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Asymptotic theory for statistics of the Poisson-Voronoi approximation

CHRISTOPH THÄLE[1] and J. E. YUKICH[2]

[1]*Faculty of Mathematics, Ruhr University Bochum, Bochum, Germany.*
E-mail: christoph.thaele@rub.de

[2]*Department of Mathematics, Lehigh University, Bethlehem, USA.*
E-mail: joseph.yukich@lehigh.edu

This paper establishes expectation and variance asymptotics for statistics of the Poisson-Voronoi approximation of general sets, as the underlying intensity of the Poisson point process tends to infinity. Statistics of interest include volume, surface area, Hausdorff measure, and the number of faces of lower-dimensional skeletons. We also consider the complexity of the so-called Voronoi zone and the iterated Voronoi approximation. Our results are consequences of general limit theorems proved with an abstract Steiner-type formula applicable in the setting of sums of stabilizing functionals.

Keywords: Combinatorial geometry, Poisson point process, Poisson-Voronoi approximation, random mosaic, stabilizing functional, stochastic geometry.

1. Main results

The Poisson-Voronoi mosaic is a classical and prominent example of a random mosaic and is used in a wide range of fields, including astronomy, biology, material sciences, and telecommunications. If $\mathcal{P}_\lambda$ is a Poisson point process on $Q := [-1/2, 1/2]^d$ whose intensity measure has density $\lambda\kappa(\,\cdot\,)$ with respect to the Lebesgue measure ($d \geq 2$, $\lambda \in (0, \infty)$ and κ is a continuous function on Q bounded away from zero and infinity), the Voronoi cell $v(x) := v(x, \mathcal{P}_\lambda)$ associated with $x \in \mathcal{P}_\lambda$ is the set of all $z \in Q$ such that the distance between z and x is less than the distance between z and any other point of $\mathcal{P}_\lambda$. Clearly, $v(x)$ is a random convex polytope and the collection of all $v(x)$ with $x \in \mathcal{P}_\lambda$ partitions Q and is called the Poisson-Voronoi mosaic of Q.

Let $A \subset Q$ be a full-dimensional admissible set whose boundary has positive and finite $(d-1)$-dimensional Hausdorff measure. Admissible sets, formally defined in Section 2, include in particular, convex sets, sets of positive reach, differentiable manifolds with smooth boundary as well as certain finite unions of such sets. Given such $A \subset Q$, the Poisson-Voronoi approximation $\mathrm{PV}_\lambda(A)$ of A is the union of all Voronoi cells $v(x)$ with $x \in A$, i.e.,

$$\mathrm{PV}_\lambda(A) := \bigcup_{x \in \mathcal{P}_\lambda \cap A} v(x)\,.$$

Typically A is an unknown set having unknown geometric characteristics such as volume or surface area. Notice that $\mathrm{PV}_\lambda(A)$ is a random polyhedral approximation of A, which closely approximates A as λ becomes large. One might expect that the volume and surface area of $\mathrm{PV}_\lambda(A)$, respectively denoted by $V_\lambda(A)$ and $S_\lambda(A)$, also closely approximate the volume and surface area of A. Our first goal is to show that this is indeed the case, though the surface area asymptotics involve a universal correction factor. For sets A which are convex or which have a smooth boundary, first-order asymptotics have been previously established in [9, 21, 25, 31]; second-order asymptotics for sets A having a smooth boundary are given in [31], while [28] provides second-order inequalities in case that A is a convex set. We extend the limit theory of these papers and obtain first- and second-order asymptotics whenever A belongs to the more general class of admissible sets. In particular we show that the variance asymptotics for $V_\lambda(A)$ are proportional to the κ-weighted surface content of A, resolving a conjecture implicit in Remark 2.2 of [25]. The approach relies on a general and far-reaching Steiner-type formula from [10], together with stabilization properties of geometric functionals of the Poisson-Voronoi mosaic.

In the sequel, we write $f(\lambda) \sim c\, g(\lambda)$ for real-valued functions f and g and constants $c \in [0,\infty)$ if $\lim_{\lambda\to\infty} f(\lambda)/g(\lambda) = c$. Throughout, we denote the s-dimensional Hausdorff measure by $\mathcal{H}^s$, $s \in [0,\infty)$. Furthermore, we say that ∂A contains a subset Γ of differentiability class C^2 with $\mathcal{H}^{d-1}(\Gamma) \in (0,\infty)$ if $\Gamma \subset \partial A$ is an open and twice differentiable $(d-1)$-dimensional sub-manifold in $\mathbb{R}^d$ in the usual sense of differential geometry. Finally, for $\gamma \in \mathbb{R}$ we define the *κ-weighted surface content*

$$\mathcal{H}^{d-1}_{\kappa,\gamma}(\partial A) := \int_{\partial A} \kappa(x)^{1-\gamma/d}\, \mathcal{H}^{d-1}(\mathrm{d}x)\,.$$

Observe that $\mathcal{H}^{d-1}_{\kappa,\gamma}(\partial A)$ reduces to the usual surface content $\mathcal{H}^{d-1}(\partial A)$ of ∂A if either $\gamma = d$ and κ is arbitrary or $\kappa \equiv 1$ and $\gamma \in \mathbb{R}$ is arbitrary.

Theorem 1.1. *There are constants $c_1, c_2 \in (0,\infty)$ depending only on the dimension d such that*

$$\mathbb{E}\, V_\lambda(A) - V(A) \sim c_1\, \lambda^{-\frac{1}{d}}\, \mathcal{H}^{d-1}(\partial A) \quad \text{and} \quad \mathbb{E}\, S_\lambda(A) \sim c_2\, \mathcal{H}^{d-1}_{\kappa,d-1}(\partial A)\,.$$

Moreover, there are constants $c_3, c_{4,1}, c_{4,2} \in [0,\infty)$ depending only on d such that

$$\mathrm{Var}[V_\lambda(A)] \sim c_3\, \lambda^{-1-\frac{1}{d}}\, \mathcal{H}^{d-1}(\partial A)$$

and

$$\mathrm{Var}[S_\lambda(A)] \sim \lambda^{-1+\frac{1}{d}} \left(c_{4,1}\mathcal{H}^{d-1}_{\kappa,2(d-1)}(\partial A) + c_{4,2}\mathcal{H}^{d-1}_{\kappa^2,d-1}(\partial A)\right).$$

If ∂A contains a subset Γ of differentiability class C^2 with $\mathcal{H}^{d-1}(\Gamma) \in (0,\infty)$, and if $\kappa \equiv 1$, then c_3 and $c_4 := c_{4,1} + c_{4,2}$ are strictly positive.

Next, we turn to other metric parameters of the Poisson-Voronoi approximation, which can be handled by our general set-up. To this end, for $\ell \in \{0,\dots,d-1\}$ denote by

$\mathrm{skel}_\ell(\mathrm{PV}_\lambda(A))$ the union of all ℓ-dimensional faces belonging to $\partial(\mathrm{PV}_\lambda(A))$, the boundary of $\mathrm{PV}_\lambda(A)$, and let $H_\lambda^{(\ell)}(A)$ be the ℓ-dimensional Hausdorff measure of $\mathrm{skel}_\ell(\mathrm{PV}_\lambda(A))$. More formally, if $\mathcal{F}_\ell(P)$ stands for the collection of ℓ-dimensional faces of a polytope P, then

$$H_\lambda^{(\ell)}(A) := \sum_{\substack{x\in\mathcal{P}_\lambda\\ x\in A}} \sum_{\substack{f\in\mathcal{F}_\ell(v(x))\\ f\subset\partial(\mathrm{PV}_\lambda(A))}} \mathcal{H}^{(\ell)}(f)\,.$$

Note that $H_\lambda^{(d-1)}(A)$ coincides with $S_\lambda(A)$ considered in Theorem 1.1.

Theorem 1.2. *Let $\ell \in \{0,\dots,d-1\}$. Then there are constants $c_5 \in (0,\infty)$ and $c_{6,1}, c_{6,2} \in [0,\infty)$ depending only on d and ℓ such that*

$$\mathbb{E}\, H_\lambda^{(\ell)}(A) \sim c_5\, \lambda^{1-\frac{1}{d}-\frac{\ell}{d}}\ \mathcal{H}^{d-1}_{\kappa,\ell}(\partial A)$$

and

$$\mathrm{Var}[H_\lambda^{(\ell)}(A)] \sim \lambda^{1-\frac{1}{d}-\frac{2\ell}{d}} \left(c_{6,1}\,\mathcal{H}^{d-1}_{\kappa,2\ell}(\partial A) + c_{6,2}\,\mathcal{H}^{d-1}_{\kappa^2,\ell}(\partial A)\right).$$

If ∂A contains a subset Γ of differentiability class C^2 with $\mathcal{H}^{d-1}(\Gamma) \in (0,\infty)$, and if $\kappa \equiv 1$, then $c_6 := c_{6,1} + c_{6,2}$ is strictly positive.

With the exception of $H_\lambda^{(0)}(A)$, the number of vertices on $\partial(\mathrm{PV}_\lambda(A))$, we have investigated only metric parameters of the Poisson-Voronoi approximation, namely the volume, the surface area and the Hausdorff measure of lower-dimensional skeletons. On the other hand, the combinatorial complexity of $\mathrm{PV}_\lambda(A)$ is also of interest. For example, it is natural to ask how many ℓ-dimensional faces ($\ell \in \{0,\dots,d-1\}$) belong to $\partial(\mathrm{PV}_\lambda(A))$. In contrast to volume and surface area, combinatorial parameters of the Poisson-Voronoi approximation have apparently not been studied in the literature. The general theory developed in Section 2 allows us to investigate such parameters. To state the result, for $\ell \in \{0,\dots,d-1\}$ we let $f_\lambda^{(\ell)}(A)$ be the number of ℓ-dimensional faces belonging to $\partial(\mathrm{PV}_\lambda(A))$. Note that $f_\lambda^{(0)}(A) = H_\lambda^{(0)}(A)$.

Theorem 1.3. *Let $\ell \in \{0,\dots,d-1\}$. Then there are constants $c_7 \in (0,\infty)$ and $c_{8,1}, c_{8,2} \in [0,\infty)$ depending only on the dimension d and on ℓ such that*

$$\mathbb{E}\, f_\lambda^{(\ell)}(A) \sim c_7\, \lambda^{1-\frac{1}{d}}\, \mathcal{H}^{d-1}_{\kappa,0}(\partial A)$$

and

$$\mathrm{Var}[f_\lambda^{(\ell)}(A)] \sim \lambda^{1-\frac{1}{d}} \left(c_{8,1}\,\mathcal{H}^{d-1}_{\kappa,0}(\partial A) + c_{8,2}\,\mathcal{H}^{d-1}_{\kappa^2,0}(\partial A)\right).$$

If ∂A contains a subset Γ of differentiability class C^2 with $\mathcal{H}^{d-1}(\Gamma) \in (0,\infty)$, and if $\kappa \equiv 1$, then $c_8 := c_{8,1} + c_{8,2}$ is strictly positive.

Next, we consider certain functionals of Voronoi cells intersecting only a part of the boundary of A. Formally, given an admissible set A and $A_0 \subset \partial A$ such that $\mathcal{H}^{d-1}(A_0) \in (0,\infty)$, define the Poissson-Voronoi zone $\mathrm{PVZ}_\lambda(A_0)$ of A_0 by

$$\mathrm{PVZ}_\lambda(A_0) := \bigcup_{\substack{x\in\mathcal{P}_\lambda \\ v(x)\cap A_0\neq\emptyset}} v(x)\,.$$

Given $\ell \in \{0,\ldots,d-1\}$, let $\widehat{f}_\lambda^{(\ell)}(A_0)$ denote the number of ℓ-dimensional faces of $\mathrm{PVZ}_\lambda(A_0)$. We emphasize that this construction is very similar to the construction of a zone in a hyperplane arrangement, see [16]. Following these classical ideas we define the complexity of $\mathrm{PVZ}_\lambda(A_0)$ as $\mathrm{Co}_\lambda(A_0) := \widehat{f}_\lambda^{(0)}(A_0) + \ldots + \widehat{f}_\lambda^{(d-1)}(A_0)$. The zone theorem in discrete geometry (see Theorem 6.4.1 in [16]) asserts that the complexity of a zone of an arbitrary hyperplane arrangement is of surface-order. Our next result shows a similar surface-order behaviour for the expectation and the variance in case of a random Poisson-Voronoi zone.

Theorem 1.4. *There are constants $c_9 \in (0,\infty)$ and $c_{10,1}, c_{10,2} \in [0,\infty)$ depending only on d such that*

$$\mathbb{E}\,\mathrm{Co}_\lambda(A_0) \sim c_9\,\lambda^{1-\frac{1}{d}}\,\mathcal{H}^{d-1}_{\kappa,0}(A_0)$$

and

$$\mathrm{Var}[\mathrm{Co}_\lambda(A_0)] \sim \lambda^{1-\frac{1}{d}}\left(c_{10,1}\,\mathcal{H}^{d-1}_{\kappa,0}(A_0) + c_{10,2}\,\mathcal{H}^{d-1}_{\kappa^2,0}(A_0)\right).$$

If A_0 contains a subset Γ of differentiability class C^2 with $\mathcal{H}^{d-1}(\Gamma) \in (0,\infty)$, and if $\kappa \equiv 1$, then $c_{10} := c_{10,1} + c_{10,2}$ is strictly positive.

Another application of our results concerns the iterated Poisson-Voronoi approximation, defined recursively as follows:

$$\mathrm{PV}_\lambda^{(1)}(A) := \mathrm{PV}_\lambda(A) \quad \text{and} \quad \mathrm{PV}_\lambda^{(n)}(A) := \mathrm{PV}_{n\lambda}(\mathrm{PV}_\lambda^{(n-1)}(A))$$

for integers $n \geq 1$ (note that the intensity used in the nth iteration is $n\lambda$, where $\lambda > 0$ is fixed). By $V_\lambda^{(n)}$, $S_\lambda^{(n)}$ and $f_\lambda^{\ell,(n)}$ we denote the volume, the surface area and the number of ℓ-dimensional faces ($\ell \in \{0,\ldots,d-1\}$) of the nth iterated Poisson-Voronoi approximation, respectively. Moreover, by $H_\lambda^{\ell,(n)}$ we indicate the ℓ-dimensional Hausdorff measure of the ℓ-skeleton of $\mathrm{PV}_\lambda^{(n)}(A)$, $\ell \in \{0,\ldots,d\}$. Note that our construction of the iterated Poisson-Voronoi approximation is close to that of so-called aggregate mosaics introduced in [29]. The expectation analysis of functionals of the iterated Poisson-Voronoi mosaic yields the following result. Variance asymptotics are less tractable and we shall omit them. For simplicity, we shall assume that the Poisson point process $\mathcal{P}_\lambda$ is homogeneous with $\kappa \equiv 1$.

