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SURFACE ORDER SCALING IN STOCHASTIC GEOMETRY[1]

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Let $\mathcal{P}_\lambda := \mathcal{P}_{\lambda\kappa}$ denote a Poisson point process of intensity $\lambda\kappa$ on $[0,1]^d$, $d \geq 2$, with κ a bounded density on $[0,1]^d$ and $\lambda \in (0,\infty)$. Given a closed subset $\mathcal{M} \subset [0,1]^d$ of Hausdorff dimension $(d-1)$, we consider general statistics $\sum_{x\in\mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda, \mathcal{M})$, where the score function ξ vanishes unless the input x is close to $\mathcal{M}$ and where ξ satisfies a weak spatial dependency condition. We give a rate of normal convergence for the rescaled statistics $\sum_{x\in\mathcal{P}_\lambda} \xi(\lambda^{1/d}x, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\mathcal{M})$ as $\lambda \to \infty$. When $\mathcal{M}$ is of class C^2, we obtain weak laws of large numbers and variance asymptotics for these statistics, showing that growth is surface order, that is, of order $\mathrm{Vol}(\lambda^{1/d}\mathcal{M})$. We use the general results to deduce variance asymptotics and central limit theorems for statistics arising in stochastic geometry, including Poisson–Voronoi volume and surface area estimators, answering questions in Heveling and Reitzner [*Ann. Appl. Probab.* **19** (2009) 719–736] and Reitzner, Spodarev and Zaporozhets [*Adv. in Appl. Probab.* **44** (2012) 938–953]. The general results also yield the limit theory for the number of maximal points in a sample.

1. Main results.

1.1. *Introduction.* Let $\mathcal{P}_\lambda := \mathcal{P}_{\lambda\kappa}$ denote a Poisson point process of intensity $\lambda\kappa$ on $[0,1]^d$, $d \geq 2$, with κ a bounded density on $[0,1]^d$ and $\lambda \in (0,\infty)$. Letting $\xi(\cdot,\cdot)$ be a Borel measurable $\mathbb{R}$-valued function defined on pairs $(x, \mathcal{X})$, with $\mathcal{X} \subset \mathbb{R}^d$ finite and $x \in \mathcal{X}$, functionals in stochastic geometry may often be represented as linear statistics $\sum_{x\in\mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda)$. Here, $\xi(x, \mathcal{P}_\lambda)$ represents the contribution from x, which in general, depends on $\mathcal{P}_\lambda$. It is often more natural to consider rescaled statistics

$$H^\xi(\mathcal{P}_\lambda) := \sum_{x\in\mathcal{P}_\lambda} \xi(\lambda^{1/d}x, \lambda^{1/d}\mathcal{P}_\lambda). \tag{1.1}$$

Laws of large numbers, variance asymptotics and asymptotic normality as $\lambda \to \infty$ for such statistics are established in [6, 18–20, 22] with limits governed by the behavior of ξ at a point inserted into the origin of a homogeneous Poisson point process. The sums $H^\xi(\mathcal{P}_\lambda)$ exhibit growth of order $\mathrm{Vol}_d((\lambda^{1/d}[0,1])^d) = \lambda$, the d-dimensional volume measure of the set carrying the scaled input $\lambda^{1/d}\mathcal{P}_\lambda$. This

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gives the limit theory for score functions of nearest neighbor distances, Voronoi tessellations, percolation and germ grain models [6, 18, 20]. Problems of interest sometimes involve $\mathbb{R}$-valued score functions ξ of three arguments, with the third being a set $\mathcal{M} \subset \mathbb{R}^d$ of Hausdorff dimension $(d-1)$, and where scores $\xi(\lambda^{1/d}x, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\mathcal{M})$ vanish unless x is close to $\mathcal{M}$. This gives rise to

$$H^\xi(\mathcal{P}_\lambda, \mathcal{M}) := \sum_{x \in \mathcal{P}_\lambda} \xi\big(\lambda^{1/d}x, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\mathcal{M}\big). \tag{1.2}$$

