisfies exponential stabilization in the wide sense. This however follows from the above proof that ζ is exponentially stabilizing in the wide sense (the proof is easier now because the boundary of A corresponds to the hyperplane $\mathbb{R}^d$).

We claim that non-degeneracy (1.12) holds. But this too follows from simple modifications of the proof of non-degeneracy of ζ. As in Remark (iii) following Theorem 1.2, we need to show that (1.12) holds with the cube Q_r replaced by a space-time cylinder $C_r := [-r^{1/d}, r^{1/d}]^d \times [0, \infty)$. For simplicity of exposition only, we show non-degeneracy for $d = 1$, but the approach extends to all dimensions.

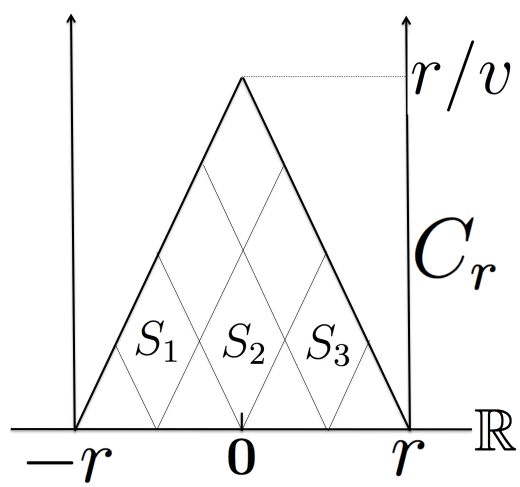

Figure 2: Space-time cylinder C_r

Referring to Figure 2, we consider the event E_1 that $\text{card}(\hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap \mathrm{S}_1) = \text{card}(\hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap \mathrm{S}_2) = 1$ and $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap S_3 = \emptyset$. Let E be the event that $\hat{\mathcal{P}}_\lambda^{\beta\Psi}$ puts no points in $([-r, r] \times [0, r/v]) \setminus (S_1 \cup S_2 \cup S_3)$ (we don't care about the point configuration in the set $[-r, r] \times (r/v, \infty)$). On $E \cap E_1$ we have that

$$V_r := \sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap C_r} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\Psi})$$

contributes a value of 2 to the total sum $\sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap C_t} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\Psi})$. Let E_2 be the event that $\text{card}(\hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap \mathrm{S}_1) = \text{card}(\hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap \mathrm{S}_2) = \text{card}(\hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap \mathrm{S}_3) = 1$. On $E \cap E_2$ we have that V_r contributes a value of 3 to the total sum $\sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap C_t} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\Psi})$.

Referring to Lemma 2.2, we put $m = 4$, $F_i = S_i$, $k_i = 1$, $1 \leq i \leq 3$, $F_4 = ([-r, r] \times [0, r/v]) \setminus (S_1 \cup S_2 \cup S_3)$. When r is large, we notice that $\text{Vol}_d(F_i)$ are large for all $1 \leq i \leq 4$, so

$$\mathbb{P}\left[\bigcap_{i=1}^{4} \left\{\text{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i\right) \geq 1\right\}\right] > 0.$$

Thus there exist $k_i \geq 1$, $1 \leq i \leq 4$, such that (2.8) holds. Repeating the argument from (2.8) to (2.10), we obtain

$$\mathbb{E}\,\mathrm{Var}[V_r|\ \mathcal{P}^{\beta\Psi} \cap Q_r^c] \geq \frac{1}{4} e^{-2\tau r} \mathbb{P}\left(\bigcap_{i=1}^{4} \left\{ \mathrm{card}\left(\mathcal{P}^{\beta\Psi} \cap F_i \right) = k_i \right\} \right) =: b_0.$$

Hence (2.8) ensures that condition (1.12) holds. Thus, ν satisfies all conditions of Theorem 1.2 and so Theorem 2.6 follows. $\square$

3 Auxiliary results

Before proving our main theorems and to keep our presentation as self-contained as possible, we recall some relevant concepts developed in [38].

(i) **Control of spatial dependencies of Gibbs point processes**. Recall that $\mathcal{P}^{\beta\Psi}$ is an admissible point process, i.e., $\Psi \in \mathbf{\Psi}^*$ and $(\tau, \beta) \in \mathcal{R}^{\Psi}$. As shown in the perfect simulation techniques of [38], the process has spatial dependencies which can be controlled by the size of the so-called *ancestor clans.* The ancestor clans are backwards in time oriented percolation clusters, where two nodes in space time are linked with a directed edge if one is the ancestor of the other. *The acceptance status of a point at x depends on points in the ancestor clan.*

As seen at (3.6) of [38], the perfect simulation is designed to ensure that the ancestor clans have exponentially decaying spatial diameter. In fact, one of the main points is to use perfect simulation on the cube Q_λ to first show the existence of $\mathcal{P}_\lambda^{\beta\Psi}$ for any $\lambda \in [1, \infty)$ and to then extend the definition of $\mathcal{P}_\lambda^{\beta\Psi}$ to a point process on all of $\mathbb{R}^d$, thus arriving at $\mathcal{P}^{\beta\Psi}$. The technical details of the perfect simulation are not relevant here. More relevant is that the ancestor clans give us control on the spatial dependencies of $\mathcal{P}^{\beta\Psi}$; control of these spatial dependencies together with exponential stabilization of score functions allows control of the spatial dependency of $\xi(x, \mathcal{P}_\lambda^{\beta\Psi})$ as seen in the next paragraphs.

More precisely, let $A_B^{\beta\Psi}(t)$ be the ancestor clan in $\mathcal{P}^{\beta\Psi}$ of the set $B \subset \mathbb{R}^d$ at time t, i.e, it is the union of ancestor clans of all points $x \in \mathcal{P}^{\beta\Psi} \cap B$. If B_1 and B_2 are subsets of $\mathbb{R}^d$ such that $A_{B_1}^{\beta\Psi}(t) \cap A_{B_2}^{\beta\Psi}(t) = \emptyset$, then statistics of $\mathcal{P}^{\beta\Psi} \cap B_1$ and $\mathcal{P}^{\beta\Psi} \cap B_2$ are independent random variables. The size of $A_B^{\beta\Psi}(t)$ is controlled as follows. For all $(\tau, \beta) \in \mathcal{R}^{\Psi}$, there is a constant $c := c(\tau, \beta) \in (0, \infty)$ such that for all $t \in (0, \infty)$, $M \in (0, \infty)$, and $B \subset \mathbb{R}^d$ we have

$$\mathbb{P}[\mathrm{diam}(A_B^{\beta\Psi}(t)) \geq M + \mathrm{diam}(B)] \leq c(1 + \mathrm{vol}(B)) \exp(-M/c). \tag{3.1}$$

Let $A_{B,\lambda}^{\beta\Psi}$ be the ancestor clan in $\mathcal{P}_\lambda^{\beta\Psi}$ of the set B. Since $\mathrm{diam}(A_{B,\lambda}^{\beta\Psi}(t)) \leq \mathrm{diam}(A_B^{\beta\Psi}(t))$, the bound (3.1) also holds for $A^{\beta\Psi}_{B,\lambda}$, i.e., for all $\lambda \in [1,\infty)$, $B \subset Q_\lambda$ we have

$$\mathbb{P}[\mathrm{diam}(A^{\beta\Psi}_{B,\lambda}(t)) \geq M + \mathrm{diam}(B)] \leq c(1+\mathrm{vol}(B))\exp(-M/c).$$

Put for all $\rho \in (0,\infty)$

$$d(\rho) := \limsup_{\lambda\to\infty} \sup_{B\subset Q_\lambda,\ \mathrm{diam}(B)\leq \rho/2} \mathbb{P}[\mathrm{diam}(A^{\beta\Psi}_{B,\lambda}) \geq \rho].$$

Then we have

$$d(\rho) \leq c(1+(\rho/2)^d v_d)\exp(-\rho/2c). \tag{3.2}$$

(ii) **Score functions with deterministic range of dependency.** Given the radius of stabilization $R^\xi(x,\mathcal{P}^{\beta\Psi}_\lambda)$, let $D(x,\mathcal{P}^{\beta\Psi}_\lambda)$ be the diameter of the ancestor clan of the stabilization ball $B_{R^\xi(x,\mathcal{P}^{\beta\Psi}_\lambda)}(x)$. For all $\rho \in (0,\infty)$, consider score functions on points having ancestor clan diameter at most ρ:

$$\xi(x,\mathcal{P}^{\beta\Psi}_\lambda\backslash\{x\};\rho) := \xi(x,\mathcal{P}^{\beta\Psi}_\lambda\backslash\{x\})\mathbf{1}(D(x,\mathcal{P}^{\beta\Psi}_\lambda)\leq\rho).$$

We study the following functional, the analog of $W(\rho)$ on page 704 of [5]:

$$W_\lambda(\rho) := \sum_{x\in\mathcal{P}^{\beta\Psi}_\lambda} \xi(x,\mathcal{P}^{\beta\Psi}_\lambda\backslash\{x\};\rho). \tag{3.3}$$

When sets A and B are separated by a Euclidean distance greater than 2ρ, then the random variables $\sum_{x\in\mathcal{P}^{\beta\Psi}_\lambda\cap A}\xi(x,\mathcal{P}^{\beta\Psi}_\lambda\backslash\{x\};\rho)$ and $\sum_{x\in\mathcal{P}^{\beta\Psi}_\lambda\cap B}\xi(x,\mathcal{P}^{\beta\Psi}_\lambda\backslash\{x\};\rho)$ depend on disjoint and hence independent portions of the birth and death process $(\varrho(t))_{t\in\mathbb{R}}$ in the construction of $\mathcal{P}^{\beta\Psi}_\lambda$. We make heavy use of this in the proofs of Theorems 1.2 and 1.3.