Theorem 1.5. *Suppose that $\kappa \equiv 1$ and let c_1 and c_2 be the constants from Theorem 1.1, c_5 the constant from Theorem 1.2, and c_7 the constant from Theorem 1.3. Put $c_{2,n} := 1 + c_2 + c_2^2 + \ldots + c_2^{n-1}$ for integers $n \geq 1$. Then*

$$\begin{aligned}
\mathbb{E}\, V_\lambda^{(n)} - V(A) &\sim c_1\, c_{2,n}\, \lambda^{-\frac{1}{d}}\, \mathcal{H}^{d-1}(\partial A)\,, \\
\mathbb{E}\, S_\lambda^{(n)} - S(A) &\sim c_2\, c_{2,n}\, \lambda^{-\frac{1}{d}}\, \mathcal{H}^{d-1}_{\kappa,d-1}(\partial A)\,, \\
\mathbb{E}\, H_\lambda^{\ell,(n)} &\sim c_5\, c_{2,n}\, \lambda^{1-\frac{1}{d}-\frac{\ell}{d}}\, \mathcal{H}^{d-1}_{\kappa,\ell}(\partial A)\,, \\
\mathbb{E}\, f_\lambda^{\ell,(n)} &\sim c_7\, c_{2,n}\, \lambda^{1-\frac{1}{d}}\, \mathcal{H}^{d-1}_{\kappa,0}(\partial A)\,.
\end{aligned}$$

Remarks. **(i)** *Theorem 1.1 (related work).* The set $\mathrm{PV}_\lambda(A)$ was introduced in [13] where it was shown that $\lim_{\lambda\to\infty} \mathrm{Vol}(A\Delta A_\lambda) = 0$ almost surely, but only when $d = 1$. This almost sure limit was extended in [21] to all dimensions $d \geq 1$. When $\mathcal{P}_\lambda$ denotes a homogeneous Poisson point process on $\mathbb{R}^d$ having intensity λ, we have that $V_\lambda(A)$ is an unbiased estimator of $V(A)$ (cf. [25]), which makes $\mathrm{PV}_\lambda(A)$ of interest in image analysis, non-parametric statistics, and quantization; see also Section 1 of [13] and Section 1 of [9].

(ii) *Invariance of limits with respect to geometry.* The common thread linking our results is that the first- and second-order asymptotic behavior of our functionals are geometry independent. By this we mean that the mean and variance asymptotics are not influenced by the precise geometric structure of the given admissible set A, but are rather controlled only by the κ-weighted surface content of A.

(iii) *The constants in Theorems 1.1 – 1.5.* The explicit dependency of the constants $c_i, i \geq 1$, in Theorems 1.1 – 1.5 on the dimension d and the parameter ℓ is given explicitly in the general results of Section 2, especially the upcoming limits (2.16) and (2.17). More precisely, let $\mathcal{P}_1^{\mathrm{hom}}$ be a homogeneous Poisson point process on $\mathbb{R}^d$ of unit intensity and put $\mathbb{R}_+^{d-1} := \mathbb{R}^{d-1} \times \mathbb{R}^+$. Let

$$\mathrm{PV}(\mathbb{R}_+^{d-1}) := \bigcup_{x \in \mathcal{P}_1^{\mathrm{hom}} \cap \mathbb{R}_+^{d-1}} v(x)$$

be the Poisson-Voronoi approximation of $\mathbb{R}_+^{d-1}$. Then the general results show that the expectation and variance asymptotics are controlled by the κ-weighted surface content of A as well as by the expected behavior of metric and combinatorial parameters of the simpler object $\mathrm{PV}(\mathbb{R}_+^{d-1})$. Finding explicit numerical values for the constants $c_i, i \geq 1$, arising in expectation and variance asymptotics is a separate problem which we do not tackle here.

(iv) *Extensions of Theorems 1.1 – 1.5.* By Theorem 2.1 below, the expectation asymptotics in Theorems 1.1 – 1.5 may be upgraded to a weak law of large numbers holding in the L^1- and L^2-sense.

(v) *General surface-order results.* Although Theorems 1.1 – 1.5 only deal with statistics of the Poisson-Voronoi approximation, we emphasize that they follow from general theorems (presented in Section 2 below) for general surface-order stabilizing functionals. These general theorems are applicable in a wider context, establishing, for example, expectation and variance asymptotics for the number of maximal points in a random sample, as described in Remark (iii) after Theorem 2.2.

The rest of this paper is structured as follows. In Section 2 we make precise our framework, in particular we introduce the class of admissible sets and score functions. We also state there two general theorems which yield Theorems 1.1 – 1.5. Their proofs form the content of Section 3, while Section 4 contains the proofs of Theorems 1.1 – 1.5. Section 5 establishes the asserted variance lower bounds in Theorems 1.1 – 1.4.

2. Framework and general theorems

Let $\mathcal{P}_\lambda$ denote a Poisson point process on $\mathbb{R}^d$ for some $d \geq 2$ whose intensity measure has density $\lambda\kappa$ with respect to the Lebesgue measure on $\mathbb{R}^d$, where $\lambda \in (0,\infty)$ but now κ is a bounded function on $\mathbb{R}^d$ not necessarily bounded away from zero. Furthermore, let $A \subset \mathbb{R}^d$ be a closed set such that its boundary ∂A has finite $(d-1)$-dimensional Hausdorff measure. We consider in this section general statistics of the form

$$\sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda, \partial A)\,, \tag{2.1}$$

where ξ is a certain score function, which associates to a point $x \in \mathcal{P}_\lambda$ a real number, which is allowed to depend on the surrounding point configuration $\mathcal{P}_\lambda$ as well as on the set A via its boundary ∂A. To introduce a re-scaled version and to simplify notation we use the abbreviation $\xi_\lambda(x, \mathcal{P}_\lambda, \partial A) := \xi(\lambda^{1/d}x, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}(\partial A))$ and define

$$H^\xi(\mathcal{P}_\lambda, \partial A) := \sum_{x \in \mathcal{P}_\lambda} \xi_\lambda(x, \mathcal{P}_\lambda, \partial A)\,. \tag{2.2}$$

The focus of this paper is on score functions which depend on the geometry of the set A in that $\xi(x, \mathcal{P}_\lambda, \partial A)$ decays with the distance of x to ∂A. Moreover, we require ξ to satisfy a weak spatial dependency condition.

To make the framework precise we first introduce terminology, including the collection $\mathbf{A}(d)$ of admissible sets $A \subset \mathbb{R}^d$ as well as the collection Ξ of admissible score functions. The reader may wonder about our choice of admissible sets. The admissible sets described below have the attractive feature that their so-called extended support measures are 'well-behaved' and satisfy a Steiner-type formula (2.3), which is a far reaching consequence of the classic Steiner formula. This key formula, proved in [10], essentially replaces the co-area formula applicable in the surface-order asymptotics of functionals of sets A having a smooth boundary of bounded curvature [31].

A Steiner-type formula. Let $A \subset \mathbb{R}^d$ be a non-empty closed set and denote by $\mathrm{exo}(A)$ the exoskeleton of A, that is, the set of all $x \in \mathbb{R}^d \setminus A$ which do not have a unique nearest point in A. Then Theorem 1G in [8] says that $\mathcal{H}^d(\mathrm{exo}(A)) = 0$. Thus, $\mathcal{H}^d$-almost every point x in $\mathbb{R}^d \setminus A$ has a unique nearest point in A, denoted by $\pi_A(x)$. The (reduced) normal bundle $N(A) \subset \mathbb{R}^d \times \mathbb{S}^{d-1}$ of A is given by

$$N(A) := \left\{ \left(\pi_A(x), \frac{x - \pi_A(x)}{\|x - \pi_A(x)\|} \right) : x \in \mathbb{R}^d \setminus \big(A \cup \mathrm{exo}(A) \big) \right\} ,$$

where here and below $\| \cdot \|$ stands for the usual Euclidean distance and $\mathbb{S}^{d-1}$ stands for the Euclidean unit sphere in $\mathbb{R}^d$. Lemma 2.3 in [10] implies that $N(A)$ is a countably $(d-1)$-rectifiable subset of $\mathbb{R}^d \times \mathbb{S}^{d-1}$ in the sense of Federer [7, Paragraph 3.2.14].

Let A be as above. The reach function of A is a strictly positive function on $N(A)$ defined as

$$\delta(A, x, n) := \inf\{ r \geq 0 : x + rn \in \mathrm{exo}(A) \}$$

for all $(x, n) \in N(A)$. The reach of A is

$$\mathrm{reach}(A) := \inf\{ \delta(A, x, n) : (x, n) \in N(A) \}$$

with the convention that $\mathrm{reach}(A) = +\infty$ if $\delta(A, x, n) = +\infty$ for all $(x, n) \in N(A)$. The set A is said to be of positive reach if $\mathrm{reach}(A) \in (0, +\infty]$. In particular, if A is convex, then $\mathrm{reach}(A) = +\infty$, and vice versa. We also remark that any compact d-manifold with C^2-smooth boundary has positive reach, cf. [10].

If A^* denotes the closure of the complement of A, we see that $N(\partial A) := N(A) \cup N(A^*)$ and we define the extended normal bundle of A as $N_e(A) := N(A) \cup TN(A^*)$, where T is the reflection map $T : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}^d \times \mathbb{S}^{d-1}$, $(x, n) \mapsto (x, -n)$. Further, denote the reach function of A in this context by $\delta^+(A, \cdot\,, \cdot\,) \in [0, +\infty]$ and define the interior reach function $\delta^-(A, x, n) := -\delta(A^*, x, -n) \in [-\infty, 0]$ for $(x, n) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$.

From Theorem 5.2 in [10] we know that for each A as above there exist uniquely determined signed measures $\nu_0, \ldots, \nu_{d-1}$ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$, the so-called extended support measures of A, vanishing outside of $N_e(A)$, such that the Steiner-type formula

$$\int_{\mathbb{R}^d \setminus \partial A} f(x) \, \mathrm{d}x = \sum_{j=0}^{d-1} \omega_{d-j} \int_{N_e(A)} \int_{\delta^-(A,x,n)}^{\delta^+(A,x,n)} r^{d-j-1} f(x + rn) \, \mathrm{d}r \nu_j(\mathrm{d}(x, n)) \qquad (2.3)$$

holds for any non-negative measurable bounded function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support. Here, for integers $j \geq 0$, $\omega_j = j\kappa_j := 2\pi^{j/2}/\Gamma(j/2)$ stands for the surface content of the j-dimensional unit sphere. The signed measures $\nu_0, \ldots, \nu_{d-1}$ encode in some sense the singularities of the boundary of A. Although this is not visible in our notation, we emphasize that the measures $\nu_0, \ldots, \nu_{d-1}$ depend on A.

Admissible sets. Following [10], we denote by

$$\partial^+ A := \{ x \in \partial A : (x, n) \in N(A) \text{ for some } n \in \mathbb{S}^{d-1} \}$$

the positive boundary of A and define $\mathrm{Nor}(A,x) := \{n \in \mathbb{S}^{d-1} : (x,n) \in N(A)\}$ for $x \in \partial^+ A$. The normal cone at $x \in \partial^+ A$ is then $\mathrm{nor}(A,x) := \{an : a \geq 0,\ n \in \mathrm{Nor}(A,x)\}$ and we put

$$\partial^{++}A := \{x \in \partial^+ A : \dim(\mathrm{nor}(A,x)) = 1\}\,, \tag{2.4}$$

where $\dim(B)$ denotes the dimension of the affine hull of a set $B \subset \mathbb{R}^d$. Clearly, $\partial^{++}A$ is the disjoint union of $\partial^1 A$ and $\partial^2 A$, where

$$\partial^k A = \{x \in \partial^{++}A : \mathrm{card}(\mathrm{Nor}(A,x)) = k\}\,, \qquad k \in \{1,2\}\,. \tag{2.5}$$

Let us recall from [14] that a closed set $A \subset \mathbb{R}^d$ is called *gentle* if

(i) $\mathcal{H}^{d-1}(N_e(A) \cap (B \times \mathbb{S}^{d-1})) < \infty$ for all bounded Borel sets $B \subset \mathbb{R}^d$,
(ii) for $\mathcal{H}^{d-1}$-almost all $x \in \partial A$ there are non-degenerate balls B_1 and B_2 containing x and satisfying $B_1 \subset A$ and $\mathrm{int}(B_2) \subset \mathbb{R}^d \setminus A$, where $\mathrm{int}(B_2)$ stands for the interior of B_2.

These assumptions ensure, for example that $\mathcal{H}^{d-1}(\partial A \setminus \partial^+ A) = 0$, cf. Equation (5) in [14]. The positive boundary of any closed subset of $\mathbb{R}^d$ is $(\mathcal{H}^{d-1}, d-1)$-rectifiable [10] and thus the boundary of every gentle set is $(\mathcal{H}^{d-1}, d-1)$-rectifiable, too. In other words, there are Lipschitz maps $f_i : \mathbb{R}^{d-1} \to \mathbb{R}^d,\ i = 1,2,\ldots$ such that $\mathcal{H}^{d-1}(\partial A \setminus \cup_{i\geq 1} f_i(\mathbb{R}^{d-1})) = 0$, see e.g. [7, Paragraph 3.2.14]. In particular, at $\mathcal{H}^{d-1}$-almost every $x \in \partial A$ there is a unique tangent hyperplane denoted by $T_x := T_x(\partial A)$.

Moreover, we recall from [14] that the extended support measures ν_j of gentle sets have locally finite total variation measures $|\nu_j|$ for all $j \in \{0, \ldots, d-1\}$. In particular, $|\nu_j|(N_e(A)) < \infty$ if A is compact.

We now define the class $\mathbf{A}(d)$ of *admissible* sets to be the class of compact sets $A \subset \mathbb{R}^d$ which are gentle, regular closed and satisfy $\mathcal{H}^{d-1}(\partial^2 A) = 0$. (Recall that a set is regular closed if it coincides with the closure of its interior.) Here, the assumption that $\mathcal{H}^{d-1}(\partial^2 A) = 0$ simplifies the structure of the measure ν_{d-1}, to be exploited later. Regularity excludes sets with lower-dimensional 'tentacles' attached (e.g. a ball with attached line segments).