Here, $\mathcal{M}$ might represent the boundary of the support of κ or more generally, the boundary of a fixed body, as would be the case in volume and surface integral estimators. We show that modifications of the methods used to study (1.1) yield the limit theory of (1.2), showing that the scaling is surface order, that is, $H^\xi(\mathcal{P}_\lambda, \mathcal{M})$ is order $\mathrm{Vol}_{d-1}(\lambda^{1/d}(\mathcal{M} \cap [0,1]^d)) = \Theta(\lambda^{(d-1)/d})$. The general limit theory for (1.2), as given in Section 1.2, yields variance asymptotics and central limit theorems for the Poisson–Voronoi volume estimator, answering questions posed in [12, 26]. We introduce a surface area estimator induced by Poisson–Voronoi tessellations and we use the general theory to obtain its consistency and variance asymptotics. Finally, the general theory yields the limit theory for the number of maximal points in random sample, including variance asymptotics and rates of normal convergence, extending [2]–[5]. See Section 2 for details. We anticipate further applications to germ-grain and continuum percolation models, but postpone treatment of this.

1.2. *General results.* We first introduce terminology, cf. [6, 18–20, 22]. Let $\mathbb{M}(d)$ denote the collection of closed sets $\mathcal{M} \subset [0,1]^d$ having finite $(d-1)$-dimensional Hausdorff measure. Elements of $\mathbb{M}(d)$ may or may not have boundary and are endowed with the subset topology of $\mathbb{R}^d$. Let $\mathbb{M}_2(d) \subset \mathbb{M}(d)$ denote those $\mathcal{M} \in \mathbb{M}(d)$ which are C^2, orientable submanifolds. Given $\mathcal{M} \in \mathbb{M}(d)$, almost all points $x \in [0,1]^d$ are uniquely represented as

$$x := y + t\mathbf{u}_y, \tag{1.3}$$

where $y := y_x \in \mathcal{M}$ is the closest point in $\mathcal{M}$ to x, $t := t_x \in \mathbb{R}$ and $\mathbf{u}_y$ is a fixed direction (see, e.g., Theorem 1G of [11], [13]); $\mathbf{u}_y$ coincides with the unit outward normal to $\mathcal{M}$ at y when $\mathcal{M} \in \mathbb{M}_2(d)$. We write $x = (y_x, t_x) := (y, t)$ and shorthand $(y, 0)$ as y when the context is clear. To avoid pathologies, we assume $\mathcal{H}^{d-1}(\mathcal{M} \cap \partial([0,1]^d)) = 0$. Here, $\mathcal{H}^{d-1}$ denotes $(d-1)$-dimensional Hausdorff measure, normalized to coincide with Vol_{d-1} on hyperplanes.

Let $\xi(x, \mathcal{X}, \mathcal{M})$ be a Borel measurable $\mathbb{R}$-valued function defined on triples $(x, \mathcal{X}, \mathcal{M})$, where $\mathcal{X} \subset \mathbb{R}^d$ is finite, $x \in \mathcal{X}$, and $\mathcal{M} \in \mathbb{M}(d)$. If $x \notin \mathcal{X}$, we shorthand $\xi(x, \mathcal{X} \cup \{x\}, \mathcal{M})$ as $\xi(x, \mathcal{X}, \mathcal{M})$. Let $S := S(\mathcal{M}) \subset [0,1]^d$ be the set of points admitting the unique representation (1.3) and put $S' := \{(y_x, t_x)\}_{x \in S}$. If $(y, t) \in S'$, then we put $\xi((y,t), \mathcal{X}, \mathcal{M}) = \xi(x, \mathcal{X}, \mathcal{M})$ where $x = y + t\mathbf{u}_y$, otherwise we put $\xi((y,t), \mathcal{X}, \mathcal{M}) = 0$.