It is also useful to consider sums of scores with respect to the global point process $\mathcal{P}^{\beta\Psi}$, namely

$$\hat{W}_\lambda := \sum_{x\in\mathcal{P}^{\beta\Psi}_\lambda}\xi(x,\mathcal{P}^{\beta\Psi}\backslash\{x\});\quad \hat{W}_\lambda(\rho) := \sum_{x\in\mathcal{P}^{\beta\Psi}_\lambda}\xi(x,\mathcal{P}^{\beta\Psi}\backslash\{x\};\rho).$$

(iii) **Wide sense stabilization of ξ on $\mathcal{P}^{\beta\Psi}_\lambda$.** If ξ is a stabilizing functional in the wide sense, then

$$Q(\rho) := \limsup_{\lambda\to\infty}\sup_{x\in Q_\lambda}\mathbb{P}[R^\xi(x,\mathcal{P}^{\beta\Psi}_\lambda) > \rho | \mathcal{P}^{\beta\Psi}_\lambda\{x\}=1] \to 0,$$

as $\rho\to\infty$. If ξ is *exponentially stabilizing* in the wide sense (1.10), then by (1.11) there is a constant $c \in (0,\infty)$ such that

$$Q(\rho) \leq c\exp(-\rho/c). \tag{3.4}$$

Notice that for any $\rho \in (0, \infty)$ we have

$$\begin{aligned} & \mathbb{P}[D(x, \mathcal{P}_\lambda^{\beta\Psi}) \geq \rho | \mathcal{P}_\lambda^{\beta\Psi}\{x\} = 1] \\ \leq \ & \mathbb{P}[D(x, \mathcal{P}_\lambda^{\beta\Psi}) \geq \rho, R^\xi(x, \mathcal{P}_\lambda^{\beta\Psi}) \leq \rho/2 | \mathcal{P}_\lambda^{\beta\Psi}\{x\} = 1] \\ & + \mathbb{P}[R^\xi(x, \mathcal{P}_\lambda^{\beta\Psi}) \geq \rho/2 | \mathcal{P}_\lambda^{\beta\Psi}\{x\} = 1]. \end{aligned}$$

Bounding the first term on the right hand side by (3.2) and the second by (3.4), we obtain whenever $\rho \in [c' \ln \lambda, \infty)$ and c' is large that there is c_1 such that $\mathbb{P}[D(x, \mathcal{P}_\lambda^{\beta\Psi}) \geq \rho | \mathcal{P}_\lambda^{\beta\Psi}\{x\} = 1] \leq c_1 \exp(-\rho/c_1)$ whenever $\rho \in [c' \ln \lambda, \infty)$. Thus, for any $L \in [1, \infty)$, there is c large enough so that if $\rho \in [c \ln \lambda, \infty)$, then

$$\mathbb{P}[\hat{W}_\lambda \neq \hat{W}_\lambda(\rho)] \leq \lambda^{-L} \tag{3.5}$$

and

$$\mathbb{P}[W_\lambda \neq W_\lambda(\rho)] \leq \lambda^{-L}. \tag{3.6}$$

4 Variance and moment bounds

Let r satisfy non-degeneracy (1.12) and let $\rho \in [r, \infty)$. Find a maximal collection of disjoint cubes $Q_{i,r} := Q_{i,r,\rho} \subset Q_\lambda, i \in I$, with $\mathrm{Vol}_d Q_{i,r} = r$, and which are separated by a distance at least 4ρ and which are at least a distance 2ρ from ∂Q_λ. Notice that $n(\rho, Q_\lambda) := \mathrm{card}(I) = \lfloor c'\lambda/\rho^d \rfloor$, c' a constant. Let $\mathcal{F}_i$ be the smallest sigma algebra making the mappings $\mathcal{P}^{\beta\Psi} \mapsto \mathrm{card}(\mathcal{P}^{\beta\Psi} \cap \Theta)$, for all Borel sets $\Theta \subset Q_{i,r}^c$, measurable (see [21], page 12).

Lemma 4.1 *Let $q \in [1, \infty)$. If ξ satisfies the moment condition* (1.7) *for some $q' \in (q, \infty)$ then there are constants $\lambda_0 \in (0, \infty)$ and $c \in (0, \infty)$ such that for all $\lambda \geq \lambda_0$ and $\rho \in [1, \infty)$*

$$\max\{\|W_\lambda\|_q, \|W_\lambda(\rho)\|_q\} \leq c\lambda \tag{4.1}$$

and

$$\sup_{i \in I} \max\{\|\mathbb{E}\,[W_\lambda|\mathcal{F}_i]\|_q, \|\mathbb{E}\,[W_\lambda(\rho)|\mathcal{F}_i]\|_q\} \leq c\lambda.$$

Identical bounds hold if W_λ is replaced by $\hat{W}_\lambda$.

Proof. Fix $q \in [1, \infty)$. We shall only prove $\|W_\lambda\|_q \leq c\lambda$ as the other inequalities follow similarly. Put $N := \mathrm{card}(\mathcal{P}_\lambda^{\beta\Psi})$. Minkowski's inequality gives

$$\|W_\lambda\|_q \leq \sum_{j=0}^{\infty} \| \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \backslash \{x\}) \mathbf{1}(\lambda\tau 2^j \leq N \leq \lambda\tau 2^{j+1}) \|_q$$

$$\leq \sum_{j=0}^{\infty} \| \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi},\ N \leq \lambda\tau 2^{j+1}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \backslash \{x\}) \mathbf{1}(N \geq \lambda\tau 2^j) \|_q.$$

Let $s \in (1, \infty)$ be such that $qs < q'$. Let $1/s + 1/t = 1$, i.e., s and t are conjugate exponents. Hölder's inequality gives

$$\|W_\lambda\|_q \leq \sum_{j=0}^{\infty} \left[\mathbb{E} \, (\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi},\ N \leq \lambda\tau 2^{j+1}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \backslash \{x\}))^{qs} \right]^{1/qs} (\mathbb{P}[N \geq \lambda\tau 2^j])^{1/qt}.$$

Since $\mathcal{P}_\lambda^{\beta\Psi}$ is Poisson-like, we have that N is stochastically dominated by a Poisson random variable $\mathrm{Po}(\lambda\tau)$ with parameter $\lambda\tau$. Recalling the definition of w_q at (1.7), we obtain

$$\|W_\lambda\|_q \leq \sum_{j=0}^{\infty} \| \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}, N \leq \lambda\tau 2^{j+1}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \backslash \{x\}) \|_{qs} (\mathbb{P}[\mathrm{Po}(\lambda\tau) \geq \lambda\tau 2^j])^{1/qt}$$

$$\leq 6\lambda\tau w_{qs} + \sum_{j=2}^{\infty} \lambda\tau 2^{j+1} w_{qs} (\mathbb{P}[\mathrm{Po}(\lambda\tau) - \lambda\tau \geq \lambda\tau(2^j - 1)])^{1/qt},$$

using Minkowski's inequality another time. For $j \geq 2$, we have that $\mathbb{P}[\mathrm{Po}(\lambda\tau) - \lambda\tau \geq \lambda\tau(2^j - 1)]$ decays exponentially fast in 2^j by standard tail probabilities for the Poisson random variable. This shows that the infinite sum is $O(\lambda\tau)$, concluding the proof. □

We put

$$\tilde{W}_\lambda(\rho) := \sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \setminus \{x\}; \rho).$$

Lemma 4.2 *Given a set $G \subset \mathbb{R}^d$ we let $\mathcal{G}_G$ (respectively $\tilde{\mathcal{G}}_G$) be the sigma algebra generated by $\mathcal{P}^{\beta\Psi} \cap G$ (respectively $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap G$). Assume that ξ satisfies condition* (1.10).
(a) If ξ satisfies the moment condition (1.7) *for some $q \in (2, \infty)$, then there exist constants λ_0 and c such that for all $\lambda \in [\lambda_0, \infty)$, $\rho \in [c \ln \lambda, \infty)$ and all Borel sets $G \subset \mathbb{R}^d$,*

$$|\mathbb{E} \operatorname{Var}[\hat{W}_\lambda(\rho)|\mathcal{G}_G] - \mathbb{E} \operatorname{Var}[\hat{W}_\lambda|\mathcal{G}_G]| \leq \lambda^{-1} \tag{4.2}$$

and

$$|\mathbb{E} \operatorname{Var}[W_\lambda(\rho)|\mathcal{G}_G] - \mathbb{E} \operatorname{Var}[W_\lambda|\mathcal{G}_G]| \leq \lambda^{-1}. \tag{4.3}$$

(b) If ξ satisfies the moment condition (1.21) *for some $q \in (2, \infty)$ then there exist constants $\lambda_0 \in (0, \infty)$ and $c \in (0, \infty)$ such that for all $\lambda \in [\lambda_0, \infty)$, $\rho \in [c \ln \lambda, \infty)$ and all Borel sets $G \subset \tilde{S}_\lambda$,*

$$|\mathbb{E} \operatorname{Var}[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{G}}_G] - \mathbb{E} \operatorname{Var}[\tilde{W}_\lambda|\tilde{\mathcal{G}}_G]| \leq \lambda^{-1}.$$

Proof. (a) Using the generic formula $\mathrm{Var}[X|\mathcal{A}] = \mathbb{E}\,[X^2|\mathcal{A}] - (\mathbb{E}\,[X|\mathcal{A}])^2$, valid for any random variable X and sigma algebra $\mathcal{A}$, we have

$$\mathbb{E}\,\mathrm{Var}[\hat{W}_\lambda(\rho)|\mathcal{G}_G] = \mathbb{E}\,\left[\mathbb{E}\,[\hat{W}_\lambda^2(\rho)|\mathcal{G}_G] - (\mathbb{E}\,[\hat{W}_\lambda(\rho)|\mathcal{G}_G)^2\right]$$

and

$$\mathbb{E}\,\mathrm{Var}[\hat{W}_\lambda|\mathcal{G}_G] = \mathbb{E}\,\left[\mathbb{E}\,[\hat{W}_\lambda^2|\mathcal{G}_G] - (\mathbb{E}\,[\hat{W}_\lambda|\mathcal{G}_G)^2\right].$$

If both differences

$$|\mathbb{E}\,[\mathbb{E}\,[\hat{W}_\lambda^2(\rho)|\mathcal{G}_G] - \mathbb{E}\,[\hat{W}_\lambda^2|\mathcal{G}_G]]| \tag{4.4}$$

and

$$|\mathbb{E}\,[\mathbb{E}\,[\hat{W}_\lambda(\rho)|\mathcal{G}_G]^2 - \mathbb{E}\,[\hat{W}_\lambda|\mathcal{G}_G]^2]]| \tag{4.5}$$

are less than $\lambda^{-1}/2$ then $\mathbb{E}\,\mathrm{Var}[\hat{W}_\lambda(\rho)|\mathcal{G}_G]$ differs from $\mathbb{E}\,\mathrm{Var}[\hat{W}_\lambda|\mathcal{G}_G]$ by less than λ^{-1}.