The class of gentle and compact sets is rather general and the support measures ν_j of such sets simplify to well known objects in special situations. We introduce the following classes of sets:

- $\mathcal{K}^d$ is the class of convex bodies in $\mathbb{R}^d$, i.e., compact convex sets $A \subset \mathbb{R}^d$ with non-empty interior,
- $\mathcal{R}^d$ is the convex ring, consisting of finite unions of convex bodies in $\mathbb{R}^d$,
- $\mathcal{M}^d$ denotes the class of compact d-dimensional manifolds in $\mathbb{R}^d$ with twice differentiable boundary,
- $\mathcal{P}^d$ is the family of compact sets $A \subset \mathbb{R}^d$ with positive reach having non-empty interior,
- $\mathcal{UP}^d$ denotes the class of all subsets $A = A_1 \cup \ldots \cup A_n \subset \mathbb{R}^d$, $n \geq 1$, for sets $A_1, \ldots, A_n \in \mathcal{P}^d$ and such that $\bigcap_{i\in I} A_i \in \mathcal{P}^d$ for any $I \subset \{1, \ldots, n\}$.

These classes satisfy the inclusions: $\mathcal{K}^d \subset \mathcal{P}^d$, $\mathcal{K}^d \subset \mathcal{R}^d$, $\mathcal{P}^d \subset \mathcal{UP}^d$, $\mathcal{M}^d \subset \mathcal{P}^d$ and $\mathcal{R}^d \subset \mathcal{UP}^d$. If $A \in \mathcal{K}^d$ then the extended support measures ν_j are related to the generalized curvature measures of A considered in convex geometry, cf. [26]. A similar comment applies if $A \in \mathcal{P}^d$ is a set with positive reach, for which curvature measures have been introduced in [6]. In both cases it holds that $\partial^+ A = \partial A$. If $A \in \mathcal{K}^d$ then A satisfies $\mathcal{H}^{d-1}(\partial^2 A) = 0$. The set classes $\mathcal{K}^d$ and $\mathcal{P}^d$ only contain gentle sets. For the set classes $\mathcal{R}^d$ and $\mathcal{UP}^d$ curvature measures are defined by additive extension, while for $\mathcal{M}^d$ curvature measures are defined via classical differential-geometric methods, see Section 3 in [10] for a detailed discussion. Moreover, for sets $A \in \mathcal{UP}^d$ we have that $\mathcal{H}^{d-1}(\partial A \setminus \partial^+ A) = 0$ (see [10, p. 251]). Furthermore, if $A \in \mathcal{R}^d$ is regular closed, then A is gentle according to [14, Proposition 2]. Additionally, many $\mathcal{UP}^d$-sets (namely those admitting a so-called non-osculating representation) are gentle by Proposition 3 in [14].

Admissible score functions. We next consider the collection Ξ of admissible score functions. By this we mean the collection of all real-valued Borel measurable functions $\xi(x, \mathcal{X}, \partial A)$ defined on triples $(x, \mathcal{X}, \partial A)$, where $\mathcal{X} \subset \mathbb{R}^d$ is locally finite, $x \in \mathcal{X}$, $A \in \mathbf{A}(d)$, and such that ξ is translation and rotation invariant. By the latter two properties we respectively mean that $\xi(x, \mathcal{X}, \partial A) = \xi(x + z, \mathcal{X} + z, \partial A + z)$ and that $\xi(x, \mathcal{X}, \partial A) = \xi(\vartheta x, \vartheta \mathcal{X}, \vartheta(\partial A))$ for all $z \in \mathbb{R}^d$, rotations $\vartheta \in SO(d)$ and input $(x, \mathcal{X}, \partial A)$. If $x \notin \mathcal{X}$, we abbreviate $\xi(x, \mathcal{X} \cup \{x\}, \partial A)$ by $\xi(x, \mathcal{X}, \partial A)$.

We recall now the concept of a stabilizing functional which was introduced in [22, 23, 24] after earlier works [12, 15]; see also the surveys [27, 30]. Roughly speaking, a functional stabilizes if its value at a given point only depends on a local random neighborhood and is unaffected by changes in point configurations outside of it. Following [31] we need to go a step further in the standard framework to account for the dependency of functionals $\xi \in \Xi$ on surfaces.

To make this precise, denote by $B_r(x)$ the Euclidean ball of radius $r \in (0, \infty)$ and centre $x \in \mathbb{R}^d$ and by $\mathcal{P}_\tau^{\mathrm{hom}}$ a homogeneous Poisson point processes on $\mathbb{R}^d$ of intensity $\tau \in (0, \infty)$. Say that $\xi \in \Xi$ is homogeneously stabilizing if for all $\tau \in (0, \infty)$ and all $(d-1)$-dimensional hyperplanes H, there is an almost surely finite random variable $R := R(\xi, \mathcal{P}_\tau^{\mathrm{hom}}, H)$ depending on ξ, $\mathcal{P}_\tau^{\mathrm{hom}}$ and H, the so-called radius of stabilization, such that

$$\xi(\mathbf{0}, \mathcal{P}_\tau^{\mathrm{hom}} \cap B_R(\mathbf{0}), H) = \xi(\mathbf{0}, (\mathcal{P}_\tau^{\mathrm{hom}} \cap B_R(\mathbf{0})) \cup \mathcal{A}, H) \tag{2.6}$$

for all locally finite sets $\mathcal{A} \subset B_R(\mathbf{0})^c$, where $\mathbf{0}$ stands for the origin in $\mathbb{R}^d$. Given (2.6), the definition of ξ extends to Poisson input on all of $\mathbb{R}^d$, that is

$$\xi(\mathbf{0}, \mathcal{P}_\tau^{\mathrm{hom}}, H) = \lim_{r \to \infty} \xi(\mathbf{0}, \mathcal{P}_\tau^{\mathrm{hom}} \cap B_r(\mathbf{0}), H) \,.$$

Given $A \in \mathbf{A}(d)$, say that ξ is exponentially stabilizing with respect to the pair $(\mathcal{P}_\lambda, \partial A)$ if for all $x \in \mathbb{R}^d$ there is a random variable $R := R(\xi, x, \mathcal{P}_\lambda, \partial A)$, also called a radius of stabilization, taking values in $[0, \infty)$ with probability one, such that

$$\xi_\lambda(x, \mathcal{P}_\lambda \cap B_{\lambda^{-1/d} R}(x), \partial A) = \xi_\lambda(x, (\mathcal{P}_\lambda \cap B_{\lambda^{-1/d} R}(x)) \cup \mathcal{A}, \partial A) \tag{2.7}$$

for all locally finite $\mathcal{A} \subset \mathbb{R}^d \setminus B_{\lambda^{-1/d}R}(x)$, and the tail probability satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\lambda>0, x\in\mathbb{R}^d} \mathbb{P}[R(\xi, x, \mathcal{P}_\lambda, \partial A) > t] < 0\,.$$

Surface-order growth for the sums (2.2) involves finiteness of the integrated score $\xi_\lambda(x+r\lambda^{-1/d}n, \mathcal{P}_\lambda, \partial A)$ over $r \in \mathbb{R}$. Thus, it is natural to require the following condition, see [31]. Given $A \in \mathbf{A}(d)$ and $p \in [1,\infty)$, say that ξ satisfies the *p-th moment condition* with respect to ∂A if there is a bounded integrable function $G^{\xi,p} := G^{\xi,p,\partial A} : \mathbb{R} \to \mathbb{R}^+$ with $\int_{-\infty}^{\infty} r^{d-1}(G^{\xi,p}(r))^{1/p}\,\mathrm{d}r < \infty$ and such that for all $r \in \mathbb{R}$ we have

$$\sup_{z\in\mathbb{R}^d\cup\emptyset}\ \sup_{(x,n)\in N_e(A)}\ \sup_{\lambda>0} \mathbb{E}\,|\xi_\lambda(x+r\lambda^{-1/d}n, \mathcal{P}_\lambda \cup z, \partial A)|^p \le G^{\xi,p}(|r|)\,. \tag{2.8}$$

Given $A \in \mathbf{A}(d)$, recall for $\mathcal{H}^{d-1}$-almost all $x \in \partial A$ that $T_x := T_x(\partial A)$ is the unique hyperplane tangent to ∂A at x. For $x \in \partial A$, we put $H_x := T_{\mathbf{0}}(\partial A - x)$. The score ξ is said to be *well approximated by $\mathcal{P}_\lambda$ input on half-spaces* if for all $A \in \mathbf{A}(d)$, almost all $x \in \partial A$, and all $w \in \mathbb{R}^d$, we have

$$\lim_{\lambda\to\infty} \mathbb{E}\,|\xi(w, \lambda^{1/d}(\mathcal{P}_\lambda - x), \lambda^{1/d}(\partial A - x)) - \xi(w, \lambda^{1/d}(\mathcal{P}_\lambda - x), H_x)| = 0\,. \tag{2.9}$$

General theorems giving first- and second-order asymptotics. The results asserted in Section 1 are consequences of general limit theorems giving expectation and variance asymptotics for the statistics (2.2). We first describe the general theory and then, in Section 4, show how to deduce the assertions of Section 1. The general limit theorems given here extend Theorems 1.1 and 1.2 in [31] to the class of admissible sets and they yield the first- and second-order asymptotics for statistics of other surfaces, as discussed in Remark (iii) below.

For a score function $\xi \in \Xi$ we put

$$\mu(\xi, \partial A) := \int_{\partial^1 A} \int_{-\infty}^{\infty} \mathbb{E}\xi(\mathbf{0} + sn, \mathcal{P}^{\mathrm{hom}}_{\kappa(x)}, \mathbb{R}^{d-1})\,\kappa(x)\,\mathrm{d}s\,\mathcal{H}^{d-1}(\mathrm{d}x)\,, \tag{2.10}$$

where n is the unique unit normal at $\mathbf{0}$ with respect to $\mathbb{R}^{d-1}$. We now state a general result giving expectation asymptotics for sums of score functions. Let $\mathcal{C}(\partial A)$ denote the set of functions on $\mathbb{R}^d$ which are continuous at all points $x \in \partial A$.

Theorem 2.1. *Let $A \in \mathbf{A}(d)$ and $\kappa \in \mathcal{C}(\partial A)$. Suppose that $\xi \in \Xi$ is homogeneously stabilizing (2.6), satisfies the moment condition (2.8) for some $p \in [1,\infty)$, and is well approximated by $\mathcal{P}_\lambda$ input on half-spaces as at (2.9). Then for $m \in \{1,2\}$, we have the following weak law of large numbers:*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} H^\xi(\mathcal{P}_\lambda, \partial A) = \mu(\xi, \partial A) \ \text{ in } \ L^m\,. \tag{2.11}$$

Next, we turn to variance asymptotics and define for $x, x' \in \mathbb{R}^d$, $\tau \in (0,\infty)$, and all $(d-1)$-dimensional hyperplanes H,

$$\begin{aligned} c^{\xi}(x, x'; \mathcal{P}_\tau^{\mathrm{hom}}, H) := &\mathbb{E}\, \xi(x, \mathcal{P}_\tau^{\mathrm{hom}} \cup \{x'\}, H)\xi(x', \mathcal{P}_\tau^{\mathrm{hom}} \cup \{x\}, H) \\ &- \mathbb{E}\, \xi(x, \mathcal{P}_\tau^{\mathrm{hom}}, H)\mathbb{E}\, \xi(x', \mathcal{P}_\tau^{\mathrm{hom}}, H)\,. \end{aligned}$$

Moreover, define $\sigma^2(\xi, \partial A)$ by

$$\begin{aligned} &\sigma^2(\xi, \partial A) := \mu(\xi^2, \partial A) + \\ &\int_{\partial^1 A} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c^{\xi}(\mathbf{0} + rn, p + sn; \mathcal{P}_{\kappa(x)}^{\mathrm{hom}}, \mathbb{R}^{d-1}) \kappa(x)^2 \mathrm{d}s \mathrm{d}r \mathrm{d}p \mathcal{H}^{d-1}(\mathrm{d}x)\,. \end{aligned} \tag{2.12}$$

The following general result gives variance asymptotics for sums of score functions.

Theorem 2.2. *Let $A \in \mathbf{A}(d)$ and $\kappa \in \mathcal{C}(\partial A)$. We assume that $\xi \in \Xi$ is homogeneously stabilizing (2.6), exponentially stabilizing (2.7), satisfies the moment condition (2.8) for some $p \in (2,\infty)$ and is well approximated by $\mathcal{P}_\lambda$ input on half-spaces as at (2.9). Then*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\, \mathrm{Var}[H^{\xi}(\mathcal{P}_\lambda, \partial A)] = \sigma^2(\xi, \partial A)\,. \tag{2.13}$$

Some of the applications presented in Section 1 require the limit theory for the non re-scaled sums $\sum_{x\in\mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda, \partial A)$. To state the result in this case, call a score function ξ *homogeneous of order* $\gamma \in \mathbb{R}$ if for all $a \in (0,\infty)$,

$$\xi(ax, a\mathcal{X}, a(\partial A)) = a^{\gamma}\, \xi(x, \mathcal{X}, \partial A)\,.$$

When ξ is homogeneous of order γ it follows that

$$\sum_{x\in\mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda, \partial A) = \lambda^{-\gamma/d}\, H^{\xi}(\mathcal{P}_\lambda, \partial A)\,.$$

Homogeneity, together with the distributional identity $\mathcal{P}_{\kappa(x)} \stackrel{\mathcal{D}}{=} \kappa(x)^{-1/d}\, \mathcal{P}_1$ gives

$$\begin{aligned} \mu(\xi, \partial A) &= \int_{-\infty}^{\infty} \mathbb{E}\xi(\mathbf{0} + sn, \mathcal{P}_1^{\mathrm{hom}}, \mathbb{R}^{d-1})\, \mathrm{d}s \int_{\partial^1 A} \kappa(x)^{1-\gamma/d}\, \mathcal{H}^{d-1}(\mathrm{d}x) \\ &= \int_{-\infty}^{\infty} \mathbb{E}\xi(\mathbf{0} + sn, \mathcal{P}_1^{\mathrm{hom}}, \mathbb{R}^{d-1})\, \mathrm{d}s \cdot \mathcal{H}_{\kappa,\gamma}^{d-1}(\partial A) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \sigma^2(\xi, \partial A) = &\int_{-\infty}^{\infty} \mathbb{E}\xi^2(\mathbf{0} + sn, \mathcal{P}_1^{\mathrm{hom}}, \mathbb{R}^{d-1})\, \mathrm{d}s \int_{\partial^1 A} \kappa(x)^{1-2\gamma/d}\, \mathcal{H}^{d-1}(\mathrm{d}x) \\ &+ \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c^{\xi}(\mathbf{0} + rn, p + sn; \mathcal{P}_1^{\mathrm{hom}}, \mathbb{R}^{d-1})\, \mathrm{d}s \mathrm{d}r \mathrm{d}p \int_{\partial^1 A} \kappa(x)^{2-2\gamma/d}\, \mathcal{H}^{d-1}(\mathrm{d}x) \end{aligned}$$

$$= \int_{-\infty}^{\infty} \mathbb{E}\xi^2(\mathbf{0} + sn, \mathcal{P}_1^{\mathrm{hom}}, \mathbb{R}^{d-1})\,\mathrm{d}s \cdot \mathcal{H}^{d-1}_{\kappa,2\gamma}(\partial A) + \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c^{\xi}(\mathbf{0} + rn, p + sn; \mathcal{P}_1^{\mathrm{hom}}, \mathbb{R}^{d-1})\,\mathrm{d}s\mathrm{d}r\mathrm{d}p \cdot \mathcal{H}^{d-1}_{\kappa^2,\gamma}(\partial A)\,. \tag{2.15}$$