We assume ξ is translation invariant, that is, for all $z \in \mathbb{R}^d$ and input $(x, \mathcal{X}, \mathcal{M})$ we have $\xi(x, \mathcal{X}, \mathcal{M}) = \xi(x+z, \mathcal{X}+z, \mathcal{M}+z)$. Given $\lambda \in [1, \infty)$, define dilated scores ξ_λ by

$$\xi_\lambda(x, \mathcal{X}, \mathcal{M}) := \xi(\lambda^{1/d}x, \lambda^{1/d}\mathcal{X}, \lambda^{1/d}\mathcal{M}), \tag{1.4}$$

so that (1.2) becomes

$$H^\xi(\mathcal{P}_\lambda, \mathcal{M}) := \sum_{x \in \mathcal{P}_\lambda} \xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M}). \tag{1.5}$$

We recall two weak spatial dependence conditions for ξ. For $\tau \in (0, \infty)$, $\mathcal{H}_\tau$ denotes the homogeneous Poisson point process of intensity τ on $\mathbb{R}^d$. For all $x \in \mathbb{R}^d$, $r \in (0, \infty)$, let $B_r(x) := \{w \in \mathbb{R}^d : \|x - w\| \le r\}$, where $\|\cdot\|$ denotes Euclidean norm. Let $\mathbf{0}$ denote a point at the origin of $\mathbb{R}^d$. Say that ξ is *homogeneously stabilizing* if for all $\tau \in (0, \infty)$ and all $(d-1)$-dimensional hyperplanes $\mathbb{H}$, there is $R := R^\xi(\mathcal{H}_\tau, \mathbb{H}) \in (0, \infty)$ a.s. (a radius of stabilization) such that

$$\xi(\mathbf{0}, \mathcal{H}_\tau \cap B_R(\mathbf{0}), \mathbb{H}) = \xi(\mathbf{0}, (\mathcal{H}_\tau \cap B_R(\mathbf{0})) \cup \mathcal{A}, \mathbb{H}) \tag{1.6}$$

for all locally finite $\mathcal{A} \subset B_R(\mathbf{0})^c$. Given (1.6), the definition of ξ extends to infinite Poisson input, that is, $\xi(\mathbf{0}, \mathcal{H}_\tau, \mathbb{H}) = \lim_{r\to\infty} \xi(\mathbf{0}, \mathcal{H}_\tau \cap B_r(\mathbf{0}), \mathbb{H})$.

Given $\mathcal{M} \in \mathbb{M}(d)$, say that ξ is *exponentially stabilizing* with respect to the pair $(\mathcal{P}_\lambda, \mathcal{M})$ if for all $x \in \mathbb{R}^d$ there is a radius of stabilization $R := R^\xi(x, \mathcal{P}_\lambda, \mathcal{M}) \in (0, \infty)$ a.s. such that

$$\xi_\lambda(x, \mathcal{P}_\lambda \cap B_{\lambda^{-1/d}R}(x), \mathcal{M}) = \xi_\lambda(x, (\mathcal{P}_\lambda \cap B_{\lambda^{-1/d}R}(x)) \cup \mathcal{A}, \mathcal{M}) \tag{1.7}$$

for all locally finite $\mathcal{A} \subset \mathbb{R}^d \setminus B_{\lambda^{-1/d}R}(x)$, and the tail probability $\tau(t) := \tau(t, \mathcal{M}) := \sup_{\lambda>0, x\in\mathbb{R}^d} P[R(x, \mathcal{P}_\lambda, \mathcal{M}) > t]$ satisfies $\limsup_{t\to\infty} t^{-1}\log\tau(t) < 0$.

Surface order growth for the sums at (1.5) should involve finiteness of the integrated score $\xi_\lambda((y,t), \mathcal{P}_\lambda, \mathcal{M})$ over $t \in \mathbb{R}$. Thus, it is natural to require the following condition. Given $\mathcal{M} \in \mathbb{M}(d)$ and $p \in [1, \infty)$, say that ξ *satisfies the p moment condition with respect to* $\mathcal{M}$ if there is a bounded integrable function $G^{\xi,p} := G^{\xi,p,\mathcal{M}} : \mathbb{R} \to \mathbb{R}^+$ such that for all $u \in \mathbb{R}$

$$\sup_{z\in\mathbb{R}^d\cup\varnothing} \sup_{y\in\mathcal{M}} \sup_{\lambda>0} \mathbb{E}|\xi_\lambda((y, \lambda^{-1/d}u), \mathcal{P}_\lambda \cup z, \mathcal{M})|^p \le G^{\xi,p}(|u|). \tag{1.8}$$