Notice that (4.4) may be bounded by $(2\lambda)^{-1}$ since it equals $\mathbb{E}\,[\hat{W}_\lambda^2(\rho) - \hat{W}_\lambda^2]$, which by Hölder's inequality is bounded by the product of $\|\hat{W}_\lambda^2(\rho) - \hat{W}_\lambda^2\|_{q/2}$ and a power of $\mathbb{P}[\hat{W}_\lambda \neq \hat{W}_\lambda(\rho)]$. The first term is $O(\lambda^2)$ by (4.1) whereas the latter is small by (3.5), the choice of ρ, and the arbitrariness of L.

Likewise (4.5) can be bounded by $\lambda^{-1}/2$ since

$$\begin{aligned}
&|\mathbb{E}\,[\mathbb{E}\,[\hat{W}_\lambda(\rho)|\mathcal{G}_G]^2 - \mathbb{E}\,[\hat{W}_\lambda|\mathcal{G}_G]^2]]| \\
&= |\mathbb{E}\,(\mathbb{E}\,[\hat{W}_\lambda(\rho)|\mathcal{G}_G] + \mathbb{E}\,[\hat{W}_\lambda|\mathcal{G}_G])(\mathbb{E}\,[\hat{W}_\lambda(\rho)|\mathcal{G}_G] - \mathbb{E}\,[\hat{W}_\lambda|\mathcal{G}_G])| \\
&\le C\lambda \|\mathbb{E}\,[\hat{W}_\lambda(\rho)|\mathcal{G}_G] - \mathbb{E}\,[\hat{W}_\lambda|\mathcal{G}_G]\|_2 \\
&\le C\lambda \sqrt{\mathbb{E}\,\left(\mathbb{E}\,\left((\hat{W}_\lambda(\rho) - \hat{W}_\lambda)^2|\mathcal{G}_G\right)\right)} = C\lambda\sqrt{\mathbb{E}\,(\hat{W}_\lambda(\rho) - \hat{W}_\lambda)^2},
\end{aligned}$$

where the first inequality follows by the Cauchy-Schwarz inequality and Lemma 4.1 and where the second inequality follows by the conditional Jensen inequality. Using Hölder's inequality and the bound (3.5), we get that (4.5) is bounded by $\lambda^{-1}/2$, concluding the proof of (4.2). The proofs of (4.3) and part (b) follow the proof of (a) verbatim. □

Proof of Theorem 2.4. We take $\tilde{S}_T := [0, T] \times \mathbb{D}$ in Theorem 1.3 and let r be the length of I. Let $n(\rho, r, \tilde{S}_T)$ be the maximum number of subsets $S_i \subset \tilde{S}_T$ of the form $(I + t_i) \times \mathbb{D}, t_i \in \mathbb{R}^+$, in $\tilde{S}_T$ which are separated by 4ρ with $\rho = \Theta(\ln T)$. Then $\mathrm{Vol}_{d+1}(\tilde{S}_T) = \Theta(T)$ and $n(\rho, r, \tilde{S}_T) = \Theta(T(\ln T)^{-1})$. Let $\tilde{\mathcal{P}}_T^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap \tilde{S}_T$ in accordance with (1.13). We show that (1.14) is satisfied for all S_i, $1 \le i \le n(\rho, r, \tilde{S}_T)$ and then apply Theorem 1.3 to $\tilde{\mathcal{P}}_T^{\beta\Psi}$. Since the conditional distribution $\tilde{W}_T|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0, T] \setminus I) \times \mathbb{D}\}$ is non-degenerate, we have

$$\mathbb{E}\,\mathrm{Var}[\tilde{W}_T|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0, T] \setminus I) \times \mathbb{D}\}] := d_0 > 0.$$

Recalling the definition of D in subsection 3 (ii), for $J \subset [0,T] \times \mathbb{D}$, we define

$$M(J) := \int_J \xi((t,\mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi}) \mathbf{1}(D((t,\mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi}) \le \rho) \tilde{\mathcal{P}}_T^{\beta\Psi}(dt, d\mathbf{s}).$$

Then

$$\begin{aligned}
& \mathbb{E}\,\mathrm{Var}[M(\tilde{S}_T)|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{\tilde{S}_T \backslash S_i\}] \\
= \; & \mathbb{E}\,\mathrm{Var}[M(S_i^{2\rho})|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{\tilde{S}_T \backslash S_i\}] \\
= \; & \mathbb{E}\,\mathrm{Var}[M((I \times \mathbb{D})^{2\rho})|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0,T] \setminus I) \times \mathbb{D}\}] \qquad \text{(by translation invariance of } \xi) \\
= \; & \mathbb{E}\,\mathrm{Var}[M(\tilde{S}_T)|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0,T] \setminus I) \times \mathbb{D}\}] \ge d_0 - O(T^{-1}),
\end{aligned}$$

where the inequality is due to Lemma 4.2(b). Using Lemma 4.2(b) again, we conclude that, for T large,

$$\mathbb{E}\,\mathrm{Var}[\tilde{W}_T|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{\tilde{S}_T \backslash S_i\}] \ge d_0 - O(T^{-1}) =: b_0.$$

All conditions of Theorem 1.3 are satisfied and it follows from (1.24) that

$$d_K\left(\frac{\tilde{W}_T - \mathbb{E}\,\tilde{W}_T}{\sqrt{\mathrm{Var}\tilde{W}_T}}, N(0,1)\right) = O((\ln T)^2 \mathrm{Vol}(\tilde{S}_T) n(\rho, r, \tilde{S}_T)^{-3/2}) = O((\ln T)^{3.5} T^{-1/2}),$$

completing the proof. □

Lemma 4.3 *Assume that ξ is translation invariant and the moment condition* (1.7) *holds for some $q \in (2,\infty)$. Under conditions* (1.10) *and* (1.12) *there exist constants $\lambda_0 \in (0,\infty)$ and $c \in (0,\infty)$ such that for all $\lambda \in [\lambda_0, \infty)$ and all $\rho \in [c \ln \lambda, \infty)$ we have*

$$\mathrm{Var}[W_\lambda(\rho)] \ge c^{-1} b_0 \lambda \rho^{-d}; \;\; \mathrm{Var}[\hat{W}_\lambda(\rho)] \ge c^{-1} b_0 \lambda \rho^{-d}. \tag{4.6}$$

Proof. We only prove the first inequality as the second follows from identical methods. Let $c \ge 2/c'$ such that Lemma 4.2(a) holds, where c' is the constant such that the cardinality of I is $\lfloor c'\lambda/\rho^d \rfloor$. Let $\mathcal{F}$ be the sigma algebra generated by $\mathcal{P}^{\beta\Psi} \cap (\bigcup_{i \in I} Q_{i,r})^c$. By the conditional variance formula

$$\mathrm{Var}[W_\lambda(\rho)] = \mathrm{Var}[\mathbb{E}\,[W_\lambda(\rho)|\mathcal{F}]] + \mathbb{E}\,\mathrm{Var}[W_\lambda(\rho)|\mathcal{F}] \ge \mathbb{E}\,\mathrm{Var}[W_\lambda(\rho)|\mathcal{F}].$$

Let $C_i := \{x \in \mathbb{R}^d : \; d(x, Q_{i,r}) \le \rho\}$. Then the C_i are separated by 2ρ because the $Q_{i,r}$ are separated by at least 4ρ (this is the reason why we chose the 4ρ separation in the first place). Also, the C_i are contained in Q_λ.

For each $i \in I$ the sum $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap C_i} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \backslash \{x\}; \rho)$ depends on points distant at most ρ from C_i. Thus the random variable $\mathbb{E}\,[W_\lambda(\rho)|\mathcal{F}]$ is a sum of conditionally independent random variables since the C_i are separated by 2ρ. Thus we obtain

$$\begin{aligned}\mathbb{E}\,\mathrm{Var}[W_\lambda(\rho)|\mathcal{F}] &= \mathbb{E}\,\mathrm{Var}[\sum_{x\in\mathcal{P}_\lambda^{\beta\Psi}} \xi(x,\mathcal{P}_\lambda^{\beta\Psi}\backslash\{x\};\rho)|\mathcal{F}] \\ &= \mathbb{E}\sum_{i\in I}\mathrm{Var}[\sum_{x\in\mathcal{P}_\lambda^{\beta\Psi}\cap C_i} \xi(x,\mathcal{P}_\lambda^{\beta\Psi}\backslash\{x\};\rho)|\mathcal{F}]. \qquad (4.7)\end{aligned}$$

Recall that $E^\epsilon = \{x \in \mathbb{R}^d : \ d(x,E) < \epsilon\}$ for any set E and $\epsilon > 0$. For all $i \in I$, the restrictions of $\mathcal{F}$ and $\mathcal{F}_i$ to C_i^ρ coincide. For $x \in C_i$, we have that $\xi(x,\mathcal{P}_\lambda^{\beta\Psi};\rho)$ depends only on points in C_i^ρ and so we may thus replace $\mathcal{F}$ with $\mathcal{F}_i$. Since $\mathcal{P}_\lambda^{\beta\Psi}$ and $\mathcal{P}^{\beta\Psi}$ coincide on C_i^ρ we may also replace $\xi(x,\mathcal{P}_\lambda^{\beta\Psi};\rho)$ with $\xi(x,\mathcal{P}^{\beta\Psi};\rho)$. Also, we may replace the range of summation $x \in \mathcal{P}_\lambda^{\beta\Psi} \cap C_i$ by $x \in \mathcal{P}_\lambda^{\beta\Psi}$ because the conditional sum

$$\sum_{x\in\mathcal{P}^{\beta\Psi}\cap C_i^c\cap Q_\lambda} \xi(x,\mathcal{P}^{\beta\Psi}\backslash\{x\};\rho)|\mathcal{F}_i$$

is constant (indeed, if $x \in C_i^c$, then $\xi(x,\mathcal{P}^{\beta\Psi}\backslash x;\rho)$ won't be affected by points in $Q_{i,r}$).