Consequently, with $\mu(\xi, \partial A)$ and $\sigma^2(\xi, \partial A)$ as in (2.14) and (2.15) respectively, we have under the conditions of Theorems 2.1 and 2.2 that

$$\lim_{\lambda\to\infty} \lambda^{-(d-1-\gamma)/d} \sum_{x\in\mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda, \partial A) = \mu(\xi, \partial A) \tag{2.16}$$

in L^m for $m \in \{1, 2\}$, and

$$\lim_{\lambda\to\infty} \lambda^{-(d-1-2\gamma)/d}\,\mathrm{Var} \sum_{x\in\mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda, \partial A) = \sigma^2(\xi, \partial A)\,. \tag{2.17}$$

Remarks. **(i)** *Convergence of random measures.* The methods presented here also yield expectation and variance asymptotics for integrals of the empirical measures

$$\sum_{x\in\mathcal{P}_\lambda} \xi_\lambda(x, \mathcal{P}_\lambda, \partial A)\delta_x$$

against elements of $\mathcal{C}(\partial A)$ (here, δ_x stands for the unit-mass Dirac measure at x). The details of this extension are straightforward and may be found in e.g. [30], which deals with volume-order asymptotics for sums of score functions.

(ii) *Central limit theorems.* Say that ξ decays *exponentially fast* with respect to the distance to ∂A if for all $p \in [1, \infty)$ the function $G^{\xi,p}$ defined at (2.8) satisfies

$$\limsup_{|u|\to\infty} |u|^{-1} \log G^{\xi,p}(|u|) < 0\,. \tag{2.18}$$

Let $\Phi(\,\cdot\,)$ denote the distribution function of a standard normal random variable. If $\xi \in \Xi$ decays exponentially fast as in (2.18) and if ξ satisfies the moment condition (2.8) with $p = 3$, then by Theorem 1.3 of [31], the statistics (2.2) satisfy a central limit theorem

$$\sup_{x\in\mathbb{R}} \Big| P\Big[\frac{H^\xi(\mathcal{P}_\lambda, \partial A) - \mathbb{E}\,H^\xi(\mathcal{P}_\lambda, \partial A)}{\sqrt{\mathrm{Var}[H^\xi(\mathcal{P}_\lambda, \partial A)]}}\Big] - \Phi(x)\Big| \le r(\lambda)$$

with rate function

$$r(\lambda) := c(\log\lambda)^{3d+1}\lambda^{(d-1)/d}(\mathrm{Var}[H^\xi(P\lambda, \partial A)])^{3/2}\,,$$

where $c > 0$ is a constant not depending on λ. In particular, if $\sigma^2(\xi, \partial A)$ is strictly positive, then $r(\lambda) = c(\log\lambda)^{3d+1}\lambda^{-(d-1)/2d}$. This is the case for the examples in Section 1, provided that $\kappa \equiv 1$ and that ∂A contains a C^2-smooth subset with positive $\mathcal{H}^{d-1}$-measure.

(iii) *Further applications of general results.* Theorems 2.1 and 2.2 have applications to statistics of surfaces going beyond those arising in Poisson-Voronoi approximation. For instance, these general theorems provide the limit theory for functionals of surfaces of germ-grain models, including for example the limit theory for the number of exposed tangent points to Boolean models, as described in Section 3.2 of [19]. Another application of the general theory involves the number of *maximal points* in a sample, which goes as follows. A point $x \in \mathcal{P}_\lambda$ is called maximal if the Minkowski sum $(\mathbb{R}_+)^d \oplus x$ contains no other point of $\mathcal{P}_\lambda$ besides x, i.e., if $((\mathbb{R}_+)^d \oplus x) \cap \mathcal{P}_\lambda = \{x\}$. The number M_λ of maximal points of $\mathcal{P}_\lambda$ has attracted considerable interest in the literature, see [1, 2, 3, 5, 11, 31]. These works restrict to domains A that are either piecewise linear, convex or smooth. We may use Theorems 2.1 and 2.2 to unify and extend these results to domains A which are admissible sets, as illustrated by the following statement, whose proof follows from modifications of the proof of Theorem 2.5 in [31] and is left to the reader. Let κ be a density supported on $A := \{(u, v) \in \mathbb{R}^{d-1} \times \mathbb{R} : u \in D,\ 0 \le v \le f(u)\}$, where $D \subset \mathbb{R}^{d-1}$ and $f : D \to \mathbb{R}$, and assume that A is an admissible set, i.e., $A \in \mathbf{A}(d)$. We further assume that the partial derivatives of f exist a.e. and are bounded away from zero and infinity. If $\mathcal{P}_\lambda$ is a Poisson point process whose intensity measure has density $\lambda\kappa$ with respect to Lebesgue measure then there are constants $c_{11} \in (0, \infty)$ and $c_{12} \in [0, \infty)$ depending only on d, κ and A such that

$$\mathbb{E}\, M_\lambda \sim c_{11} \lambda^{1-\frac{1}{d}} \qquad \text{and} \qquad \mathrm{Var}[M_\lambda] \sim c_{12} \lambda^{1-\frac{1}{d}}\,.$$

3. Proofs of Theorems 2.1 and 2.2

To keep the paper self-contained, we give three preparatory lemmas pertaining to the re-scaled scores $\xi_\lambda, \lambda > 0$. These are re-formulations of Lemmas 3.1 – 3.3 in [31], which we adopt to our more general set-up. The following lemmas do not require continuity of κ but instead use that a.e. $x \in \mathbb{R}^d$ is a Lebesgue point of κ, that is to say

$$\frac{1}{\varepsilon^d} \int_{B_\varepsilon(x)} |\kappa(y) - \kappa(x)| \, \mathrm{d}y$$

tends to zero as $\varepsilon \downarrow 0$. Given $x \in \partial^1 A$, with $\partial^1 A$ defined at (2.5), recall that $H_x := T_{\mathbf{0}}(\partial A - x)$ is the unique tangent hyperplane to $\partial A - x$ at $\mathbf{0}$ with unit normal $n(x)$. Let $\mathbf{0}_x$ denote a point at the origin of H_x.

Lemma 1. *Fix $A \in \mathbf{A}(d)$. Assume that ξ is homogeneously stabilizing as at (2.6), satisfies the moment condition (2.8) for some $p \in (1, \infty)$ and is well approximated by $\mathcal{P}_\lambda$ input on half-spaces (2.9). Then for all $x \in \partial^1 A$, $w \in \mathbb{R}^d$, and $r \in \mathbb{R}$ we have*

$$\lim_{\lambda \to \infty} \mathbb{E}\, \xi_\lambda(x + r\lambda^{-1/d} n(x) + \lambda^{-1/d} w, \mathcal{P}_\lambda, \partial A) = \mathbb{E}\, \xi(\mathbf{0}_x + rn(x) + w, \mathcal{P}^{\mathrm{hom}}_{\kappa(x)}, H_x)\,. \quad (3.1)$$

Lemma 2. *Fix $A \in \mathbf{A}(d)$. Assume that ξ is homogeneously stabilizing as at (2.6), satisfies the moment condition (2.8) for some $p \in (2,\infty)$, and is well approximated by $\mathcal{P}_\lambda$ input on half-spaces (2.9). Given $x \in \partial^1 A$, $v \in \mathbb{R}^d$, and $r \in \mathbb{R}$, we put for $\lambda \in (0,\infty)$,*

$$\begin{aligned} X_\lambda &:= \xi_\lambda(x + r\lambda^{-1/d}n(x), \mathcal{P}_\lambda \cup \{x + r\lambda^{-1/d}n(x) + \lambda^{-1/d}v\}, \partial A)\,, \\ Y_\lambda &:= \xi_\lambda(x + r\lambda^{-1/d}n(x) + \lambda^{-1/d}v, \mathcal{P}_\lambda \cup \{x + r\lambda^{-1/d}n(x)\}, \partial A)\,, \\ X &:= \xi(\mathbf{0}_x + rn(x), \mathcal{P}^{\mathrm{hom}}_{\kappa(x)} \cup \{\mathbf{0}_x + rn(x) + v\}, H_x)\,, \\ Y &:= \xi(\mathbf{0}_x + rn(x) + v, \mathcal{P}^{\mathrm{hom}}_{\kappa(x)} \cup \{\mathbf{0}_x + rn(x)\}, H_x)\,. \end{aligned}$$

Then $\lim_{\lambda\to\infty} \mathbb{E}\,[X_\lambda Y_\lambda] = \mathbb{E}\,[XY]$.

Lemma 3. *Fix $A \in \mathbf{A}(d)$. Let ξ be exponentially stabilizing as at (2.7) and assume the moment condition (2.8) holds for some $p \in (2,\infty)$. Then there is a constant $C \in (0,\infty)$ such that for all $w, v \in \mathbb{R}^d$ and $\lambda \in (0,\infty)$, we have*

$$\begin{aligned} \big|\mathbb{E}\,\xi_\lambda(w, \mathcal{P}_\lambda \cup \{w + \lambda^{-1/d}v\}, \partial A)&\xi_\lambda(w + \lambda^{-1/d}v, \mathcal{P}_\lambda \cup \{w\}, \partial A) \\ &- \mathbb{E}\,\xi_\lambda(w, \mathcal{P}_\lambda, \partial A)\mathbb{E}\,\xi_\lambda(w + \lambda^{-1/d}v, \mathcal{P}_\lambda, \partial A)\big| \\ \le C(\mathbb{E}\,\xi_\lambda(w, &\mathcal{P}_\lambda \cup \{w + \lambda^{-1/d}v\}, \partial A)^p)^{1/p} \\ &\times (\mathbb{E}\,\xi_\lambda(w + \lambda^{-1/d}v, \mathcal{P}_\lambda \cup \{w\}, \partial A)^p)^{1/p}\,\exp(-C^{-1}\|v\|)\,. \end{aligned}$$

In particular, there is a constant $c \in (0,\infty)$ such that if $w = x + r\lambda^{-1/d}n(x)$, then

$$\begin{aligned} \big|\mathbb{E}\,\xi_\lambda(w, \mathcal{P}_\lambda \cup \{w + \lambda^{-1/d}v\}, \partial A)&\xi_\lambda(w + \lambda^{-1/d}v, \mathcal{P}_\lambda \cup \{w\}, \partial A) \\ - \mathbb{E}\,\xi_\lambda(w, \mathcal{P}_\lambda, \partial A)\mathbb{E}\,\xi_\lambda(w + \lambda^{-1/d}v, &\mathcal{P}_\lambda, \partial A)\big| \le c\,G^{\xi,p}(|r|)^{1/p}\exp(-c^{-1}\|v\|)\,. \end{aligned}$$

Proof. The first asserted inequality follows as in either Lemma 4.2 of [20] or Lemma 4.1 of [4]. The second assertion follows from the first assertion together with the moment condition (2.8). □

Given these auxiliary lemmas, we may now prove the general results.

Proof of Theorem 2.1. To show (2.11), it is enough to show the expectation asymptotics

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\,\mathbb{E}\sum_{x\in\mathcal{P}_\lambda} \xi_\lambda(x, \mathcal{P}_\lambda, \partial A) = \mu(\xi, \partial A) \tag{3.2}$$

and then follow the method of proof of Theorem 1.1 of [31] to deduce L^m-convergence for $m \in \{1,2\}$.

To show (3.2), we first apply the Mecke identity [26, Theorem 3.2.5] for Poisson point processes to obtain

$$\lambda^{-(d-1)/d}\,\mathbb{E}\sum_{x\in\mathcal{P}_\lambda}\xi_\lambda(x,\mathcal{P}_\lambda,\partial A)=\lambda^{-(d-1)/d}\int_{\mathbb{R}^d}\mathbb{E}\xi_\lambda(x,\mathcal{P}_\lambda,\partial A)\,\lambda\kappa(x)\,\mathrm{d}x$$
$$=\lambda^{1/d}\int_{\mathbb{R}^d}\mathbb{E}\xi_\lambda(x,\mathcal{P}_\lambda,\partial A)\,\kappa(x)\,\mathrm{d}x\,;$$

recall that we write $\xi_\lambda(x,\mathcal{P}_\lambda,\partial A)$ instead of $\xi_\lambda(x,\mathcal{P}_\lambda\cup\{x\},\partial A)$ if $x\notin\mathcal{P}_\lambda$. We now use the Steiner-type formula (2.3) to re-write the last integral as

$$\lambda^{1/d}\sum_{j=0}^{d-1}\omega_{d-j}\int_{N_e(A)}\int_{T(x,n)}r^{d-j-1}\,\mathbb{E}\xi_\lambda(x+rn,\mathcal{P}_\lambda,\partial A)\,\kappa(x+rn)\,\mathrm{d}r\,\nu_j(\mathrm{d}(x,n))\,,$$

where for fixed $(x,n)\in N_e(A)$, $T(x,n):=[\delta^-(A,x,n),\delta^+(A,x,n)]$. Upon the substitution $r=\lambda^{-1/d}r'$ we obtain that $\lambda^{-(d-1)/d}\,\mathbb{E}\sum_{x\in\mathcal{P}_\lambda}\xi_\lambda(x,\mathcal{P}_\lambda,\partial A)$ equals

$$\sum_{j=0}^{d-1}\omega_{d-j}\lambda^{-(d-1-j)/d}\int_{N_e(A)}\int_{\lambda^{1/d}T(x,n)}(r')^{d-j-1}\,\mathbb{E}\xi_\lambda(x+\lambda^{-1/d}r'n,\mathcal{P}_\lambda,\partial A)$$
$$\times\,\kappa(x+\lambda^{-1/d}r'n)\,\mathrm{d}r'\,\nu_j(\mathrm{d}(x,n))\,. \tag{3.3}$$