Say that ξ decays *exponentially fast* with respect to the distance to $\mathcal{M}$ if for all $p \in [1, \infty)$

$$\limsup_{|u|\to\infty} |u|^{-1}\log G^{\xi,p}(|u|) < 0. \tag{1.9}$$

Next, given $\mathcal{M} \in \mathbb{M}_2(d)$ and $y \in \mathcal{M}$, let $\mathbb{H}(y, \mathcal{M})$ be the $(d-1)$-dimensional hyperplane tangent to $\mathcal{M}$ at y. Put $\mathbb{H}_y := \mathbb{H}(\mathbf{0}, \mathcal{M} - y)$. The score ξ is *well-approximated by $\mathcal{P}_\lambda$ input on half-spaces* if for all $\mathcal{M} \in \mathbb{M}_2(d)$, all $y \in \mathcal{M}$, and all

$w \in \mathbb{R}^d$, we have

$$\lim_{\lambda\to\infty} \mathbb{E}\big|\xi\big(w, \lambda^{1/d}(\mathcal{P}_\lambda - y), \lambda^{1/d}(\mathcal{M} - y)\big) - \xi\big(w, \lambda^{1/d}(\mathcal{P}_\lambda - y), \mathbb{H}_y\big)\big| = 0. \tag{1.10}$$

We now give three general limit theorems, proved in Sections 4 and 5. In Section 2, we use these results to deduce the limit theory for statistics arising in stochastic geometry. Let $\mathcal{C}(\mathcal{M})$ denote the set of functions on $[0,1]^d$ which are continuous at all points $y \in \mathcal{M}$. Let $\mathbf{0}_y$ be a point at the origin of $\mathbb{H}_y$.

THEOREM 1.1 (Weak law of large numbers). *Assume $\mathcal{M} \in \mathbb{M}_2(d)$ and $\kappa \in \mathcal{C}(\mathcal{M})$. If ξ is homogeneously stabilizing* (1.6), *satisfies the moment condition* (1.8) *for some $p > 1$, and is well-approximated by $\mathcal{P}_\lambda$ input on half-spaces* (1.10), *then*

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} H^\xi(\mathcal{P}_\lambda, \mathcal{M}) &= \mu(\xi, \mathcal{M}) \\ &:= \int_{\mathcal{M}} \int_{-\infty}^{\infty} \mathbb{E}\xi\big((\mathbf{0}_y, u), \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big)\kappa(y)\,du\,dy \qquad \text{in } L^p.\end{aligned} \tag{1.11}$$

Next, for $x, x' \in \mathbb{R}^d$, $\tau \in (0,\infty)$, and all $(d-1)$-dimensional hyperplanes $\mathbb{H}$ we put

$$\begin{aligned}&c^\xi\big(x, x'; \mathcal{H}_\tau, \mathbb{H}\big) \\ &\quad := \mathbb{E}\xi\big(x, \mathcal{H}_\tau \cup x', \mathbb{H}\big)\xi\big(x', \mathcal{H}_\tau \cup x, \mathbb{H}\big) - \mathbb{E}\xi(x, \mathcal{H}_\tau, \mathbb{H})\mathbb{E}\xi\big(x', \mathcal{H}_\tau, \mathbb{H}\big).\end{aligned}$$

Put for all $\mathcal{M} \in \mathbb{M}_2(d)$

$$\begin{aligned}\sigma^2(\xi, \mathcal{M}) &:= \mu(\xi^2, \mathcal{M}) \\ &\quad + \int_{\mathcal{M}} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c^\xi\big((\mathbf{0}_y, u), (z, s); \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big) \\ &\qquad \times \kappa(y)^2\,du\,ds\,dz\,dy.\end{aligned} \tag{1.12}$$

THEOREM 1.2 (Variance asymptotics). *Assume $\mathcal{M} \in \mathbb{M}_2(d)$ and $\kappa \in \mathcal{C}(\mathcal{M})$. If ξ is homogeneously stabilizing* (1.6), *exponentially stabilizing* (1.7), *satisfies the moment condition* (1.8) *for some $p > 2$, and is well-approximated by $\mathcal{P}_\lambda$ input on half-spaces* (1.10), *then*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} \operatorname{Var}\big[H^\xi(\mathcal{P}_\lambda, \mathcal{M})\big] = \sigma^2(\xi, \mathcal{M}) \in [0,\infty). \tag{1.13}$$

Let $N(0, \sigma^2)$ denote a mean zero normal random variable with variance σ^2 and let $\Phi(t) := P[N(0,1) \le t]$, $t \in \mathbb{R}$, be the distribution function of the standard normal.