This yields

$$\mathbb{E}\,\mathrm{Var}[W_\lambda(\rho)|\mathcal{F}] = \mathbb{E}\sum_{i\in I}\mathrm{Var}[\sum_{x\in\mathcal{P}_\lambda^{\beta\Psi}} \xi(x,\mathcal{P}^{\beta\Psi}\backslash x;\rho)|\mathcal{F}_i]. \qquad (4.8)$$

By Lemma 4.2(a) for all $i \in I$,

$$\mathbb{E}\,\mathrm{Var}[\sum_{x\in\mathcal{P}_\lambda^{\beta\Psi}} \xi(x,\mathcal{P}^{\beta\Psi}\setminus x;\rho)|\mathcal{F}_i] \geq b_0/2.$$

Thus

$$\mathrm{Var}[W_\lambda(\rho)] \geq \mathbb{E}\,\mathrm{Var}[W_\lambda(\rho)|\mathcal{F}] \geq \mathbb{E}\sum_{i\in I} b_0/2 \geq b_0 c^{-1}\lambda\rho^{-d}.$$

□

Roughly speaking, the factor $\lambda\rho^{-d}$ in (4.6) is the cardinality of I, the index set of cubes of volume r, separated by 4ρ, and having the property that the total score on each cube has positive variability. For score functions which may not be translation invariant and/or are defined on a subset $\tilde{S}_\lambda$ of $\mathbb{R}^d$, we have the following analog of Lemma 4.3. Recall the definition of $n(\rho,r,\tilde{S}_\lambda)$ right after (1.14).

Lemma 4.4 *Assume the moment condition* (1.21) *holds for some* $q \in (2,\infty)$. *Under conditions* (1.10) *and* (1.14) *there exist constants* $\lambda_0 \in (0,\infty)$ *and* $c \in (0,\infty)$ *such that for all* $\lambda \in [\lambda_0,\infty)$ *and all* $\rho \in [c\ln\lambda,\infty)$ *we have*

$$\mathrm{Var}[\tilde{W}_\lambda(\rho)] \geq c^{-1} b_0 n(\rho,r,\tilde{S}_\lambda).$$

Proof. We follow the proof of Lemma 4.3. We write $\{\tilde{Q}_{i,r}: \ i \in \tilde{I}\} := \mathcal{C}(\rho, r, \tilde{S}_\lambda)$, the collection of cubes defined after (1.14). Let $\tilde{\mathcal{F}}_\lambda$ be the sigma algebra generated by $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap (\bigcup_{i\in\tilde{I}} \tilde{Q}_{i,r})^c$. By the conditional variance formula

$$\mathrm{Var}[\tilde{W}_\lambda(\rho)] = \mathrm{Var}[\mathbb{E}\,[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{F}}_\lambda]] + \mathbb{E}\,\mathrm{Var}[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{F}}_\lambda] \geq \mathbb{E}\,\mathrm{Var}[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{F}}_\lambda].$$

For $i \in \tilde{I}$, let $\tilde{C}_i := \{x \in \tilde{S}_\lambda: \ d(x, \tilde{Q}_{i,r}) \leq \rho\}$. Then the $\tilde{C}_i$ are separated by 2ρ because the $\tilde{Q}_{i,r}$ are separated by at least 4ρ. Also, the $\tilde{C}_i$ are contained in $\tilde{S}_\lambda$.

For each $i \in \tilde{I}$ the sum $\sum_{x\in\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\tilde{C}_i} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi}\backslash\{x\}; \rho)$ depends on points distant at most ρ from $\tilde{C}_i$. Thus $\mathbb{E}\,[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{F}}_\lambda]$ is a sum of conditionally independent random variables since the $\tilde{C}_i$ are separated by 2ρ. Thus we obtain the analog of (4.7), namely

$$\mathbb{E}\,\mathrm{Var}[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{F}}_\lambda] = \mathbb{E}\sum_{i\in\tilde{I}} \mathrm{Var}[\sum_{x\in\tilde{\mathcal{P}}_\lambda^{\beta\Psi}\cap\tilde{C}_i} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi}\backslash\{x\}; \rho)|\tilde{\mathcal{F}}_\lambda].$$

Let $\tilde{\mathcal{F}}_{\lambda,i}$ be the sigma algebra generated by $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{Q}_{i,r}$. For all $i \in \tilde{I}$, the restrictions of $\tilde{\mathcal{F}}_\lambda$ and $\tilde{\mathcal{F}}_{\lambda,i}$ to $\tilde{C}_i^\rho \cap \tilde{S}_\lambda$ coincide.

As in the proof of Lemma 4.3, we obtain the analog of (4.8), namely

$$\mathbb{E}\,\mathrm{Var}[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{F}}_\lambda] = \mathbb{E}\sum_{i\in\tilde{I}} \mathrm{Var}[\sum_{x\in\tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}^{\beta\Psi}\backslash\{x\}; \rho)|\tilde{\mathcal{F}}_{\lambda,i}].$$

If $\lambda \in [\lambda_0, \infty)$ and if λ_0 is large enough, then by Lemma 4.2(b) for all $i \in \tilde{I}$,

$$\mathbb{E}\,\mathrm{Var}[\sum_{x\in\tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}^{\beta\Psi}\backslash\{x\}; \rho)|\tilde{\mathcal{F}}_{\lambda,i}] \geq b_0/2.$$

Thus

$$\mathrm{Var}[\tilde{W}_\lambda(\rho)] \geq \mathbb{E}\,\mathrm{Var}[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{F}}_\lambda] \geq \mathbb{E}\sum_{i\in\tilde{I}} b_0/2 \geq b_0 \cdot \mathrm{card}(\tilde{\mathrm{I}})/2.$$

□

Lemma 4.5 *If the moment condition* (1.7) *holds for some* $q \in (2, \infty)$ *then* $|\mathrm{Var}W_\lambda - \mathrm{Var}\hat{W}_\lambda| = o(\lambda)$.

Proof. Put $\rho = c \ln \lambda$, c large. By (4.3) and (4.2) with $G = \emptyset$ we have $|\mathrm{Var}W_\lambda(\rho) - \mathrm{Var}W_\lambda| = o(1)$ and $|\mathrm{Var}\hat{W}_\lambda(\rho) - \mathrm{Var}\hat{W}_\lambda| = o(1)$. So it is enough to prove $|\mathrm{Var}W_\lambda(\rho) - \mathrm{Var}\hat{W}_\lambda(\rho)| = o(\lambda)$. We have

$$|\mathrm{Var}W_\lambda(\rho) - \mathrm{Var}\hat{W}_\lambda(\rho)| \leq \mathrm{Var}(W_\lambda(\rho) - \hat{W}_\lambda(\rho)) + 2\mathrm{cov}(W_\lambda(\rho) - \hat{W}_\lambda(\rho), \hat{W}_\lambda(\rho)).$$

The scores $\xi(x, \mathcal{P}_\lambda^{\beta\Psi}; \rho)$ and $\xi(x, \mathcal{P}^{\beta\Psi}; \rho)$ coincide when $x \in Q_\lambda$ is distant at least ρ from ∂Q_λ. Thus $W_\lambda(\rho) - \hat{W}_\lambda(\rho) = U_\lambda - V_\lambda$, where

$$U_\lambda := \sum_{x\in\mathcal{P}_\lambda^{\beta\Psi}\cap(\partial Q_\lambda)^\rho} \xi(x, \mathcal{P}_\lambda^{\beta\Psi}; \rho); \quad V_\lambda := \sum_{x\in\mathcal{P}^{\beta\Psi}\cap(\partial Q_\lambda)^\rho} \xi(x, \mathcal{P}^{\beta\Psi}; \rho).$$

Lemma 4.1 with $q = 2$ and $q' > 2$ ensures $\mathrm{Var}U_\lambda$ and $\mathrm{Var}V_\lambda$ are both of order $O((\mathrm{Vol}(\partial Q_\lambda)^\rho)^2)$. These bounds and the formula $\mathrm{Var}[U_\lambda - V_\lambda] = \mathrm{Var}U_\lambda + \mathrm{Var}V_\lambda - 2\mathrm{Cov}[U_\lambda, V_\lambda]$ shows that $\mathrm{Var}[U_\lambda - V_\lambda] = o(\lambda)$. By the Cauchy-Schwarz inequality and Lemma 4.1, we obtain $\mathrm{cov}[W_\lambda(\rho) - \hat{W}_\lambda(\rho), \hat{W}_\lambda(\rho)] = o(\lambda)$ as well. □

We need one more lemma. It shows that if fluctuations of $\hat{W}_\lambda$ are not of volume order then they are necessarily at most of surface order and vice versa. A version of this dichotomy appears in the statistical physics literature [24] and also in [7]. We do not have any natural examples of $\hat{W}_\lambda$ which are defined on all of Q_λ and which have fluctuations at most of surface order. However, when ancestor clans and stabilization radii have slowly decaying tails we expect that $\mathrm{Var}\hat{W}_\lambda$ behaves less like a sum of i.i.d. random variables and more like a sum of random variables with very long range dependencies, presumably giving rise to smaller fluctuations. When the score at x is allowed to depend on nearby point configurations as well as on nearby scores, then Martin and Yalcin [24] establish conditions giving surface order fluctuations.