To simplify the notation, write r for r'. By the moment assumption (2.8) with $p=1$, we conclude that, for each $j\in\{0,\dots,d-1\}$, the integrand is bounded by the product $|r|^{d-j-1}\,G^{\xi,1}(|r|)\,\|\kappa\|_\infty$, implying that

$$\left|\int_{N_e(A)}\int_{\lambda^{1/d}T(x,n)}r^{d-j-1}\,\mathbb{E}\xi_\lambda(x+\lambda^{-1/d}rn,\mathcal{P}_\lambda,\partial A)\,\kappa(x+\lambda^{-1/d}rn)\,\mathrm{d}r\,\nu_j(\mathrm{d}(x,n))\right|$$

$$\le\int_{N_e(A)}\int_{-\infty}^{\infty}r^{d-j-1}\,G^{\xi,1}(|r|)\,\|\kappa\|_\infty\,\mathrm{d}r\,|\nu_j|(\mathrm{d}(x,n))$$

$$=\|\kappa\|_\infty\,|\nu_j|(N_e(A))\int_{-\infty}^{\infty}r^{d-j-1}\,G^{\xi,1}(|r|))\,\mathrm{d}r\,.$$

The integral $\int_{-\infty}^{\infty}r^{d-j-1}\,G^{\xi,1}(|r|))\,\mathrm{d}r$ is finite by assumption. Moreover, $\|\kappa\|_\infty<\infty$ by assumption and $|\nu_j|(N_e(A))<\infty$ since $A\in\mathbf{A}(d)$. Consequently, taking the limit in (3.3) as $\lambda\to\infty$, it follows by the dominated convergence theorem that only the term $j=d-1$ remains:

$$\lim_{\lambda\to\infty}\lambda^{-(d-1)/d}\,\mathbb{E}\sum_{x\in\mathcal{P}_\lambda}\xi_\lambda(x,\mathcal{P}_\lambda,\partial A)$$
$$=2\int_{N_e(A)}\int_{-\infty}^{\infty}\lim_{\lambda\to\infty}\mathbb{E}\xi_\lambda(x+\lambda^{-1/d}rn,\mathcal{P}_\lambda,\partial A) \tag{3.4}$$
$$\times\,\kappa(x+\lambda^{-1/d}rn)\mathbf{1}(r\in\lambda^{1/d}T(x,n))\,\mathrm{d}r\,\nu_{d-1}(\mathrm{d}(x,n))\,.$$

Here, we use the identity $\omega_1 = 2$ and we also use that $\lim_{\lambda\to\infty} \lambda^{1/d} T(x,n) = (-\infty,\infty)$, which holds by construction of $N_e(A)$, where the exoskeleton has been excluded. By continuity of κ on ∂A, we have $\lim_{\lambda\to\infty} \kappa(x+\lambda^{-1/d} rn) = \kappa(x)$. Finally, consider the limit

$$\lim_{\lambda\to\infty} \mathbb{E}\xi_\lambda(x+\lambda^{-1/d} rn, \mathcal{P}_\lambda, \partial A)\,.$$

To identify it, we use translation invariance and the definition of ξ_λ, and write

$$\begin{aligned}\xi_\lambda(x+\lambda^{-1/d} rn, \mathcal{P}_\lambda, \partial A) &= \xi_\lambda(\mathbf{0}_x + \lambda^{-1/d} rn, \mathcal{P}_\lambda - x, \partial A - x)\\ &= \xi(\mathbf{0}_x + rn, \lambda^{1/d}(\mathcal{P}_\lambda - x), \lambda^{1/d}(\partial A - x))\,.\end{aligned}$$

The measure ν_{d-1} concentrates, according to the discussion around Proposition 4.1 of [10, Section 4], on the subset ∂A^{++} of the boundary ∂A where the normal cone is one dimensional; recall (2.4). Moreover, since $A \in \mathbf{A}(d)$, the measure ν_{d-1} in fact concentrates on the subset $\partial^1 A \subset \partial^{++} A$ (see (2.5)), that is to say, on points of the boundary having a unique normal vector or tangent hyperplane as in the case of a smooth surface.

Since ξ is well approximated by input on half-spaces, Lemma 1 implies for all $(x,n) \in N_e(A)$ with $x \in \partial^1 A$, that the expectation of the latter expression converges to

$$\lim_{\lambda\to\infty} \mathbb{E}\,\xi(\mathbf{0}_x + rn, \lambda^{1/d}(\mathcal{P}_\lambda - x), \lambda^{1/d}(\partial A - x)) = \mathbb{E}\xi(\mathbf{0}_x + rn, \mathcal{P}^{\mathrm{hom}}_{\kappa(x)}, H_x)\,.$$

Thus, we obtain from (3.4),

$$\begin{aligned}&\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} \mathbb{E} \sum_{x\in\mathcal{P}_\lambda} \xi_\lambda(x, \mathcal{P}_\lambda, \partial A)\\ &= 2\int_{N_e(A)} \int_{-\infty}^{\infty} \mathbb{E}\xi(\mathbf{0}_x + rn, \mathcal{P}^{\mathrm{hom}}_{\kappa(x)}, H_x)\kappa(x)\,\mathrm{d}r\,\nu_{d-1}(\mathrm{d}(x,n))\,.\end{aligned} \tag{3.5}$$

Now, we simplify the last integral and show that it coincides with $\mu(\xi, \partial A)$, as given in (3.2). First, recall that there is a unique unit normal vector $n(x)$ at each $x \in \partial^1 A$ and define a measure μ_{d-1} on $N(A)$ by

$$\mu_{d-1}(\,\cdot\,) = \frac{1}{2}\int_{\partial^1 A} \mathbf{1}((x,n(x)) \in \,\cdot\,)\,\mathcal{H}^{d-1}(\mathrm{d}x)\,.$$

Since $A \in \mathbf{A}(d)$ it follows by Corollary 2.5 and Proposition 4.1 in [10] that

$$\mu_{d-1}(\,\cdot\,) = \frac{1}{2}\int_{N(A)} \mathbf{1}((x,n) \in \,\cdot\,)\,H_0(x,n)\,\mathcal{H}^{d-1}(\mathrm{d}(x,n))\,,$$

where $H_0(x,n)$ is a certain function depending on the so-called generalized principal curvatures of A, see Equations (2.13) and (2.24) in [10]. Next, write

$$\begin{aligned}\int_{N_e(A)} f(x,n)\,\nu_{d-1}(\mathrm{d}(x,n)) &= \int_{N(A)} f(x,n)\,\mu_{d-1}(\mathrm{d}(x,n))\\ &\quad + \int_{T(N(A^*))} f(x,n)\,\mu_{d-1}(\mathrm{d}(x,n)) - \int_{N(A)\cap T(N(A^*))} f(x,n)\,\mu_{d-1}(\mathrm{d}(x,n))\,.\end{aligned}$$

According to the discussion before Theorem 5.2 in [10], given a measurable function f on $\mathbb{R}^d \times \mathbb{S}^{d-1}$, we can split the integral over $N_e(A)$ in (3.5) into three parts. The projection map $\pi_1 : N(A) \to \mathbb{R}^d$, $(x,n) \mapsto x$ has Jacobian also given by $H_0(x,n)$ for $\mathcal{H}^{d-1}$-almost all $(x,n) \in N(A)$, see [10, Section 3]. Combining these facts with the area formula [7, Paragraph 3.2.3] applied to π_1 in each of the three resulting integrals, which can be combined to a single integral over $\partial^1 A$, we find that

$$2\int_{N_e(A)}\int_{-\infty}^{\infty} \mathbb{E}\xi(\mathbf{0}_x + rn, \mathcal{P}^{\mathrm{hom}}_{\kappa(x)}, H_x)\kappa(x)\,\mathrm{d}r\,\nu_{d-1}(\mathrm{d}(x,n))$$

$$= \int_{\partial^1 A}\int_{-\infty}^{\infty} \mathbb{E}\xi(\mathbf{0}_x + rn(x), \mathcal{P}^{\mathrm{hom}}_{\kappa(x)}, H_x)\,\kappa(x)\,H_0(x,n(x))\,H_0(x,n(x))^{-1}\,\mathrm{d}r\,\mathcal{H}^{d-1}(\mathrm{d}x)$$

$$= \int_{\partial^1 A}\int_{-\infty}^{\infty} \mathbb{E}\xi(\mathbf{0}_x + rn(x), \mathcal{P}^{\mathrm{hom}}_{\kappa(x)}, H_x)\,\kappa(x)\,\mathrm{d}r\,\mathcal{H}^{d-1}(\mathrm{d}x)\,,$$

where we also have used the explicit representation of the measure μ_{d-1} as well as the fact that $\mathcal{H}^{d-1}(\partial A^{++} \setminus \partial^1 A) = 0$, which holds because $A \in \mathbf{A}(d)$. Since ξ is invariant under rotations we may replace H_x by $\mathbb{R}^{d-1}$ and $\mathbf{0}_x + rn(x)$ by $\mathbf{0} + rn$ to obtain (3.2) from (3.4), as desired. □

Proof of Theorem 2.2. Applying the Mecke formula for Poisson point processes we get

$$\lambda^{-(d-1)/d}\mathrm{Var}[H^\xi(\mathcal{P}_\lambda, \partial A)] = \lambda^{1/d}\int_{\mathbb{R}^d} \mathbb{E}\,\xi_\lambda(x, \mathcal{P}_\lambda, \partial A)^2\kappa(x)\,\mathrm{d}x \qquad (3.6)$$

$$+\lambda^{1+1/d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} I_1\,\kappa(x)\kappa(w)\,\mathrm{d}w\,\mathrm{d}x\,,$$

where

$$I_1 := \mathbb{E}\,\xi_\lambda(x, \mathcal{P}_\lambda \cup \{w\}, \partial A)\xi_\lambda(w, \mathcal{P}_\lambda \cup \{x\}, \partial A) - \mathbb{E}\,\xi_\lambda(x, \mathcal{P}_\lambda, \partial A)\mathbb{E}\,\xi_\lambda(w, \mathcal{P}_\lambda, \partial A)\,.$$

The proof of Theorem 2.1 shows that the first integral in (3.6) converges to

$$\int_{\partial^1 A}\int_{-\infty}^{\infty} \mathbb{E}\xi(\mathbf{0}_x + rn(x), \mathcal{P}^{\mathrm{hom}}_{\kappa(x)}, \mathbb{R}^{d-1})^2\,\kappa(x)\,\mathrm{d}r\,\mathcal{H}^{d-1}(\mathrm{d}x) = \mu(\xi^2, \partial A)\,.$$

To complete the proof we show that the second integral in (3.6) converges to the quadruple integral in (2.12). We re-write the integral with respect to x according to the generalized Steiner formula (2.3), using the notation already introduced in the proof of Theorem 2.1. Furthermore, for all $(x,n) \in N_e(A)$, let $H(x,n)$ denote the hyperplane orthogonal to n and containing x. Given $(x,n) \in N_e(A)$, we re-write the integral with respect to w

as the iterated integral over $H(x,n)$ and $\mathbb{R}$. This gives

$$\lambda^{1+1/d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} I_1\,\kappa(x)\kappa(w)\,\mathrm{d}x\mathrm{d}w$$

$$= \lambda^{1+1/d}\sum_{j=1}^{d-1}\omega_{d-j}\int_{(x,n)\in N_e(A)}\int_{r\in T(x,n)}\int_{v\in H(x,n)}\int_{s\in\mathbb{R}} r^{d-1-j}\,I_2 \\ \times\kappa(x+rn)\kappa((x+rn)+(v+sn))\,\mathrm{d}s\mathrm{d}v\mathrm{d}r\nu_j(\mathrm{d}(x,n))$$

with I_2 equal to

$$\mathbb{E}\,\xi_\lambda(x+rn,\mathcal{P}_\lambda\cup\{(x+rn)+(v+sn)\},\partial A)\xi_\lambda((x+rn)+(v+sn),\mathcal{P}_\lambda\cup\{x+rn\},\partial A) \\ -\mathbb{E}\,\xi_\lambda(x+rn,\mathcal{P}_\lambda,\partial A)\mathbb{E}\,\xi_\lambda((x+rn)+(v+sn),\mathcal{P}_\lambda,\partial A)\,.$$

We change variables by putting $s=\lambda^{-1/d}s'$, $r=\lambda^{-1/d}r'$ and $v=\lambda^{-1/d}v'$. This transforms the differential $\lambda^{1+1/d}\,\mathrm{d}s\mathrm{d}v\mathrm{d}r\nu_j(\mathrm{d}(x,n))$ into

$$\mathrm{d}s'\mathrm{d}v'\mathrm{d}r'\nu_j(\mathrm{d}(x,n))\,,\qquad j\in\{1,\ldots,d-1\}$$

and I_2 into I_3 given by

$$I_3 := \mathbb{E}\,\xi_\lambda(x+\lambda^{-1/d}r'n,\mathcal{P}_\lambda\cup\{(x+\lambda^{-1/d}r'n)+(\lambda^{-1/d}v'+\lambda^{-1/d}s'n)\},\partial A) \\ \times\xi_\lambda((x+\lambda^{-1/d}r'n)+(\lambda^{-1/d}v'+\lambda^{-1/d}s'n),\mathcal{P}_\lambda\cup\{x+\lambda^{-1/d}r'n\},\partial A) \\ -\mathbb{E}\,\xi_\lambda(x+\lambda^{-1/d}r'n,\mathcal{P}_\lambda,\partial A)\mathbb{E}\,\xi_\lambda((x+\lambda^{-1/d}r'n)+(\lambda^{-1/d}v'+\lambda^{-1/d}s'n),\mathcal{P}_\lambda,\partial A)\,.$$

To simplify the notation we shall write s, r and v for s', r' and v', respectively. Then

$$\lambda^{1+1/d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} I_1\,\kappa(x)\kappa(w)\,\mathrm{d}x\mathrm{d}w$$

$$= \sum_{j=1}^{d-1}\lambda^{-(d-1-j)/d}\omega_{d-j}\int_{N_e(A)}\int_{\lambda^{1/d}T(x,n)}\int_{H(x,n)}\int_{\mathbb{R}} r^{d-1-j}\,I_3 \\ \times\kappa(x+\lambda^{-1/d}rn)\kappa((x+\lambda^{-1/d}rn)+(\lambda^{-1/d}v+\lambda^{-1/d}sn))\,\mathrm{d}s\mathrm{d}v\mathrm{d}r\nu_j(\mathrm{d}(x,n))\,. \tag{3.7}$$