THEOREM 1.3 (Rate of convergence to the normal). *Assume $\mathcal{M} \in \mathbb{M}(d)$. If ξ is exponentially stabilizing (1.7) and satisfies exponential decay (1.9) for some $p > q$, $q \in (2,3]$, then there is a finite constant $c := c(d, \xi, p, q)$ such that for all $\lambda \geq 2$*

$$\sup_{t\in\mathbb{R}}\left|P\left[\frac{H^{\xi}(\mathcal{P}_\lambda,\mathcal{M}) - \mathbb{E}[H^{\xi}(\mathcal{P}_\lambda,\mathcal{M})]}{\sqrt{\mathrm{Var}[H^{\xi}(\mathcal{P}_\lambda,\mathcal{M})]}} \leq t\right] - \Phi(t)\right| \leq c(\log\lambda)^{dq+1}\lambda^{(d-1)/d}\big(\mathrm{Var}\big[H^{\xi}(\mathcal{P}_\lambda,\mathcal{M})\big]\big)^{-q/2}. \tag{1.14}$$

In particular, if $\sigma^2(\xi,\mathcal{M}) > 0$, then putting $q = 3$ yields a rate of convergence $O((\log\lambda)^{3d+1}\lambda^{-(d-1)/2d})$ to the normal distribution.

REMARKS. (i) (*Simplification of limits.*) If $\xi(x, \mathcal{X}, \mathcal{M})$ is invariant under rotations of $(x, \mathcal{X}, \mathcal{M})$, then the limit $\mu(\xi, \mathcal{M})$ at (1.11) simplifies to

$$\mu(\xi,\mathcal{M}) := \int_{\mathcal{M}}\int_{-\infty}^{\infty} \mathbb{E}\xi\big((\mathbf{0},u), \mathcal{H}_{\kappa(y)}, \mathbb{R}^{d-1}\big)\,du\,\kappa(y)\,dy, \tag{1.15}$$

where $(\mathbf{0}, u) \in \mathbb{R}^{d-1} \times \mathbb{R}$. The limit (1.12) simplifies to

$$\begin{aligned}\sigma^2(\xi,\mathcal{M}) &:= \mu(\xi^2,\mathcal{M})\\ &\quad + \int_{\mathcal{M}}\int_{\mathbb{R}^{d-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^{\xi}\big((\mathbf{0},u),(z,s);\mathcal{H}_{\kappa(y)},\mathbb{R}^{d-1}\big)\\ &\qquad \times \kappa(y)^2\,du\,ds\,dz\,dy.\end{aligned} \tag{1.16}$$

If, in addition, ξ is homogeneous of order γ in the sense that for all $a \in (0,\infty)$ we have

$$\xi\big(ax, a\mathcal{X}, \mathbb{R}^{d-1}\big) = a^{\gamma}\xi\big(x, \mathcal{X}, \mathbb{R}^{d-1}\big),$$

then putting

$$\mu(\xi, d) := \int_{-\infty}^{\infty} \mathbb{E}\xi\big((\mathbf{0},u), \mathcal{H}_1, \mathbb{R}^d\big)\,du \tag{1.17}$$

we get that $\mu(\xi, \mathcal{M})$ further simplifies to

$$\mu(\xi,\mathcal{M}) := \mu(\xi, d-1)\int_{\mathcal{M}} \kappa(y)^{(d-\gamma-1)/d}\,dy. \tag{1.18}$$