Lemma 4.6 *Let ξ be translation invariant. Either $\mathrm{Var}\hat{W}_\lambda = \Omega(\lambda)$ or $\mathrm{Var}\hat{W}_\lambda = O(\lambda^{(d-1)/d})$.*

Proof. Recall the definitions of $c^\xi(x)$ and $c^\xi(x, y)$ at (1.15) and (1.16), respectively. Similar to the proof of Theorem 2.2 of [38], by the integral characterization of Gibbs point processes, as in Chapter 6.4 of [27], it follows from the Georgii-Nguyen-Zessin formula that

$$\mathrm{Var}\hat{W}_\lambda = \mathrm{Var} \sum_{x \in \mathcal{P}^{\beta\Psi}_\lambda} \xi(x, \mathcal{P}^{\beta\Psi} \setminus \{x\}) = \tau \int_{Q_\lambda} c^{\xi^2}(x) dx - \tau^2 \int_{Q_\lambda} \int_{Q_\lambda} c^\xi(x, y) dy dx.$$

Note that $c^\xi(x, y)$ decays exponentially fast with $|x - y|$, as shown in Lemmas 3.4 and 3.5 of [38]. By translation invariance of ξ and stationarity of $\mathcal{P}^{\beta\Psi}$ we get

$$\mathrm{Var}\hat{W}_\lambda = \tau c^{\xi^2}(\mathbf{0})\lambda - \tau^2 \int_{Q_\lambda} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y - x) \mathbf{1}(y \in Q_\lambda) dy dx \tag{4.9}$$

$$= \tau c^{\xi^2}(\mathbf{0})\lambda - \tau^2 \int_{Q_\lambda} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) \mathbf{1}(x + y \in Q_\lambda) dy dx := I_\lambda + II_\lambda.$$

Now

$$\lambda^{-1} II_\lambda = -\tau^2 \lambda^{-1} \int_{Q_\lambda} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) \mathbf{1}(x \in Q_\lambda - y) dy dx$$

and writing $\mathbf{1}(x \in Q_\lambda - y)$ as $1 - \mathbf{1}(x \in (Q_\lambda - y)^c)$ gives

$$\lambda^{-1} II_\lambda = -\tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) dy + \lambda^{-1}\tau^2 \int_{\mathbb{R}^d} \int_{Q_\lambda} c^\xi(\mathbf{0}, y) \mathbf{1}(x \in \mathbb{R}^d \setminus (Q_\lambda - y)) dx dy.$$

As in [24], for all $y \in \mathbb{R}^d$, put $\gamma_{Q_\lambda}(y) := \mathrm{Vol}_d(Q_\lambda \cap (\mathbb{R}^d \setminus (Q_\lambda - y)))$. Then

$$\lambda^{-1}\mathrm{Var}\hat{W}_\lambda = \lambda^{-1}I_\lambda + \lambda^{-1}II_\lambda = \tau c^{\xi^2}(\mathbf{0}) - \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y)dy + \lambda^{-1}\tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y)\gamma_{Q_\lambda}(y)dy. \tag{4.10}$$

Now we assert that

$$\lim_{\lambda\to\infty} \lambda^{-1} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y)\gamma_{Q_\lambda}(y)dy = 0. \tag{4.11}$$

Indeed, by Lemma 1 of [24], we have $\lambda^{-1}\gamma_{Q_\lambda}(y) \to 0$ and since $\lambda^{-1}c^\xi(\mathbf{0}, y)\gamma_{Q_\lambda}(y)$ is dominated by $c^\xi(\mathbf{0}, y)$, which decays exponentially fast, the result follows by the dominated convergence theorem. Collecting terms in (4.9)-(4.11) and recalling (1.17) gives

$$\lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Var}\hat{W}_\lambda = \tau c^{\xi^2}(\mathbf{0}) - \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y)dy = \tau\sigma^2(\xi, \tau) \in [0, \infty), \tag{4.12}$$

where we note $\sigma^2(\xi, \tau)$ is finite by the exponential decay of $c^\xi(\mathbf{0}, y)$ as shown in Lemma 3.5 of [38].

It follows that if $\mathrm{Var}\hat{W}_\lambda$ is not of volume order then we have $\tau c^{\xi^2}(\mathbf{0}) - \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y)dy = 0$. Using this identity in (4.10), multiplying (4.10) by $\lambda^{1/d}$, and taking limits gives

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\mathrm{Var}\hat{W}_\lambda = \lim_{\lambda\to\infty} \tau^2\lambda^{-(d-1)/d} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y)\gamma_{Q_\lambda}(y)dy. \tag{4.13}$$

Now as in [24], we have $\lambda^{-(d-1)/d}\gamma_{Q_\lambda}(y) \le C|y|$, showing that the integrand in (4.13) is dominated by an integrable function. By Lemma 1 of [24], there is a function $\gamma : \mathbb{R}^d \to \mathbb{R}^+$ such that

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\gamma_{Q_\lambda}(y) = \gamma(y).$$

By dominated convergence we get the desired result:

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\mathrm{Var}\hat{W}_\lambda = \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y)\gamma(y)dy < \infty,$$

where once again the integral is finite by the exponential decay of $c^\xi(\mathbf{0}, y)$. □

5 Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1. Combining (4.12) and Lemma 4.5 we obtain $\lim_{\lambda\to\infty} \lambda^{-1}\mathrm{Var}W_\lambda = \tau\sigma^2(\xi, \tau)$, giving (1.18). Now assume non-degeneracy (1.12) and put $\rho = c\ln\lambda$. By Lemma 4.3 we have

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\mathrm{Var}\hat{W}_\lambda(\rho) = \infty$$

and therefore by (4.2) with $G = \emptyset$ we have $\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\mathrm{Var}\hat{W}_\lambda = \infty$. By Lemma 4.6 we have $\mathrm{Var}\hat{W}_\lambda = \Omega(\lambda)$ and Lemma 4.5 gives $\sigma^2(\xi, \tau) > 0$, as desired. □

Proof of Theorem 1.2. We use a result based on the Stein method to derive rates of normal convergence. We follow the set-up of [5], as this yields rates which are a slight improvement over the methods of [38]. Given an admissible Gibbs point process $\mathcal{P}_\lambda^{\beta\Psi}$ with both β and Ψ fixed, we shall simply write $\mathcal{P}_\lambda$ for $\mathcal{P}_\lambda^{\beta\Psi}$. Our first goal is to get rates of normal convergence for $W_\lambda(\rho)$ defined at (3.3). Then we use this to obtain rates for W_λ. Without loss of generality, we assume $p \in (2, q)$ and we show for all $\rho \in (0, \infty)$:

$$d_K\left(\frac{W_\lambda(\rho) - \mathbb{E}\, W_\lambda(\rho)}{\sqrt{\operatorname{Var}(W_\lambda(\rho))}}, N(0,1)\right) = O\left((\operatorname{Var}W_\lambda(\rho))^{-p/2}\lambda w_q^p \rho^{d(p-1)} + (\operatorname{Var}W_\lambda(\rho))^{-1/2} w_q \rho^d\right) \tag{5.1}$$

and, if (1.12) holds and if (1.7) holds for some $q \in (3, \infty)$,

$$d_K\left(\frac{W_\lambda(\rho) - \mathbb{E}\, W_\lambda(\rho)}{\sqrt{\operatorname{Var}(W_\lambda(\rho))}}, N(0,1)\right) = O\left(\rho^{2d}\lambda^{-1/2}\right). \tag{5.2}$$

The proof goes as follows. The local dependence condition LD3 of [5] requires for each $x \in Q_\lambda$ three nested neighborhoods A_x, B_x and C_x which satisfy $B_r(x) \subset A_x \subset B_x \subset C_x$ as $r \downarrow 0$ and such that the sum of scores over points in $B_r(x)$ (resp. A_x, B_x) are independent of the sum of scores over points in $(A_x^r)^c$ (resp. B_x^c, C_x^c). We claim that $W_\lambda(\rho)$ satisfies the local dependence condition LD3 with the neighborhoods $A_x := B_{2\rho}(x)$, $B_x := B_{4\rho}(x)$ and $C_x := B_{6\rho}(x)$, $x \in Q_\lambda$. Indeed, this follows immediately since $\xi(\cdot, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{\cdot\}; \rho)$ enjoys spatial independence over sets separated by more than 2ρ, as already noted in the discussion after (3.3).

It follows from Corollary 2.2 of [5] that

$$d_K\left(\frac{W_\lambda(\rho) - \mathbb{E}\, W_\lambda(\rho)}{\sqrt{\operatorname{Var}(W_\lambda(\rho))}}, N(0,1)\right) \le 48\varepsilon_3 + 160\varepsilon_4 + 2\varepsilon_5,$$

where, with $R(dx) := |\xi(x, \mathcal{P}_\lambda; \rho)|\mathcal{P}_\lambda(dx)$, $N(C_x) := B_{10\rho}(x)$, and $p \in (2, \infty)$,

$$\begin{aligned}
\varepsilon_3 &:= (\operatorname{Var}W_\lambda(\rho))^{-p/2}\mathbb{E}\int_{Q_\lambda} R(N(C_x))^{p-1}R(dx),\\
\varepsilon_4 &:= (\operatorname{Var}W_\lambda(\rho))^{-p/2}\int_{Q_\lambda} \mathbb{E}\, R(N(C_x))^{p-1}\mathbb{E}\, R(dx),\\
\varepsilon_5 &:= (\operatorname{Var}W_\lambda(\rho))^{-1/2}\sup_{x\in Q_\lambda}\mathbb{E}\, R(N(C_x)).
\end{aligned}$$

We write $G_{x,\lambda} := \{D(x, \mathcal{P}_\lambda) \leq \rho\}$. For ε_3, we have by definition of $R(dx)$ that

$$\begin{aligned}
& \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\
= \; & \mathbb{E} \int_{Q_\lambda} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})| \mathbf{1}(G_{z,\lambda}) \mathcal{P}_\lambda(dz) \right)^{p-1} |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathbf{1}(G_{x,\lambda}) \mathcal{P}_\lambda(dx) \\
\leq \; & \mathbb{E} \int_{Q_\lambda} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})| \mathcal{P}_\lambda(dz) \right)^{p-1} |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathcal{P}_\lambda(dx).
\end{aligned}$$

Hölder's inequality $(\int_D |f| \mu(dx))^{p-1} \leq \int_D |f|^{p-1} \mu(dx) \cdot \mu(D)^{p-2}$ gives that

$$\begin{aligned}
& \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\
\leq \; & \mathbb{E} \int_{Q_\lambda} \int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})|^{p-1} \mathcal{P}_\lambda(dz) \cdot \mathcal{P}_\lambda(N(C_x))^{p-2} |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathcal{P}_\lambda(dx) \\
\leq \; & \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \backslash \{x\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dx) \\
& + \mathbb{E} \int_{Q_\lambda} \int_{N(C_x) \backslash \{x\}} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})|^{p-1} \mathcal{P}_\lambda(dz) \mathcal{P}_\lambda(N(C_x))^{p-2} |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathcal{P}_\lambda(dx),
\end{aligned}$$

where we write $\int_{N(C_x)} \cdots \mathcal{P}_\lambda(dz)$ as $\int_{\{x\}} \cdots \mathcal{P}_\lambda(dz) + \int_{N(C_x) \backslash \{x\}} \cdots \mathcal{P}_\lambda(dz)$. The inequality $|a||b|^{p-1} \leq |a|^p + |b|^p$ gives

$$\begin{aligned}
& \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\
\leq \; & \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \backslash \{x\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dx) \\
& + \mathbb{E} \int_{Q_\lambda} \int_{N(C_x) \backslash \{x\}} (|\xi(z, \mathcal{P}_\lambda \backslash \{z\})|^p + |\xi(x, \mathcal{P}_\lambda \backslash \{x\})|^p) \cdot \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dz) \mathcal{P}_\lambda(dx).
\end{aligned}$$