By the second part of Lemma 3, the factor $|I_3|$ in (3.7) is dominated uniformly in λ by an integrable function of $(x,n)\in N_e(A)$, $s\in\mathbb{R}$, $v\in H(x,n)$ and $r\in\mathbb{R}$. More precisely,

$$|I_3|\le c\,G^{\xi,p}(|r|)^{1/p}\exp\big(-c^{-1}\sqrt{\|v\|^2+s^2}\big)\,,$$

where the constant c is independent of all arguments. Thus for each $j \in \{1, \dots, d-1\}$, we have

$$\Big| \int_{N_e(A)} \int_{\lambda^{1/d} T(x,n)} \int_{H(x,n)} \int_{\mathbb{R}} r^{d-1-j} \, I_3 \\ \times \kappa(x + \lambda^{-1/d} rn) \kappa((x + \lambda^{-1/d} rn) + (\lambda^{-1/d} v + \lambda^{-1/d} sn))) \, \mathrm{d}s \mathrm{d}v \mathrm{d}r \nu_j(\mathrm{d}(x,n)) \Big|$$

$$\leq c \, \|\kappa\|_\infty^2 \int_{N_e(A)} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} r^{d-1-j} G^{\xi,p}(|r|)^{1/p} \exp(-c^{-1} \sqrt{\|v\|^2 + s^2}) \\ \mathrm{d}s \mathrm{d}v \mathrm{d}r \nu_j(\mathrm{d}(x,n))$$

$$\leq c \, \|\kappa\|_\infty^2 |\nu_j|(N_e(A)) \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} r^{d-1-j} G^{\xi,p}(|r|)^{1/p} \exp(-c^{-1} \sqrt{\|v\|^2 + s^2}) \, \mathrm{d}s \mathrm{d}v \mathrm{d}r \, .$$

Notice that $|\nu_j|(N_e(A))$ and the triple integral are finite by the assumption that $A \in \mathbf{A}(d)$ and the moment condition (2.8), respectively. As in the proof of Theorem 2.1 we have $\lim_{\lambda \to \infty} \lambda^{1/d} T(x,n) = (-\infty, \infty)$. Taking the limit, as $\lambda \to \infty$, in (3.7) and applying the dominated convergence theorem, we see that only the term $j = d-1$ remains. By Fubini's theorem and Lemma 2, this gives

$$\lim_{\lambda \to \infty} \lambda^{1+1/d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} I_1 \, \kappa(x) \kappa(w) \, \mathrm{d}x \mathrm{d}w$$

$$= 2 \int_{N_e(A)} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c^{\xi}(\mathbf{0}_x + rn, v + sn; \mathcal{P}_{\kappa(x)}^{\mathrm{hom}}, \mathbb{R}^{d-1}) \, \kappa(x)^2 \, \mathrm{d}s \mathrm{d}r \mathrm{d}v \nu_{d-1}(\mathrm{d}(x,n)) \, .$$

We can now use the same arguments as in the proof of Theorem 2.1 to show that the integral reduces to the quadruple integral in (2.12). This yields (2.13), as desired. □

4. Proof of Theorems 1.1 – 1.5

We shall deduce Theorems 1.1 – 1.5 from the general Theorems 2.1 and 2.2. In each case it suffices to express the relevant statistic as a sum of score functions and to show that the score function satisfies the conditions of the general theorems. We anticipate that the expectation formula (2.10) could be evaluated explicitly for some of the score functions described below. The proof of the positivity of the constants appearing in the variance expressions is postponed to Section 5.

Proof of Theorem 1.1. We first prove the asserted results for the volume functional $V_\lambda(A)$, with $A \in \mathbf{A}(d)$. For locally finite $\mathcal{X} \subset \mathbb{R}^d$, $x \in \mathcal{X}$, define the score

$$\xi^{(1)}(x, \mathcal{X}, \partial A) := \begin{cases} \mathrm{Vol}(v(x) \cap A^c) & \text{if } v(x) \cap \partial A \neq \emptyset,\ x \in A \\ -\mathrm{Vol}(v(x) \cap A) & \text{if } v(x) \cap \partial A \neq \emptyset,\ x \in A^c \\ 0 & \text{if } v(x) \cap \partial A = \emptyset\,, \end{cases} \tag{4.1}$$

where $v(x) := v(x, \mathcal{X})$ is the Voronoi cell of x based on the point configuration $\mathcal{X}$. In view of the limits appearing in our main results we also need to define scores on hyperplanes, i.e., on $\mathbb{R}^{d-1}$. We thus put

$$\xi^{(1)}(x, \mathcal{X}, \mathbb{R}^{d-1}) := \begin{cases} \mathrm{Vol}(v(x) \cap \mathbb{R}^{d-1}_+) & \text{if } v(x) \cap \mathbb{R}^{d-1} \neq \emptyset,\ x \in \mathbb{R}^{d-1}_- \\ -\mathrm{Vol}(v(x) \cap \mathbb{R}^{d-1}_-) & \text{if } v(x) \cap \mathbb{R}^{d-1} \neq \emptyset,\ x \in \mathbb{R}^{d-1}_+ \\ 0 & \text{if } v(x) \cap \mathbb{R}^{d-1} = \emptyset\,, \end{cases} \tag{4.2}$$

where we recall $\mathbb{R}^{d-1}_+ := \mathbb{R}^{d-1} \times [0, \infty)$ and $\mathbb{R}^{d-1}_- := \mathbb{R}^{d-1} \times (-\infty, 0]$. These definitions ensure that

$$V_\lambda(A) - \mathrm{Vol}(A) = \sum_{x \in \mathcal{P}_\lambda} \xi^{(1)}(x, \mathcal{P}_\lambda, \partial A) = \lambda^{-1} \sum_{x \in \mathcal{P}_\lambda} \xi^{(1)}_\lambda(x, \mathcal{P}_\lambda, \partial A)\,,$$

where we use that $\xi^{(1)}$ is homogenous of order d. We wish to deduce the volume asymptotics for $V_\lambda(A)$ by applying the limits (2.16) and (2.17) with $\gamma = d$ and with ξ set to $\xi^{(1)}$. It suffices to show that the score $\xi^{(1)}$ is homogeneously stabilizing (2.6), exponentially stabilizing as at (2.7), satisfies the moment condition (2.8) for $p = 1$ and some $p \in (2, \infty)$, and is well approximated by $\mathcal{P}_\lambda$ input on half-spaces as at (2.9). The first three conditions have been established several times in the literature; see the proof of Theorem 2.2 of [31].

To show that $\xi^{(1)}$ is well approximated by $\mathcal{P}_\lambda$ input on half-spaces as at (2.9), it suffices to slightly modify the proof of the analogous result in Theorem 2.2 of [31]. For the sake of completeness, we provide the details as follows.

By definition of $\mathbf{A}(d)$, almost all points of ∂A belong to $\partial^1 A$ and it so suffices to show (2.9) for a fixed $y \in \partial^1 A$. Translating y to the origin, letting $\mathcal{P}_\lambda$ denote a Poisson point process on $\mathbb{R}^d$, letting ∂A denote $\partial A - y$, and using rotation invariance of $\xi^{(1)}$, it is enough to show for all $w \in \mathbb{R}^d$ that

$$\lim_{\lambda \to \infty} \mathbb{E}\,|\xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})| = 0\,,$$

where $\mathbb{R}^{d-1}$ is the unique hyperplane tangent to ∂A at the origin. Without loss of generality, we assume, locally around the origin, that $\partial A \subset \mathbb{R}^{d-1}_-$. Fix $\varepsilon > 0$ and $w \in \mathbb{R}^d$. We note that there is a constant $L \in (0, \infty)$ such that

$$\sup_{\lambda > 0} (\mathbb{E}\,[\xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A)^2])^{1/2} \leq L$$

and

$$\sup_{\lambda > 0} (\mathbb{E}\,[\xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})^2])^{1/2} \leq L\,.$$

Let $\widetilde{v}(w, \lambda^{1/d}\mathcal{P}_\lambda)$ be the union of $v(w, \lambda^{1/d}\mathcal{P}_\lambda)$ and all the Voronoi cells adjacent to $v(w, \lambda^{1/d}\mathcal{P}_\lambda)$ in the Voronoi mosaic of $\mathcal{P}_\lambda$. For all $r \in (1, \infty)$ consider the event

$$E_1(\lambda, w, r) := \{\mathrm{diam}(\widetilde{v}(w, \lambda^{1/d}\mathcal{P}_\lambda)) \le r\}\,, \tag{4.3}$$

where diam(·) stands for the diameter of the argument set. Lemma 2.2 of [18] shows there is $r_0 := r_0(\varepsilon, L)$ such that for $r \in [r_0, \infty)$ and λ large we have $\mathbb{P}(E_1(\lambda, w, r)^c) \le (\varepsilon/2L)^2$. It follows by the Cauchy-Schwarz inequality that

$$\lim_{\lambda\to\infty} \mathbb{E}\,|(\xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1}))\mathbf{1}(E_1(\lambda, w, r_0)^c)| \le \varepsilon\,.$$

By the triangle inequality and the arbitrariness of ε it is therefore enough to show that

$$\lim_{\lambda\to\infty} \mathbb{E}\,|(\xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1}))\mathbf{1}(E_1(\lambda, w, r_0))| \le \varepsilon\,. \tag{4.4}$$

By the way that y was chosen, $\mathbf{0}$ is a point in $\partial^1 A$ and thus has a unique normal vector. We first assume $w \in \mathbb{R}^{d-1}_-$; the arguments with $w \in \mathbb{R}^{d-1}_+$ are nearly identical. Moreover, we may assume $w \in \lambda^{1/d}A$ for λ large. Consider the (possibly degenerate) solid

$$\Delta_\lambda(w) := \Delta_\lambda(w, r_0) := (\mathbb{R}^{d-1}_- \setminus \lambda^{1/d}A) \cap B_{r_0}(w)\,. \tag{4.5}$$

Recalling that ∂A is $(\mathcal{H}^{d-1}, d-1)$ rectifiable, it follows that almost all of ∂A is contained in a union of C^1 sub-manifolds of $\mathbb{R}^d$ [7, Theorem 3.2.29]. Since $\mathbf{0}$ is a point of $\partial^1 A$, it follows that the maximal 'height' $h_\lambda(w, r_0)$ of the solid $\Delta_\lambda(w, r_0)$ with respect to the hyperplane $\mathbb{R}^{d-1}$ satisfies $\lim_{\lambda\to\infty} h_\lambda(w, r_0) = 0$ for fixed w and r_0 (see also the linear approximation properties of rectifiable sets summarized in Chapter 15 of [17]). Hence

$$\mathrm{Vol}(\Delta_\lambda(w, r_0)) = O(h_\lambda(w, r_0) \cdot r_0^{d-1})$$

and so for large λ we have $\mathrm{Vol}(\Delta_\lambda(w, r_0)) \le \varepsilon$. On the event $E_1(\lambda, w, r_0)$, the difference of the volumes $v(w, \lambda^{1/d}\mathcal{P}_\lambda) \cap \lambda^{1/d}A^c$ and $v(w, \lambda^{1/d}\mathcal{P}_\lambda) \cap \mathbb{R}^{d-1}_+$ is at most $\mathrm{Vol}(\Delta_\lambda(w, r_0))$. Thus for large λ we get

$$\mathbb{E}\,|(\xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \xi^{(1)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1}))\mathbf{1}(E_1(\lambda, w, r_0))|$$
$$\le \mathrm{Vol}(\Delta_\lambda(w, r_0)) \le \varepsilon\,,$$

which gives (2.9) as desired.

We now prove the asserted results for the surface area functional $S_\lambda(A)$. As in [31], given $\mathcal{X}$ locally finite and an admissible set $A \subset \mathbb{R}^d$, define for $x \in \mathcal{X} \cap A$ the area score $\xi^{(2)}(x, \mathcal{X}, \partial A)$ to be the $\mathcal{H}^{d-1}$-measure of the $(d-1)$-dimensional faces of $v(x)$ belonging to the boundary of $\bigcup_{x\in\mathcal{X}\cap A} v(x)$ (if there are no such faces or if $x \notin \mathcal{X} \cap A$, then put $\xi^{(2)}(x, \mathcal{X}, \partial A)$ to be zero). Similarly, for $x \in \mathcal{X} \cap \mathbb{R}^{d-1}_-$, put $\xi^{(2)}(x, \mathcal{X}, \mathbb{R}^{d-1})$ to be the $\mathcal{H}^{d-1}$-measure of the $(d-1)$-dimensional faces of $v(x)$ belonging to the boundary of

$\bigcup_{x\in\mathcal{X}\cap\mathbb{R}^{d-1}_-} v(x)$, otherwise $\xi^{(2)}(x,\mathcal{X},\mathbb{R}^{d-1})$ is zero. We note that $\xi^{(2)}$ is homogenous of order $d-1$ and that

$$S_\lambda(A) = \sum_{x\in\mathcal{P}_\lambda} \xi^{(2)}(x,\mathcal{P}_\lambda,\partial A)\,.$$

We wish to deduce the first- and second-order limit behavior of $S_\lambda(A)$ by applying the limits (2.16) and (2.17) with $\gamma = d-1$ and with ξ set to $\xi^{(2)}$.

It is easy to see and well-known that the score $\xi^{(2)}$ is homogeneously stabilizing (2.6), exponentially stabilizing (2.7), and satisfies the moment condition (2.8) for all $p\geq 1$; see for example the proof of Theorem 2.4 of [31]. To see that $\xi^{(2)}$ is well approximated by $\mathcal{P}_\lambda$ input on half-spaces (2.9), it suffices to follow the proof of Theorem 2.4 of [31]. For sake of completeness we include the details as follows.

Fix $\varepsilon>0$ and $w\in\mathbb{R}^d$. By the moment bounds on $\xi^{(2)}$ and the Cauchy-Schwarz inequality, it is enough to show the following counterpart to (4.4), namely to show that

$$\lim_{\lambda\to\infty}\mathbb{E}\,|(\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\lambda^{1/d}\partial A)-\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\mathbb{R}^{d-1}))\mathbf{1}(E_1(\lambda,w,r_0))|\leq C\varepsilon^{1/2}, \tag{4.6}$$

where $E_1(\lambda,w,r_0)$ is as at (4.3), and where, as above, the origin is a point of $\partial^1 A - y$. Define

$$E_0(\lambda,w,r_0) := \{\lambda^{1/d}\mathcal{P}_\lambda\cap\Delta_\lambda(w,r_0)=\emptyset\}\,,$$

where $\Delta_\lambda(w,r_0)$ is as at (4.5). The intensity measure of $\lambda^{1/d}\mathcal{P}_\lambda$ is upper bounded by $||\kappa||_\infty$, yielding for large λ that

$$\mathbb{P}[E_0(\lambda,w,r_0)^c]\leq 1-\exp(-||\kappa||_\infty\mathrm{Vol}(\Delta_\lambda(w,r_0)))\leq ||\kappa||_\infty\varepsilon\,, \tag{4.7}$$

where we used that $\mathrm{Vol}(\Delta_\lambda(w,r_0))\leq\varepsilon$.