Similarly, the variance limit $\sigma^2(\xi,\mathcal{M})$ simplifies to

$$\begin{aligned}\sigma^2(\xi,\mathcal{M}) &:= \mu(\xi^2, d-1)\int_{\mathcal{M}}\kappa(y)^{(d-\gamma-1)/d}\,dy\\ &\quad + \int_{\mathbb{R}^{d-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^{\xi}\big((\mathbf{0},u),(z,s);\mathcal{H}_1,\mathbb{R}^{d-1}\big)\,du\,ds\,dz\\ &\quad \times \int_{\mathcal{M}}\kappa(y)^{(d-2\gamma-2)/d}\,dy.\end{aligned}$$

If $\kappa \equiv 1$, then putting

$$\nu(\xi, d) := \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c^{\xi}\big((\mathbf{0}, u), (z, s); \mathcal{H}_1, \mathbb{R}^d\big)\, du\, ds\, dz \tag{1.19}$$

we get that (1.11) and (1.13), respectively, reduce to

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} H^{\xi}(\mathcal{P}_\lambda, \mathcal{M}) = \mu(\xi, d-1)\mathcal{H}^{d-1}(\mathcal{M}) \qquad \text{in } L^p \tag{1.20}$$

and

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} \operatorname{Var}\big[H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})\big] \\ &\quad = \big[\mu\big(\xi^2, d-1\big) + \nu(\xi, d-1)\big]\mathcal{H}^{d-1}(\mathcal{M}). \end{aligned} \tag{1.21}$$

(ii) (*A scalar central limit theorem.*) Under the hypotheses of Theorems 1.2 and 1.3, we obtain as $\lambda \to \infty$,

$$\lambda^{-(d-1)/2d}\big(H^{\xi}(\mathcal{P}_\lambda, \mathcal{M}) - \mathbb{E}H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})\big) \xrightarrow{\mathcal{D}} N\big(0, \sigma^2(\xi, \mathcal{M})\big). \tag{1.22}$$

In general, separate arguments are needed to show strict positivity of $\sigma^2(\xi, \mathcal{M})$.

(iii) (*Extensions to binomial input.*) By coupling $\mathcal{P}_\lambda$ and binomial input $\{X_i\}_{i=1}^n$, where $X_i, i \geq 1$, are i.i.d. with density κ, it may be shown that Theorems 1.1 and 1.2 hold for input $\{X_i\}_{i=1}^n$ under additional assumptions on ξ. See Lemma 6.1.

(iv) (*Extensions to random measures.*) Consider the random measure

$$\mu_\lambda^{\xi} := \sum_{x\in\mathcal{P}_\lambda} \xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})\delta_x,$$

where δ_x denotes the Dirac point mass at x. For $f \in B([0, 1]^d)$, the class of bounded functions on $[0, 1]^d$, we put $\langle f, \mu_\lambda^{\xi}\rangle := \int f\, d\mu_\lambda^{\xi}$. Modifications of the proof of Theorem 1.1 show that when $f \in \mathcal{C}([0, 1]^d)$, we have L^p, $p \in \{1, 2\}$, convergence

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\langle f, \mu_\lambda^{\xi}\rangle \\ &\quad = \mu(\xi, \mathcal{M}, f) \\ &\quad := \int_{\mathcal{M}} \int_{-\infty}^{\infty} \mathbb{E}\xi\big((\mathbf{0}_y, u), \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big)\kappa(y) f(y)\, du\, dy. \end{aligned} \tag{1.23}$$

Using that a.e. $x \in [0, 1]^d$ is a Lebesgue point for f, it may be shown this limit extends to $f \in B([0, 1]^d)$ (Lemma 3.5 of [18] and Lemma 3.5 of [19]). The limit (1.23) shows up in surface integral approximation as seen in Theorem 2.4 in Section 2.2.