Splitting the last integral into two integrals gives

$$\begin{aligned}
& \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\
\leq \; & \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \backslash \{x\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dx) \\
& + \mathbb{E} \int_{Q_\lambda} \int_{N(C_x) \backslash \{x\}} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dz) \mathcal{P}_\lambda(dx) \\
& + \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \backslash \{x\})|^p \mathcal{P}_\lambda(N(C_x))^{p-1} \mathcal{P}_\lambda(dx).
\end{aligned}$$

The inequality can be further bounded by

$$
\begin{aligned}
& \mathbb{E}\int_{Q_\lambda} R(N(C_x))^{p-1}R(dx) \\
\leq\ & \mathbb{E}\int_{Q_\lambda} \mathcal{P}_\lambda(N(C_x))^{p-2}|\xi(x,\mathcal{P}_\lambda\backslash\{x\})|^p\mathcal{P}_\lambda(dx) \\
& +\mathbb{E}\iint_{0<d(x,z)\leq 10\rho} |\xi(z,\mathcal{P}_\lambda\backslash\{z\})|^p\mathcal{P}_\lambda(N(C_x))^{p-2}\mathcal{P}_\lambda(dx)\mathcal{P}_\lambda(dz) \\
& +\mathbb{E}\int_{Q_\lambda} \mathcal{P}_\lambda(N(C_x))^{p-1}|\xi(x,\mathcal{P}_\lambda\backslash\{x\})|^p\mathcal{P}_\lambda(dx).
\end{aligned}
$$

Now integrating the double integral gives

$$
\begin{aligned}
& \mathbb{E}\int_{Q_\lambda} R(N(C_x))^{p-1}R(dx) \\
\leq\ & \mathbb{E}\int_{Q_\lambda} \mathcal{P}_\lambda(N(C_x))^{p-2}|\xi(x,\mathcal{P}_\lambda\backslash\{x\})|^p\mathcal{P}_\lambda(dx) \\
& +\mathbb{E}\int_{Q_\lambda} |\xi(z,\mathcal{P}_\lambda\backslash\{z\})|^p\cdot\mathcal{P}_\lambda(B_{20\rho}(z))^{p-1}\mathcal{P}_\lambda(dz) \\
& +\mathbb{E}\int_{Q_\lambda} \mathcal{P}_\lambda(N(C_x))^{p-1}|\xi(x,\mathcal{P}_\lambda\backslash\{x\})|^p\mathcal{P}_\lambda(dx).
\end{aligned}
$$

Combining integrals and using Hölder's inequality for $p_1\in(1,q/p)$ gives

$$
\begin{aligned}
& \mathbb{E}\int_{Q_\lambda} R(N(C_x))^{p-1}R(dx) \\
\leq\ & 3\mathbb{E}\int_{Q_\lambda} |\xi(z,\mathcal{P}_\lambda\backslash\{z\})|^p\mathcal{P}_\lambda(B_{20\rho}(z))^{p-1}\mathcal{P}_\lambda(dz) \\
\leq\ & 3\left\{\mathbb{E}\int_{Q_\lambda} \mathcal{P}_\lambda(B_{20\rho}(z))^{\frac{(p-1)p_1}{p_1-1}}\mathcal{P}_\lambda(dz)\right\}^{\frac{p_1-1}{p_1}}\left\{\mathbb{E}\int_{Q_\lambda} |\xi(z,\mathcal{P}_\lambda\backslash\{z\})|^{pp_1}\mathcal{P}_\lambda(dz)\right\}^{\frac{1}{p_1}} \qquad (5.3)
\end{aligned}
$$

Since $\mathcal{P}_\lambda^{\beta\Psi}$ is a Gibbs point process, we apply the Georgii-Nguyen-Zessin integral characterization of Gibbs point processes [27] to see that the conditional probability of observing an extra point of $\mathcal{P}_\lambda^{\beta\Psi}$ in the volume element dz, given that configuration without that point, equals $\exp(-\beta\Delta^\Psi(\{z\},\mathcal{P}_\lambda^{\beta\Psi}))dz\leq dz$, where $\Delta^\Psi(\{z\},\mathcal{P}_\lambda^{\beta\Psi})$ is defined at (1.3). Using that $\mathbb{E}\,\mathcal{P}_\lambda^{\beta\Psi}(dx)\leq\tau dx$, we have from (5.3) that

$$
\begin{aligned}
& \mathbb{E}\int_{Q_\lambda} R(N(C_x))^{p-1}R(dx) \\
\leq\ & 3\tau\left\{\mathbb{E}\int_{Q_\lambda} (\mathcal{P}_\lambda(B_{20\rho}(z))+1)^{\frac{(p-1)p_1}{p_1-1}}dz\right\}^{\frac{p_1-1}{p_1}}\left\{\mathbb{E}\int_{Q_\lambda} |\xi(z,\mathcal{P}_\lambda\cup\{z\})|^{pp_1}dz\right\}^{\frac{1}{p_1}} \qquad (5.4)
\end{aligned}
$$

Notice that $\mathcal{P}_\lambda(B_{20\rho}(x))$ is stochastically bounded by $\mathrm{Po}(\tau M)$with $M:=\mathrm{Vol}(B_{20\rho}(0))$, we have from Lemma 4.3 of [5] that $\mathbb{E}\{\mathcal{P}_\lambda(B_{20\rho}(x))+1\}^{(p-1)p_1/(p_1-1)}\leq c_1\rho^{d(p-1)p_1/(p_1-1)}$, giving

$$\varepsilon_3 \le 3\tau \mathrm{Var}(W_\lambda(\rho))^{-p/2} c_1^{\frac{p_1-1}{p_1}} \rho^{d(p-1)} \lambda^{(p_1-1)/p_1} \left\{ \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \cup \{x\})|^{pp_1} dx \right\}^{\frac{1}{p_1}}.$$

Then since $w_{pp_1} \le w_q$, we have

$$\varepsilon_3 \le 3\tau\lambda \mathrm{Var}(W_\lambda(\rho))^{-p/2} c_1^{\frac{p_1-1}{p_1}} w_q^p \rho^{d(p-1)}. \tag{5.5}$$

Next, we bound ε_4. To this end, let $p_2 := pp_1/(p-1)$, we again replace the indicator function with 1 and then apply Hölder's inequality to get

$$\begin{aligned}
& \int_{Q_\lambda} \mathbb{E}\, R(N(C_x))^{p-1} \mathbb{E}\, R(dx) \\
= & \int_{Q_\lambda} \mathbb{E} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})| \mathbf{1}(G_{z,\lambda}) \mathcal{P}_\lambda(dz) \right)^{p-1} \mathbb{E}\, |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathbf{1}(G_{x,\lambda}) \mathcal{P}_\lambda(dx) \\
\le & \int_{Q_\lambda} \mathbb{E} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})| \mathcal{P}_\lambda(dz) \right)^{p-1} \mathbb{E}\, |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathcal{P}_\lambda(dx) \\
\le & \int_{Q_\lambda} \mathbb{E} \left\{ \int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})|^{p-1} \mathcal{P}_\lambda(dz) \mathcal{P}_\lambda(N(C_x))^{p-2} \right\} \mathbb{E}\, |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathcal{P}_\lambda(dx).
\end{aligned}$$

This ensures

$$\begin{aligned}
& \int_{Q_\lambda} \mathbb{E}\, R(N(C_x))^{p-1} \mathbb{E}\, R(dx) \\
\le & \int_{Q_\lambda} \mathbb{E} \left\{ \int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})|^{p-1} \mathcal{P}_\lambda(B_{20\rho}(z))^{p-2} \mathcal{P}_\lambda(dz) \right\} \mathbb{E}\, |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathcal{P}_\lambda(dx) \\
\le & \int_{Q_\lambda} \left\{ \mathbb{E} \int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})|^{p_2(p-1)} \mathcal{P}_\lambda(dz) \right\}^{\frac{1}{p_2}} \\
& \left\{ \mathbb{E} \int_{N(C_x)} \mathcal{P}_\lambda(B_{20\rho}(z))^{(p-2)\frac{p_2}{p_2-1}} \mathcal{P}_\lambda(dz) \right\}^{\frac{p_2-1}{p_2}} \mathbb{E}\, |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathcal{P}_\lambda(dx).
\end{aligned} \tag{5.6}$$

Reasoning as for (5.4), we obtain from (5.6) that

$$\begin{aligned}
& \int_{Q_\lambda} \mathbb{E}\, R(N(C_x))^{p-1} \mathbb{E}\, R(dx) \\
\le & \int_{Q_\lambda} \left\{ \int_{N(C_x)} \mathbb{E}\, |\xi(z, \mathcal{P}_\lambda \cup \{z\})|^{p_2(p-1)} \tau dz \right\}^{\frac{1}{p_2}} \\
& \left\{ \int_{N(C_x)} \mathbb{E}\, (\mathcal{P}_\lambda(B_{20\rho}(z)) + 1)^{(p-2)\frac{p_2}{p_2-1}} \tau dz \right\}^{\frac{p_2-1}{p_2}} \mathbb{E}\, |\xi(x, \mathcal{P}_\lambda \backslash \{x\})| \mathcal{P}_\lambda(dx) \\
\le & \tau^2 w_{pp_1}^{p-1} c_2^{\frac{p_2-1}{p_2}} \rho^{d(p-2)} \int_{Q_\lambda} \left\{ \int_{N(C_x)} dz \right\}^{\frac{1}{p_2}} \left\{ \int_{N(C_x)} dz \right\}^{\frac{p_2-1}{p_2}} w_{pp_1} dx \\
\le & w_{pp_1}^p c_3 \lambda \rho^{d(p-1)}.
\end{aligned}$$