The two score functions $\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\lambda^{1/d}\partial A)$ and $\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\mathbb{R}^{d-1})$ coincide on the event $E_1(\lambda,w,r_0)\cap E_0(\lambda,w,r_0)$. Indeed, on this event it follows that f is a face of a boundary cell of $\lambda^{1/d}A_\lambda$ iff f is a face of a boundary cell of the Poisson-Voronoi mosaic of $\mathbb{R}^{d-1}_-$. (If f is a face of the boundary cell $v(w,\lambda^{1/d}\mathcal{P}_\lambda)$, $w\in\lambda^{1/d}A$, then f is also a face of $v(z,\lambda^{1/d}\mathcal{P}_\lambda)$ for some $z\in\lambda^{1/d}A^c$. If $\lambda^{1/d}\mathcal{P}_\lambda\cap\Delta_\lambda(w,r_0)=\emptyset$, then z must belong to $\mathbb{R}^{d-1}_+$, showing that f is face of a boundary cell of the Poisson-Voronoi mosaic of $\mathbb{R}^{d-1}_-$. The reverse implication is shown similarly.)

On the other hand, since

$$\mathbb{E}\left[(\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\lambda^{1/d}\partial A)-\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\mathbb{R}^{d-1}))^2\mathbf{1}(E_1(\lambda,w,r_0))\right]=O(1)\,,$$

and since by (4.7) we have $\mathbb{P}[E_0(\lambda,w,r_0)^c]\leq||\kappa||_\infty\varepsilon$, it follows by the Cauchy-Schwarz inequality that, as $\lambda\to\infty$,

$$\begin{aligned}&\mathbb{E}\,|(\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\lambda^{1/d}\partial A)-\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\mathbb{R}^{d-1}))\mathbf{1}(E_1(\lambda,w,r_0))|\\ &=\mathbb{E}\,|(\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\lambda^{1/d}\partial A)-\xi^{(2)}(w,\lambda^{1/d}\mathcal{P}_\lambda,\mathbb{R}^{d-1}))\\ &\qquad\qquad\times\mathbf{1}(E_1(\lambda,w,r_0))\mathbf{1}(E_0(\lambda,w,r_0)^c)|\\ &\leq C(||\kappa||_\infty\varepsilon)^{1/2}\,.\end{aligned} \tag{4.8}$$

Therefore (4.6) holds and so $\xi^{(2)}$ is well approximated by $\mathcal{P}_\lambda$ input on half-spaces as at (2.9), as desired. $\square$

Proof of Theorem 1.2. Let us first recall that the Poisson-Voronoi mosaic is a normal mosaic, see [26]. This means that with probability one each ℓ-dimensional face in $\mathrm{skel}_\ell(\mathrm{PV}_\lambda(A))$ arises as the intersection of exactly $d-\ell+1$ Voronoi cells.

Now, given $\mathcal{X}$ locally finite, $x \in \mathcal{X}$, and an admissible $A \subset \mathbb{R}^d$, define $\xi^{(3,\ell)}(x, \mathcal{X}, \partial A)$ as

$$\xi^{(3,\ell)}(x, \mathcal{X}, \partial A) := \frac{1}{d-\ell+1} \sum_{\substack{f \in \mathcal{F}_\ell(v(x)) \\ f \subset \partial(\mathrm{PV}_\lambda(A))}} \mathcal{H}^{(\ell)}(f)$$

and zero otherwise. Then,

$$H_\lambda^{(\ell)}(A) = \sum_{\substack{x \in \mathcal{P}_\lambda \\ x \in A}} \xi^{(3,\ell)}(x, \mathcal{P}_\lambda, \partial A) = \lambda^{-\ell/d} \sum_{x \in \mathcal{P}_\lambda} \xi_\lambda^{(3,\ell)}(x, \mathcal{P}_\lambda, \partial A)\,,$$

where we used that $\xi^{(3,\ell)}$ is homogeneous of order ℓ. We wish to deduce the first- and second-order limit behaviour of $H_\lambda^{(\ell)}(A)$ by applying the limits (2.16) and (2.17) with $\gamma = \ell$ and with ξ set to $\xi^{(3,\ell)}$.

The proof that $\xi^{(3,\ell)}$ is homogeneously stabilizing (2.6), exponentially stabilizing (2.7), and satisfies the moment condition (2.8) for all $p \geq 1$ follows nearly verbatim the proof that $\xi^{(2)}$ has these properties. Indeed the radius of stabilization for $\xi^{(3,\ell)}$ coincides with that of $\xi^{(2)}$.

To see that $\xi^{(3,\ell)}$ is well approximated by $\mathcal{P}_\lambda$ input on half-spaces as at (2.9), we may follow the proof that $\xi^{(2)}$ is well approximated by $\mathcal{P}_\lambda$ input on half-spaces. Notice that on the event $E_1(\lambda, w, r_0) \cap E_0(\lambda, w, r_0)$, the scores $\xi^{(3,\ell)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A)$ and $\xi^{(3,\ell)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})$ coincide. As in (4.8) we obtain

$$\begin{aligned} &\mathbb{E}\,|(\xi^{(3,\ell)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \xi^{(3,\ell)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1}))\mathbf{1}(E_1(\lambda, w, r_0))| \\ &= \mathbb{E}\,|(\xi^{(3,\ell)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \xi^{(3,\ell)}(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})) \\ &\qquad \times \mathbf{1}(E_1(\lambda, w, r_0))\mathbf{1}(E_0(\lambda, w, r_0)^c)| \leq C(||\kappa||_\infty \varepsilon)^{1/2}\,. \end{aligned}$$

This gives that $\xi^{(3,\ell)}$ satisfies (2.9) as desired. $\square$

Proof of Theorem 1.3. Given $\mathcal{X}$ locally finite, $x \in \mathcal{X}$, and $A \in \mathbf{A}(d)$, let us define the score $\xi^{(4,\ell)}(x, \mathcal{X}, \partial A)$ to be the number of ℓ-dimensional faces of $v(x) := v(x, \mathcal{X})$ belonging to $\partial(\mathrm{PV}_\lambda(A))$. Define $\xi^{(4,\ell)}(x, \mathcal{X}, \mathbb{R}^{d-1})$ similarly. Then

$$f_\lambda^\ell(A) = \sum_{x \in \mathcal{P}_\lambda} \xi_\lambda^{(4,\ell)}(x, \mathcal{P}_\lambda, \partial A)\,.$$

We shall show that $\xi^{(4,\ell)}$ satisfies the hypotheses of Theorems 2.1 and 2.2 and thus deduce Theorem 1.3 from (2.16) and (2.17) with ξ set to $\xi^{(4,\ell)}$ and γ set to zero (notice that $\xi^{(4,\ell)}$

is homogeneous of order 0). For brevity write $\xi^{(4)}$ for $\xi^{(4,\ell)}$ for fixed $\ell \in \{0,\dots,d-1\}$. Now, $\xi^{(4)}$ is homogeneously and exponentially stabilizing since its radius of stabilization coincides with that for the volume score $\xi^{(1)}$ defined in the proof of Theorem 1.1. The number $N^{(\ell)}(x,\mathcal{P}_\lambda)$ of ℓ-dimensional faces of a Poisson-Voronoi cell $v(x)$ has moments of all orders and therefore the moment condition (2.8) holds because

$$\begin{aligned}&|\xi_\lambda^{(4)}(x+\lambda^{-1/d}rn,\mathcal{P}_\lambda\cup\{z\},\partial A)|\\&\le N^{(\ell)}(x+\lambda^{-1/d}rn,\mathcal{P}_\lambda\cup\{z\})\,\mathbf{1}(v(\lambda^{1/d}x+rn,\lambda^{1/d}\mathcal{P}_\lambda)\cap\partial A\neq\emptyset)\end{aligned}$$

for $(x,n)\in N_e(A)$. The expectation of the last factor decays uniformly fast in r, giving that $\xi^{(4)}$ satisfies the moment condition (2.8) for all $p\ge 1$.

The arguments in the proof of Theorem 1.1 showing that the surface area score $\xi^{(2)}$ is well approximated by $\mathcal{P}_\lambda$ input on half-spaces extend to show that $\xi^{(4)}$ is likewise well approximated by $\mathcal{P}_\lambda$ input on half-spaces. The guiding idea is that with high probability, we have that f is a face of a Voronoi cell $v(w)$ belonging to the Poisson-Voronoi approximation of $\lambda^{1/d}(A-y)$ if and only if it belongs to the Poisson-Voronoi approximation of $\mathbb{R}_+^{d-1}$. Indeed, this happens on the high probability event that the region 'between' the boundary of the Poisson-Voronoi approximation of A and $\mathbb{R}^{d-1}$ in the neighbourhood of w, must be devoid of points, see the proof of Theorem 1.1. Thus $\xi^{(4,\ell)}$ satisfies all the hypotheses of Theorems 2.1 and 2.2 and this concludes the proof of Theorem 1.3. □

Proof of Theorem 1.4. Given $\mathcal{X}$ locally finite, $x\in\mathcal{X}$, an admissible $A\subset\mathbb{R}^d$, and $A_0\subset\partial A$, put $\xi^{(5,\ell)}(x,\mathcal{X},A_0)$ to be the number of ℓ-dimensional faces of $v(x)$ if $v(x)\cap A_0\neq\emptyset$ and zero otherwise. Define $\xi^{(5,\ell)}(x,\mathcal{X},\mathbb{R}^{d-1})$ similarly. Now, put

$$\xi^{(5)}(x,\mathcal{X},A_0):=\sum_{l=0}^{d-1}\xi^{(5,\ell)}(x,\mathcal{X},A_0)$$

and notice that

$$\mathrm{Co}_\lambda(A_0)=\sum_{x\in\mathcal{P}_\lambda}\xi^{(5)}(x,\mathcal{P}_\lambda,A_0)\,.$$

We shall show that $\xi^{(5,\ell)}$ satisfies the hypotheses of Theorems 2.1 and 2.2 and thus deduce Theorem 1.4 from (2.16) and (2.17) with ξ set to $\xi^{(5)}$ and γ set to zero (notice that $\xi^{(5)}$ is homogeneous of order 0). The score function $\xi^{(5)}$ is homogeneously stabilizing as at (2.6), exponentially stabilizing as at (2.7), and satisfies the moment condition (2.8) for all $p\ge 1$. This is because each $\xi^{(5,\ell)}$ with $\ell\in\{0,\dots,d-1\}$ has this property. Also, since each $\xi^{(5,\ell)}$ is well approximated by $\mathcal{P}_\lambda$ input on half-spaces for each $\ell\in\{0,\dots,d-1\}$, it follows that $\xi^{(5)}$ enjoys this property as well. Thus $\xi^{(5)}$ satisfies the hypotheses of Theorems 2.1 and 2.2, concluding the proof of Theorem 1.4. □

Proof of Theorem 1.5. We start with the iterated volume $V_\lambda^{(n)}$. Conditioned on $\mathrm{PV}_\lambda^{(1)}$ the first asymptotic equivalence of Theorem 1.1 yields

$$\mathbb{E}\,[V_\lambda^{(2)}-V_\lambda^{(1)}|\mathrm{PV}_\lambda^{(1)}]\sim c_1\,\lambda^{-\frac{1}{d}}\,\mathcal{H}^{d-1}(\partial(\mathrm{PV}_\lambda^{(1)}))\,.$$

Taking expectations and recalling the equivalence $\mathbb{E}\, S_\lambda(A) \sim c_2 \mathcal{H}^{d-1}(\partial A)$, we obtain

$$\mathbb{E}\, V_\lambda^{(2)} - V(A) \sim c_1\, \lambda^{-\frac{1}{d}}\, (c_2+1)\mathcal{H}^{d-1}(\partial A)\,.$$

Next,

$$\begin{aligned}\mathbb{E}\, V_\lambda^{(3)} - V(A) &= \mathbb{E}\, \mathbb{E}\, [V_\lambda^{(3)} - V_\lambda^{(2)} | \mathrm{PV}_\lambda^{(2)}] + \mathbb{E}\, \mathbb{E}\, [V_\lambda^{(2)} - V_\lambda^{(1)} | \mathrm{PV}_\lambda^{(1)}] + \mathbb{E}\, V_\lambda^{(1)} - V(A) \\ &\sim c_1 \lambda^{-\frac{1}{d}}\, c_2^2 \mathcal{H}^{d-1}(\partial A) + c_1\, \lambda^{-\frac{1}{d}}\, c_2 \mathcal{H}^{d-1}(\partial A) + c_1\, \lambda^{-\frac{1}{d}} \mathcal{H}^{d-1}(\partial A) \\ &= c_1\, c_{2,2} \lambda^{-\frac{1}{d}} \mathcal{H}^{d-1}(\partial A)\,.\end{aligned}$$

Recursively continuing this way proves the desired claim, namely

$$\mathbb{E}\, V_\lambda^{(n)} - V(A) \sim c_1\, c_{2,n}\, \lambda^{-\frac{1}{d}}\, \mathcal{H}^{d-1}(\partial A)\,.$$

The asymptotic equivalences for $\mathbb{E}\, S_\lambda^{(n)}$, $\mathbb{E}\, H_\lambda^{\ell,(n)}$ and $\mathbb{E}\, f_\lambda^{\ell,(n)}$ follow similarly. □

5. Variance lower bounds

We complete the proofs of Theorems 1.1 − 1.4 by proving positivity of the constants appearing in the variance expressions. The assumption that ∂A contains a C^2-smooth subset with positive $(d-1)$-dimensional Hausdorff measure is essential for our following arguments, but we conjecture that this condition can be relaxed. For example, in [28] the author establishes upper and lower bounds on $\mathrm{Var}[V_\lambda(A)]$ for any compact convex set A having non-empty interior, without additional smoothness assumptions. However, it is unclear (to us) whether the methods of [28] extend to the more general class of admissible sets $\mathbf{A}(d)$ as well as to the other Poisson-Voronoi statistics considered in Theorems 1.1–1.4.