Likewise, under the assumptions of Theorem 1.2, it may be shown for all $f \in B([0, 1]^d)$ that

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} \operatorname{Var}[\langle f, \mu_\lambda^{\xi}\rangle] = \sigma^2(\xi, \mathcal{M}, f),$$

where

$$
\begin{aligned}
\sigma^2(\xi, \mathcal{M}, f) := {} & \mu(\xi^2, \mathcal{M}, f^2) \\
& + \int_{\mathcal{M}} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c^{\xi}\big((\mathbf{0}_y, u), (z, s); \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big) \\
& \qquad \times \kappa(y)^2 f(y)^2 \, du \, ds \, dz \, dy.
\end{aligned}
$$

Finally, under the assumptions of Theorem 1.3, we get the rate of convergence (1.14) with $H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})$ replaced by $\langle f, \mu_\lambda^{\xi} \rangle$.

(v) (*Comparison with* [22].) Theorem 1.3 is the surface order analog of Theorem 2.1 of [22]. Were one to directly apply the latter result to $H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})$, one would get

$$
\begin{aligned}
& \sup_{t \in \mathbb{R}} \left| P\left[\frac{H^{\xi}(\mathcal{P}_\lambda, \mathcal{M}) - \mathbb{E} H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})}{\sqrt{\operatorname{Var}[H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})]}} \le t \right] - \Phi(t) \right| \\
& \qquad = O\big((\log \lambda)^{3d+1} \lambda \big(\operatorname{Var}\big[H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})\big]\big)^{-3/2}\big).
\end{aligned}
\tag{1.24}
$$

However, when $\operatorname{Var}[H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})] = \Omega(\lambda^{(d-1)/d})$, as is the case in Theorem 1.2, the right-hand side of (1.24) is $O((\log \lambda)^{3d+1} \lambda^{-(d+1)/2d})$. The reason for this suboptimal rate is that [22] considers sums of $\Theta(\lambda)$ nonnegligible contributions $\xi(x, \mathcal{P}_\lambda)$, whereas here, due to condition (1.9), the number of nonnegligible contributions is surface order, that is, of order $\Theta(\lambda^{(d-1)/d})$.

(vi) (*Comparison with* [23].) Let $\mathcal{M} \in \mathbb{M}_2(d)$. In contrast with the present paper, [23] considers statistics $H^{\xi}(\mathcal{Y}_n) := \sum_{i=1}^{n} \xi(n^{1/(d-1)} Y_i, n^{1/(d-1)} \mathcal{Y}_n)$, with input $\mathcal{Y}_n := \{Y_j\}_{j=1}^{n}$ carried by $\mathcal{M}$ rather than $[0, 1]^d$. In this set-up, $H^{\xi}(\mathcal{Y}_n)$ exhibits growth $\Theta(n)$.

2. Applications.

2.1. *Poisson–Voronoi volume estimators.* Given $\mathcal{P}_\lambda$ as in Section 1 and an unknown Borel set $A \subset [0, 1]^d$, suppose one can determine which points in the realization of $\mathcal{P}_\lambda$ belong to A and which belong to $A^c := [0, 1]^d \setminus A$. How can one use this information to establish consistent statistical estimators of geometric properties of A, including $\operatorname{Vol}(A)$ and $\mathcal{H}^{d-1}(\partial A)$? Here and henceforth, we shorthand Vol_d by Vol. In this section, we use our general results to give the limit theory for a well-known estimator of $\operatorname{Vol}(A)$; the next section proposes a new estimator of $\mathcal{H}^{d-1}(\partial A)$ and gives its limit theory as well.

For $\mathcal{X} \subset \mathbb{R}^d$ locally finite and $x \in \mathcal{X}$, let $C(x, \mathcal{X})$ denote the Voronoi cell generated by $\mathcal{X}$ and with center x. Given $\mathcal{P}_\lambda$ and a Borel set $A \subset [0, 1]^d$, the Poisson–Voronoi approximation of A is the union of Voronoi cells with centers inside A, namely

$$A_\lambda := \bigcup_{x \in \mathcal{P}_\lambda \cap A} C(x, \mathcal{P}_\lambda).$$

The set A_λ was introduced by Khmaladze and Toronjadze [16], who anticipated that A_λ should well-approximate the target A in the sense that a.s. $\lim_{\lambda\to\infty} \mathrm{Vol}(A \Delta A_\lambda) = 0$. This conjectured limit holds; as shown by [16] when $d = 1$ and by Penrose [18] for all $d = 1, 2, \dots$. Additionally, if $\mathcal{P}_\lambda$ is replaced by a homogeneous Poisson point process on $\mathbb{R}^d$ of intensity λ, then $\mathrm{Vol}(A_\lambda)$ is an unbiased estimator of $\mathrm{Vol}(A)$ (cf. [26]), rendering A_λ of interest in image analysis, nonparametric statistics and quantization, as discussed in Section 1 of [16] as well as Section 1 of Heveling and Reitzner [12].