Hence

$$\varepsilon_4 \le (\mathrm{Var}W_\lambda(\rho))^{-p/2} w_q^p c_3 \lambda \rho^{d(p-1)}, \tag{5.7}$$

showing that the bounds for ε_3 and ε_4 coincide. Turning to ε_5, we have

$$\begin{aligned}
\varepsilon_5 &\le (\mathrm{Var}W_\lambda(\rho))^{-1/2} \sup_{x\in Q_\lambda} \mathbb{E}\left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \backslash \{z\})| \mathcal{P}_\lambda(dz)\right) \\
&\le (\mathrm{Var}W_\lambda(\rho))^{-1/2} \sup_{x\in Q_\lambda} \left(\int_{N(C_x)} \mathbb{E}\,|\xi(z, \mathcal{P}_\lambda \cup \{z\})| \tau dz\right) \\
&\le \mathrm{Var}(W_\lambda(\rho))^{-1/2} \sup_{x\in Q_\lambda} \left(\int_{N(C_x)} \{\mathbb{E}\,|\xi(z, \mathcal{P}_\lambda \cup \{z\})|^{pp_1}\}^{\frac{1}{pp_1}} \tau dz\right) \\
&\le \mathrm{Var}(W_\lambda(\rho))^{-1/2} w_q c_4 \rho^d.
\end{aligned} \tag{5.8}$$

Combining estimates (5.5), (5.7) and (5.8), we get (5.1).

Assuming condition (1.7), using (4.3) with $G = \emptyset$ and Theorem 1.1, we have $\mathrm{Var}[W_\lambda(\rho)] \ge c_5\lambda$. When $p = 3$, this, together with (5.1), gives (5.2).

To complete the proof, we need to replace $W_\lambda(\rho)$ with W_λ. We rely heavily on Lemma 4.2 for this. Note for all $\epsilon_1 \in \mathbb{R}$ and $\epsilon_2 > -0.6$,

$$\begin{aligned}
&d_K(N(0,1), N(\epsilon_1, 1+\epsilon_2)) \le d_K(N(0,1), N(\epsilon_1, 1)) + d_K(N(\epsilon_1, 1), N(\epsilon_1, 1+\epsilon_2)) \\
&\le \frac{|\epsilon_1|}{\sqrt{2\pi}} + \frac{|\epsilon_2|}{\sqrt{2e\pi}}.
\end{aligned} \tag{5.9}$$

Now $d_K(X, N(0,1)) = d_K(aX, N(0,a^2)) = d_K(aX+b, N(b,a^2))$ holds for X with $\mathbb{E}\,X = 0$ and all constants a and b. Hence

$$\begin{aligned}
&d_K\left(\frac{W_\lambda - \mathbb{E}\,W_\lambda}{\sqrt{\mathrm{Var}W_\lambda}}, N(0,1)\right) = d_K\left(\frac{W_\lambda - \mathbb{E}\,W_\lambda(\rho)}{\sqrt{\mathrm{Var}W_\lambda(\rho)}}, N\left(\frac{\mathbb{E}\,W_\lambda - \mathbb{E}\,W_\lambda(\rho)}{\sqrt{\mathrm{Var}W_\lambda(\rho)}}, \frac{\mathrm{Var}W_\lambda}{\mathrm{Var}W_\lambda(\rho)}\right)\right) \\
\le\ & d_K\left(\frac{W_\lambda - \mathbb{E}\,W_\lambda(\rho)}{\sqrt{\mathrm{Var}W_\lambda(\rho)}}, N(0,1)\right) + d_K\left(N(0,1), N\left(\frac{\mathbb{E}\,W_\lambda - \mathbb{E}\,W_\lambda(\rho)}{\sqrt{\mathrm{Var}W_\lambda(\rho)}}, \frac{\mathrm{Var}W_\lambda}{\mathrm{Var}W_\lambda(\rho)}\right)\right)
\end{aligned} \tag{5.10}$$

by the triangle inequality for d_K. Now for any random variables Y and Y' we have

$$d_K(Y, N(0,1)) \le d_K(Y', N(0,1)) + \mathbb{P}[Y \ne Y'] \tag{5.11}$$

which follows from $|\mathbb{P}[Y \le t] - \Phi(t)| \le |\mathbb{P}[Y' \le t] - \Phi(t)| + |\mathbb{P}[Y' \le t] - \mathbb{P}[Y \le t]|$. We have by (5.11) and (5.9) that

$$\begin{aligned}
&d_K\left(\frac{W_\lambda - \mathbb{E}\,W_\lambda}{\sqrt{\mathrm{Var}W_\lambda}}, N(0,1)\right) \\
\le\ & \mathbb{P}[W_\lambda \ne W_\lambda(\rho)] + d_K\left(\frac{W_\lambda(\rho) - \mathbb{E}\,W_\lambda(\rho)}{\sqrt{\mathrm{Var}W_\lambda(\rho)}}, N(0,1)\right) \\
&+ \frac{1}{\sqrt{2\pi}}\left|\frac{\mathbb{E}\,W_\lambda - \mathbb{E}\,W_\lambda(\rho)}{\sqrt{\mathrm{Var}W_\lambda(\rho)}}\right| + \frac{1}{\sqrt{2e\pi}}\left|\frac{\mathrm{Var}W_\lambda - \mathrm{Var}W_\lambda(\rho)}{\mathrm{Var}W_\lambda(\rho)}\right|.
\end{aligned} \tag{5.12}$$

However, the Cauchy-Schwarz inequality ensures

$$|\mathbb{E}\, W_\lambda - \mathbb{E}\, W_\lambda(\rho)| \le \|W_\lambda - W_\lambda(\rho)\|_2 \mathbb{P}(W_\lambda \neq W_\lambda(\rho))^{1/2} \le \lambda^{-1},$$

where the last inequality is due to (4.1), (3.6) and the arbitrariness of L. Hence, it follows from (5.12) that

$$d_K\left(\frac{W_\lambda - \mathbb{E}\, W_\lambda}{\sqrt{\mathrm{Var} W_\lambda}}, N(0,1)\right) \le \lambda^{-2} + O\left((\mathrm{Var} W_\lambda)^{-p/2} \lambda (\ln \lambda)^{d(p-1)}\right),$$

where we use (3.6) with $L = 2$, (5.1) and (4.3) with $G = \emptyset$. $\square$

Proof of Theorem 1.3. The bound (1.23) follows from Lemma 4.4 and Lemma 4.2(b) with $G = \emptyset$. The proof of (1.22) follows by replacing Q_λ with $\tilde{S}_\lambda$ in the proof of (1.19), whereas (1.24) follows by combining (1.22) and (1.23). $\square$

6 Appendix

Gibbs point processes with Hamiltonians Ψ in the class $\mathbf{\Psi}^*$. Here we describe some Hamiltonians contained in the class $\mathbf{\Psi}^*$ described in Section 1.1. We follow nearly verbatim the description in [38].

(i) *Point processes with a pair potential function.* A large class of Gibbs point processes, known as pairwise interaction point processes [39], has Hamiltonian

$$\Psi(\mathcal{X}) := \sum_{i<j} \phi(\|x_i - x_j\|), \ \mathcal{X} := \{x_i\}_{i=1}^n, \tag{6.1}$$

with pair potential $\phi : [0,\infty) \to [0,\infty)$ and where $\|\cdot\|$ denotes the Euclidean norm. Such processes clearly do not include pair potentials with a negative part, but they do include for example the Strauss process, which involves perturbing a Poisson process according to an exponential of the number of pairs of points closer than a fixed cutoff r_0, with $\phi(u) = \alpha \mathbf{1}_{u \le r_0}$ for some $\alpha > 0$; see [1], [39], and Section 10.4 of Volume 2 of [14] for details.

(ii) *Area interaction point processes.* These are Gibbs-modified germ grain processes, where the grain shape is a fixed compact convex set K; see p. 9 of [19], [1], and Section 10.4 of Volume 2 of [14] for details. As in [14], these processes have Hamiltonian

$$\Psi(\mathcal{X}) = \nu\left(\bigcup_{i=1}^n (x_i \oplus K)\right) + \alpha_1 n + \alpha_2, \quad \mathcal{X} =: \{x_i\}_{i=1}^n, \tag{6.2}$$

where ν is a totally finite positive Borel regular measure.

(iii) *Point processes given by the hard-core model.* The hard-core model conditions a Poisson point process to contain no two points at distance less than $2r_0$, with $r_0 > 0$ denoting a parameter of the model. The hard-core point process has Hamiltonian

$$\Psi(\mathcal{X}) = \alpha_1 n + \alpha_2, \quad \mathcal{X} := \{x_i\}_{i=1}^n, \tag{6.3}$$

if no two points of $\mathcal{X}$ are within distance $2r_0$ and otherwise $\Psi(\mathcal{X}) = \infty$.

(iv) *Truncated Poisson processes.* The hard-core model is a particular example of a truncated Poisson process. In general, a truncated Poisson process arises by conditioning a Poisson point process on a constraint event. For example, we may fix $k \in \mathbb{N}$ and $r_0 \in (0, \infty)$ and require that no ball of radius r_0 contain more than k points from the process. In this case,

$$\Psi(\mathcal{X}) = \infty \text{ if there is } x \in \mathbb{R}^d \text{ such that } \mathrm{card}(\mathcal{X} \cap B_{r_0}(x)) > k, \tag{6.4}$$

and otherwise $\Psi(\mathcal{X}) = 0$.

Proofs of lemmas in Section 2.