In what follows we use the standard Landau notation. More precisely, for two functions $f, g : [0,\infty) \to \mathbb{R}$ we write

- $f = o(g)$ if for all $c \in (0,\infty)$ there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $|f(\lambda)| \leq c\, |g(\lambda)|$,
- $f = O(g)$ if there exists $c \in (0,\infty)$ and $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $|f(\lambda)| \leq c\, |g(\lambda)|$, and
- $f = \Omega(g)$ if there exists $c \in (0,\infty)$ and $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $|f(\lambda)| \geq c\, g(\lambda)$.

Positivity of c_3 and c_4. Positivity of c_3 is shown in Theorem 2.3 of [31] and it remains to consider c_4. For this, recall that $\Gamma \subset \partial A$ is C^2-smooth, with $\mathcal{H}^{d-1}(\Gamma) \in (0,\infty)$. Recalling $A \subset Q$, subdivide Q into cubes of edge length $l(\lambda) := (\lfloor \lambda^{1/d} \rfloor)^{-1}$. The number $L(\lambda)$ of cubes having non-empty intersection with Γ satisfies $L(\lambda) = \Omega(\lambda^{(d-1)/d})$, as otherwise the cubes would partition Γ into $o(\lambda^{(d-1)/d})$ sets, each of $\mathcal{H}^{d-1}$-measure $O((\lambda^{-1/d})^{d-1})$, which when $\lambda \to \infty$ gives $\mathcal{H}^{d-1}(\Gamma) = 0$, a contradiction.

We find a sub-collection $Q_1, \ldots, Q_M$ of the $L(\lambda)$ cubes such that $d(Q_i, Q_j) \geq 2\sqrt{d}\, l(\lambda)$ for all $i, j \leq M$, and $M = \Omega(\lambda^{(d-1)/d})$, where $d(Q_i, Q_j)$ stands for the distance between Q_i and Q_j. Rotating and translating $Q_i, 1 \leq i \leq M$, by a distance at most $(\sqrt{d}/2)\, l(\lambda)$, if necessary, we obtain a collection $\widetilde{Q}_1, \ldots, \widetilde{Q}_M$ of disjoint cubes (with faces not necessarily parallel to a coordinate plane) such that

- $d(\widetilde{Q}_i, \widetilde{Q}_j) \geq \sqrt{d}\, l(\lambda)$ for all $i, j \leq M$,
- Γ contains the centre of each $\widetilde{Q}_i$, here denoted $x_i, 1 \leq i \leq M$.

By the assumed differentiability of Γ, $\Gamma \cap \widetilde{Q}_i$ is well approximated locally around each x_i by the hyperplane $T_i := T_{x_i}$ tangent to Γ at x_i. By the C^2-assumption, the approximation is uniform over all $1 \leq i \leq M$. Making a further rotation of $\widetilde{Q}_i$, if necessary, we may assume that T_i partitions $\widetilde{Q}_i$ into congruent rectangular solids. Let T_i coincide with the hyperplane $\mathbb{R}^{d-1}$. Without loss of generality we assume $\partial A \subset \mathbb{R}^{d-1} \times (-\infty, 0]$, that is ∂A is 'beneath' T_i.

We now exhibit a configuration of Poisson points $\mathcal{P}_\lambda$ which has strictly positive probability and for which $S_\lambda(A)$ has variability bounded below by $\Omega(\lambda^{-(d-1)/d}\mathcal{H}^{d-1}(\Gamma))$. Let $\epsilon := \epsilon(\lambda) := l(\lambda)/28$ and sub-divide each $\widetilde{Q}_i, 1 \leq i \leq M$, into 28^d sub-cubes of edge length ϵ. Sub-cubes within Hausdorff distance 4ϵ of $\partial\widetilde{Q}_i$ are called 'boundary' sub-cubes; if a sub-cube is not a boundary sub-cube then we call it an interior sub-cube. If each boundary sub-cube in $\widetilde{Q}_i$ contains a point from $\mathcal{P}_\lambda$, then the geometry of the Voronoi cells with centres in $\widetilde{Q}_i$ and distant more than 4ϵ from $\partial\widetilde{Q}_i$ is not altered by point configurations outside $\widetilde{Q}_i$ (see e.g. [21]).

We assume that x_i coincides with the origin and we recall that $\partial A \subset \mathbb{R}^{d-1} \times (-\infty, 0]$ so that points near ∂A may be parametrized by a pair in $\mathbb{R}^{d-1} \times (-\infty, 0]$). By $2(\mathbb{Z}^{d-1})$ we mean the set of all points in $\mathbb{R}^{d-1}$ having integer coordinates of even parity. Consider the sub-cubes $\widetilde{Q}_i$ having the following properties:

(a) the boundary sub-cubes each contain at least one point from $\mathcal{P}_\lambda$,
(b) $\mathcal{P}_\lambda \cap B_{\epsilon/100}((\epsilon j, \pm\epsilon))$ consists of a singleton for $j \in 2(\mathbb{Z}^{d-1}), |j| \leq 10$, or
(b') $\mathcal{P}_\lambda \cap B_{\epsilon/100}((\epsilon j, \epsilon/100))$ consists of a singleton for $j \in 2(\mathbb{Z}^{d-1}), |j| \leq 10$ and also $\mathcal{P}_\lambda \cap B_{\epsilon/100}((\epsilon j, -\epsilon/100))$ consists of a singleton for $j = 0$ and $j \in 2(\mathbb{Z}^{d-1}) + 1, |j| \leq 10$,
(c) $\mathcal{P}_\lambda$ puts no other points in $\widetilde{Q}_i$.

(We remark that the choice of the constants 28 and 100 is arbitrary and that we could have used any sufficiently large number.) Events (b) and (b') happen with the same probability, which is small but bounded away from zero uniformly in λ, since $\kappa \equiv 1$.

Re-labelling if necessary, let $I := \{1, \ldots, K\}$ be the indices of cubes $\widetilde{Q}_i$ having properties (a)-(c). It is easily checked that the probability a given $\widetilde{Q}_i, 1 \leq i \leq M$, satisfies property (a) is strictly positive, uniformly in λ. This is also true for properties (b)-(c), showing that

$$\mathbb{E}\, K = \Omega(\lambda^{(d-1)/d}). \tag{5.1}$$

Abusing notation, let $\mathcal{Q} := \bigcup_{i=1}^{K} \widetilde{Q}_i$ and put $\mathcal{Q}^c := [0,1]^d \setminus \mathcal{Q}$. Let $\mathcal{F}_\lambda$ be the σ-algebra determined by the random set I, the positions of points of $\mathcal{P}_\lambda$ in all boundary sub-cubes, and the positions of points $\mathcal{P}_\lambda$ in $\mathcal{Q}^c$. Let $U_i, 1 \leq i \leq M$, be the union of the interior sub-cubes in $\widetilde{Q}_i$. If $d = 2$, we notice that if (b) happens, then the surface $\partial A_\lambda \cap U_i$ contains nearly horizontal edges and the total length of these edges is generously bounded above by 30ϵ. Indeed, if (b) happens, the 11 cells centered at the points in $\mathcal{P}_\lambda \cap B_{\epsilon/100}((\epsilon j, -\epsilon))$, $j \in \{0, \pm 2, \pm 4, \ldots, \pm 10\}$, contribute to $\partial(PV_\lambda(A))$ a length roughly bounded by the width of U_i plus some negligible corrections. On the other hand, if (b') happens then $\partial A_\lambda \cap U_i$ contains 10 sharp peaks, with abscissas roughly equal to $\{\pm 1, \pm 3, \ldots, \pm 9\}$. In fact, it is easily checked that $\partial A_\lambda \cap U_i$ contains at least 18 'long', nearly vertical edges of length at least 2ϵ, giving a total edge length of at least 36ϵ. A similar situation holds in higher dimensions $d \geq 3$.

Conditional on $\mathcal{F}_\lambda$, $\mathcal{H}^{d-1}(\partial A_\lambda \cap \widetilde{Q}_i)$ has variability $\Omega(\epsilon^{2(d-1)}) = \Omega(\lambda^{-2+2/d})$, uniformly in $i \in I$, that is

$$\mathrm{Var}[\mathcal{H}^{d-1}(\partial A_\lambda \cap \widetilde{Q}_i)|\mathcal{F}_\lambda] = \Omega(\lambda^{-2+2/d}), \qquad i \in I. \tag{5.2}$$

By the conditional variance formula,

$$\begin{aligned}\mathrm{Var}[S_\lambda(A)] &= \mathrm{Var}[\mathbb{E}\, S_\lambda(A)|\mathcal{F}_\lambda]] + \mathbb{E}\,[\mathrm{Var}[S_\lambda(A)|\mathcal{F}_\lambda]] \\ &\geq \mathbb{E}\,[\mathrm{Var}[S_\lambda(A))|\mathcal{F}_\lambda]] \\ &= \mathbb{E}\,[\mathrm{Var}[\mathcal{H}^{d-1}(\partial A_\lambda \cap \mathcal{Q}) + \mathcal{H}^{d-1}(\partial A_\lambda \cap \mathcal{Q}^c)|\mathcal{F}_\lambda]]\,.\end{aligned}$$

Given $\mathcal{F}_\lambda$, the Poisson-Voronoi mosaic of $\mathcal{P}_\lambda$ admits variability only inside $\mathcal{Q}$, that is to say, given $\mathcal{F}_\lambda$, we have $\mathcal{H}^{d-1}(\partial A_\lambda \cap \mathcal{Q}^c)$ is constant. Thus

$$\begin{aligned}\mathrm{Var}[S_\lambda(A)] &\geq \mathbb{E}\,[\mathrm{Var}[\mathcal{H}^{d-1}(\partial A_\lambda \cap \mathcal{Q})|\mathcal{F}_\lambda]] \\ &= \mathbb{E}\,[\mathrm{Var}[\sum_{i \in I} \mathcal{H}^{d-1}(\partial A_\lambda \cap \widetilde{Q}_i)|\mathcal{F}_\lambda]] \\ &= \mathbb{E} \sum_{i \in I} \mathrm{Var}[\mathcal{H}^{d-1}(\partial A_\lambda \cap \widetilde{Q}_i)|\mathcal{F}_\lambda]\,,\end{aligned} \tag{5.3}$$

since, given $\mathcal{F}_\lambda$, $\mathcal{H}^{d-1}(\partial A_\lambda \cap \widetilde{Q}_i), i \in I$, are independent. By (5.1) and (5.2), we have

$$\mathrm{Var}[S_\lambda(A)] \geq c\, \lambda^{-2+2/d}\, \mathbb{E}\,[K] = \Omega(\lambda^{-(d-1)/d})$$

with some finite constant $c \in (0, \infty)$, concluding the proof that c_4 is positive.

Positivity of c_6 and c_8. The general idea is to show that configuration (b') generates a surface which has more variability (both in terms of complexity and measure) than the surface generated by configuration (b). The details go as follows. For $\ell \in \{0, 1, \ldots, d-1\}$ and $i \in I$, put $S_{\ell,i} := (\mathrm{skel}_\ell(\mathrm{PV}_\lambda(A))) \cap U_i$, noting that $\partial(PV_\lambda(A)) \cap U_i = S_{d-1,i}$ (recall the notation introduced in the discussion around Equation (5.1)). Let $S_{\ell,i}(b)$ be the ℓ-dimensional skeleton arising from configuration (b) and define $S_{\ell,i}(b')$ similarly. Henceforth without loss of generality we fix $i = 1$ and write S_ℓ for $S_{\ell,1}$. Observe that

$S_{d-1}(b)$ consists of a single $(d-1)$-dimensional facet f which is nearly a hypercube of dimension $d-1$ (and nearly a horizontal edge when $d=2$). Also, $S_{d-2}(b)$ is the union of $2(d-1)$ faces, each of which is nearly a hypercube of dimension $d-2$.

On the other hand, $S_{d-1}(b')$ is the union of $(d-1)$-dimensional facets $F_j, 1 \leq j \leq 2(d-1)$, whose union forms the boundary of a solid hyper-pyramid in $\mathbb{R}^d$ whose base is a translate, up to a negligible perturbation, of $S_{d-1}(b)$. The boundary of the surface $\cup_{j=1}^{2(d-1)} F_j$ is of dimension $d-2$ and is the union of $2(d-1)$ faces, each of which is nearly a hypercube of dimension $d-2$. In fact, the boundary of the surface $\cup_{j=1}^{2(d-1)} F_j$ is a translate, also up to a negligible perturbation, of $S_{d-2}(b)$; we thus denote the boundary of $\cup_{j=1}^{2(d-1)} F_j$ by $\tilde{S}_{d-2}(b')$. In other words we have that

$$S_{d-1}(b) = S_{d-2}(b) \cup (\mathrm{int} f)$$

and

$$S_{d-1}(b') = \tilde{S}_{d-2}(b') \cup \Big(\bigcup_{j=1}^{2(d-1)} F_j \setminus \tilde{S}_{d-2}(b) \Big).$$

Now, $S_{d-2}(b)$ and $\tilde{S}_{d-2}(b')$ are indistinguishable from the viewpoint of their combinatorial complexity, as measured by their lower-dimensional skeletons. Moreover, they are nearly indistinguishable from a measure theoretic point of view, since the $\mathcal{H}^\ell$-measure of their ℓ-skeletons nearly coincide (modulo negligible corrections). On the other hand, the open facet $\mathrm{int} f$ differs significantly from $\cup_{j=1}^{2(d-1)} F_j \setminus \tilde{S}_{d-2}(b)$ in terms of both combinatorial complexity and measure. Indeed, the $\mathcal{H}^\ell$-measure of the ℓ-skeleton of the latter (facets of a pyramid) is strictly larger than the $\mathcal{H}^\ell$-measure of $\mathrm{int} f$ (the base of the pyramid). Likewise, for $\ell \in \{0, 1, \ldots, d-2\}$, the single facet $\mathrm{int} f$ has no ℓ-dimensional faces, whereas $\cup_{j=1}^{2(d-1)} F_j \setminus \tilde{S}_{d-2}(b)$ has a non-zero number of ℓ-dimensional faces. These arguments apply to all skeletons $S_{d-1,i}(b), i \in I$. By following nearly verbatim the arguments showing that c_4 is positive, we get that c_6 and c_8 are positive.

Positivity of c_{10}. We have that $\mathrm{Co}_\lambda(A_0)$ is defined in terms of $f_\lambda^{(\ell)}(A), \ell \in \{0, 1, \ldots, d-1\}$, and it suffices to note that configuration (b') leads to a complexity which is strictly larger than the complexity arising from configuration (b). We now follow the arguments that c_4 is strictly positive. □

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