Proof of Lemma 2.1. We first assume that $\tau \mathrm{Vol}_d(B) < 1$. The Georgii-Nguyen-Zessin formula for Gibbs point processes [27] states that the conditional probability of observing an extra point of $\mathcal{P}^{\beta\Psi}$ in the volume element dz, given that configuration without that point, is $\exp(-\beta\Delta^\Psi(\{z\}, \mathcal{P}^{\beta\Psi}))dz$, where $\Delta^\Psi(\{z\}, \mathcal{P}^{\beta\Psi})$ is defined at (1.3). Hence, for any $A \in \mathcal{G}_{B^c}$ we have

$$\begin{aligned}
& \mathbb{P}\left[\mathcal{P}^{\beta\Psi} \cap B \neq \emptyset, A\right] \\
\leq\ & \mathbb{E} \int_B \mathbf{1}(A) \mathcal{P}^{\beta\Psi}(dz) \\
=\ & \tau \mathbb{E} \int_B \mathbf{1}(A) \exp(-\beta\Delta^\Psi(\{z\}, \mathcal{P}^{\beta\Psi}))dz \\
\leq\ & \tau \mathbb{E} \int_B \mathbf{1}(A) dz \\
=\ & \tau \mathrm{Vol}_d(B) \mathbb{P}[A],
\end{aligned}$$

where the first equality follows from the Georgii-Nguyen-Zessin formula. The inequality can be rewritten as $\mathbb{P}\left[\mathcal{P}^{\beta\Psi} \cap B \neq \emptyset \middle| A\right] \leq \tau \mathrm{Vol}_d(B)$ which ensures

$$\mathbb{P}\left[\mathcal{P}^{\beta\Psi} \cap B = \emptyset \middle| A\right] \geq 1 - \tau \mathrm{Vol}_d(B). \tag{6.5}$$

Now, for general B with $\mathrm{Vol}_d(B) < \infty$, we take an integer $k > \tau \mathrm{Vol}_d(B)$ and partition

B into $B_1, \ldots, B_k$ such that $\mathrm{Vol}_d(B_i) = \mathrm{Vol}_d(B)/k$, $1 \le i \le k$. For any set $A \in \mathcal{G}_{B^c}$, setting $B_0 = \emptyset$, we have

$$\begin{aligned}
& \mathbb{P}\left[\mathcal{P}^{\beta\Psi} \cap B = \emptyset \middle| A\right] \\
= \; & \mathbb{P}\left[\mathcal{P}^{\beta\Psi} \cap B_1 = \emptyset, \ldots, \mathcal{P}^{\beta\Psi} \cap B_k = \emptyset \middle| A\right] \\
= \; & \prod_{i=0}^{k-1} \mathbb{P}\left[\mathcal{P}^{\beta\Psi} \cap B_{i+1} = \emptyset \middle| \mathcal{P}^{\beta\Psi} \cap B_0 = \emptyset, \ldots, \mathcal{P}^{\beta\Psi} \cap B_i = \emptyset, A\right] \\
\ge \; & (1 - \tau \mathrm{Vol}_d(B)/k)^k,
\end{aligned}$$

where the inequality follows from (6.17). The proof is completed by letting $k \to \infty$. □

Proof of Lemma 2.2. Recall $F := \cup_{i=1}^m F_i$. It suffices to prove for $G_m \in \sigma\left(\mathcal{P}^{\beta\Psi} \cap F^c\right)$, that

$$\begin{aligned}
& \mathbb{P}\left[G_m \cap \bigcap_{i=1}^m \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_i) = l_i\}\right] \\
& \ge e^{-\tau \mathrm{Vol}_d(F)} \mathbb{P}\left[G_m \cap \bigcap_{i=1}^m \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_i) = k_i\}\right].
\end{aligned} \tag{6.6}$$

We first assume $m = 1$. If $l_1 = 0$, then (6.18) follows from Lemma 2.1 with $A := G_m$ there. It remains to show (6.18) for $0 < l_1 \le k_1$. For a positive integer n, we partition F_1 into n disjoint subsets A_{nj} such that $\mathrm{Vol}_d(A_{nj}) = \mathrm{Vol}_d(F_1)/n$, $1 \le j \le n$. Let $\Omega_n := \bigcap_{1 \le j \le n} \{\mathrm{card}(A_{nj} \cap \mathcal{P}^{\beta\Psi}) \le 1\}$. Then

$$\mathbb{P}[\Omega_n^c] \le \sum_{j=1}^n \mathbb{P}[\mathrm{card}(A_{nj} \cap \mathcal{P}^{\beta\Psi}) \ge 2] \le \sum_{j=1}^n \mathbb{P}[\mathrm{card}(A_{nj} \cap \tilde{\mathcal{P}}_\tau) \ge 2] = O(n^{-1}), \tag{6.7}$$

where the second inequality follows since $\mathcal{P}^{\beta\Psi}$ is stochastically dominated by the reference process $\tilde{\mathcal{P}}_\tau$. Let $\mathcal{H}_n := \{I : \ I \subset \{1, \ 2, \ \ldots, \ n\}, \ \mathrm{card}(I) = l_1\}$ and for $I \in \mathcal{H}_n$, define $A_{nI} := \bigcup_{j \in I} A_{nj}$ and

$$E_{nI} := \{\mathcal{P}^{\beta\Psi} \cap (F_1 \setminus A_{nI}) = \emptyset\}, \ E'_{nI} := \bigcap_{j \in I} \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap A_{nj}) = 1\}.$$

Note that the event $E_{nI} \cap E'_{nI}$ happens iff $\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_1) = l_1$ and each of $A_{nj}, j \in I$, hosts exactly one Gibbs point. We now claim that

$$\mathbb{P}[G_1 \cap E'_{nI}] \le e^{\tau \mathrm{Vol}_d(F_1)} \mathbb{P}[E_{nI} \cap G_1 \cap E'_{nI}]. \tag{6.8}$$

If $\mathbb{P}[G_1 \cap E'_{nI}] = 0$ then clearly (6.20) holds. For $\mathbb{P}[G_1 \cap E'_{nI}] > 0$, applying Lemma 2.1 again, we have

$$\mathbb{P}[E_{nI} | G_1 \cap E'_{nI}] \ge e^{-\tau \mathrm{Vol}_d(F_1 \setminus A_{nI})} \ge e^{-\tau \mathrm{Vol}_d(F_1)},$$

which implies (6.20). Now,

$$
\begin{aligned}
&\mathbb{P}[G_1 \cap \Omega_n \cap \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_1) = k_1\}] \\
&\leq \sum_{I \in \mathcal{H}_n} \mathbb{P}\left[G_1 \cap E'_{nI}\right] \\
&\leq e^{\tau \mathrm{Vol}_d(F_1)} \sum_{I \in \mathcal{H}_n} \mathbb{P}\left[E_{nI} \cap G_1 \cap E'_{nI}\right] \\
&\leq e^{\tau \mathrm{Vol}_d(F_1)} \mathbb{P}\left[G_1 \cap \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_1) = l_1\}\right],
\end{aligned}
\tag{6.9}
$$

where the second inequality holds by (6.20). Letting $n \to \infty$ in (6.21) and recalling (6.19), we obtain (6.18) when $m = 1$.

For $m \geq 2$, we apply induction on m and assume that (6.18) holds for $m - 1$. That is, for every $G_{m-1} \in \sigma\left(\mathcal{P}^{\beta\Psi} \cap \left(\cup_{i=1}^{m-1} F_i\right)^c\right)$,

$$
\begin{aligned}
&\mathbb{P}\left[G_{m-1} \cap \bigcap_{i=1}^{m-1} \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_i) = l_i\}\right] \\
&\geq e^{-\tau \mathrm{Vol}_d\left(\cup_{i=1}^{m-1} F_i\right)} \mathbb{P}\left[G_{m-1} \cap \bigcap_{i=1}^{m-1} \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_i) = k_i\}\right].
\end{aligned}
\tag{6.10}
$$

For $G_m \in \sigma\left(\mathcal{P}^{\beta\Psi} \cap F^c\right)$, write $G_1 := G_m \cap \bigcap_{i=1}^{m-1} \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_i) = l_i\}$ and $G_{m-1} := G_m \cap \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_m) = k_m\}$. Note that $G_{m-1} \in \sigma(\mathcal{P}^{\beta\Psi} \cap (\cup_{i=1}^{m-1} F_i)^c)$. Then

$$
\begin{aligned}
&\mathbb{P}\left[G_m \bigcap \cap_{i=1}^{m} \{\mathrm{card}(\cap F_i) = l_i\}\right] \\
&= \mathbb{P}\left[G_1 \cap \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_m) = l_m\}\right] \\
&\geq e^{-\tau \mathrm{Vol}_d(F_m)} \mathbb{P}\left[G_1 \cap \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_m) = k_m\}\right] \\
&= e^{-\tau \mathrm{Vol}_d(F_m)} \mathbb{P}\left[G_{m-1} \bigcap \cap_{i=1}^{m-1} \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_i) = l_i\}\right] \\
&\geq e^{-\tau \mathrm{Vol}_d(F_m)} e^{-\tau \mathrm{Vol}_d\left(\cup_{i=1}^{m-1} F_i\right)} \mathbb{P}\left[G_{m-1} \bigcap \cap_{i=1}^{m-1} \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_i) = k_i\}\right] \\
&= e^{-\tau \mathrm{Vol}_d(F)} \mathbb{P}\left[G_m \bigcap \cap_{i=1}^{m} \{\mathrm{card}(\mathcal{P}^{\beta\Psi} \cap F_i) = k_i\}\right],
\end{aligned}
$$

where the first inequality is from (6.18) with $m = 1$ and the second inequality is from the induction assumption (6.22). □

Proof of Lemma 2.3. Let $E := \{Y \in A_1 \cup A_2\}$, $\mathcal{G} = \sigma\{E\}$ and $F_E(x)$ be the conditional distribution function of Y given E. Without loss of generality, we assume that $\inf\{|x - \mathbb{E}(Y|E)| : x \in A_1\} \geq \frac{1}{2} d(A_1, A_2)$. By the conditional variance formula,

$$
\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(\mathbb{E}(Y|\mathcal{G})) + \mathbb{E}\,\mathrm{Var}(Y|\mathcal{G}) \geq \mathrm{Var}(Y|E)\mathbb{P}[E] \\
&= \int_{A_1 \cup A_2} (x - \mathbb{E}(Y|E))^2 dF_E(x) \mathbb{P}[E] \\
&\geq \frac{1}{4} d(A_1, A_2)^2 \mathbb{P}[Y \in A_1] \geq \frac{1}{4} d(A_1, A_2)^2 (p_1 \wedge p_2),
\end{aligned}
$$

as required. □

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