Heuristically, $\mathrm{Vol}(A_\lambda) - \mathbb{E}\mathrm{Vol}(A_\lambda)$ involves cell volumes $\mathrm{Vol}(C(x, \mathcal{P}_\lambda))$, $x \in \mathcal{P}_\lambda$, within $O(\lambda^{-1/d})$ of ∂A. The number of such terms is of surface order, that is there are roughly $O(\lambda^{(d-1)/d})$ such terms, each contributing roughly $O(\lambda^{-2})$ toward the total variance. Were the terms spatially independent, one might expect that as $\lambda \to \infty$,

$$\lambda^{(d+1)/2d}\big(\mathrm{Vol}(A_\lambda) - \mathbb{E}\mathrm{Vol}(A_\lambda)\big) \xrightarrow{\mathcal{D}} N\big(0, \sigma^2\big), \tag{2.1}$$

as conjectured in Remark 2.2 of [26]. We use Theorems 1.2–1.3 to prove this conjecture and to obtain a closed form expression for σ^2 when $\partial A \in \mathbb{M}_2(d)$; we find rates of normal convergence for $(\mathrm{Vol}(A_\lambda) - \mathbb{E}\mathrm{Vol}(A_\lambda))/\sqrt{\mathrm{Var}\,\mathrm{Vol}(A_\lambda)}$ assuming only $\partial A \in \mathbb{M}(d)$. This adds to Schulte [27], who for $\kappa \equiv 1$ and A compact, convex, shows that $(\mathrm{Var}\,\mathrm{Vol}(A_\lambda))^{-1/2}(\mathrm{Vol}(A_\lambda) - \mathbb{E}\mathrm{Vol}(A_\lambda))$ is asymptotically normal, $\lambda \to \infty$. We obtain analogous limits for $\mathrm{Vol}(A \Delta A_\lambda)$. In addition to the standing assumption $\|\kappa\|_\infty < \infty$, we assume everywhere in this section that κ is bounded away from zero on $[0, 1]^d$.

THEOREM 2.1. *If $\partial A \in \mathbb{M}(d)$, then*

$$\sup_{t\in\mathbb{R}} \left| P\left[\frac{\mathrm{Vol}(A_\lambda) - \mathbb{E}\mathrm{Vol}(A_\lambda)}{\sqrt{\mathrm{Var}\,\mathrm{Vol}(A_\lambda)}} \le t\right] - \Phi(t)\right| = O\big((\log\lambda)^{3d+1}\lambda^{-2-1/d}\big(\mathrm{Var}\,\mathrm{Vol}(A_\lambda)\big)^{-3/2}\big)$$

and

$$\sup_{t\in\mathbb{R}} \left| P\left[\frac{\mathrm{Vol}(A \Delta A_\lambda) - \mathbb{E}\mathrm{Vol}(A \Delta A_\lambda)}{\sqrt{\mathrm{Var}\,\mathrm{Vol}(A \Delta A_\lambda)}} \le t\right] - \Phi(t)\right| = O\big((\log\lambda)^{3d+1}\lambda^{-2-1/d}\big(\mathrm{Var}\,\mathrm{Vol}(A \Delta A_\lambda)\big)^{-3/2}\big).$$

The rate of convergence is uninformative without lower bounds on $\mathrm{Var}\,\mathrm{Vol}(A_\lambda)$ and $\mathrm{Var}\,\mathrm{Vol}(A \Delta A_\lambda)$. Schulte [27] shows $\mathrm{Var}\,\mathrm{Vol}(A_\lambda) = \Omega(\lambda^{-(d+1)/d})$ when A is compact and convex. The next result provides lower bounds when ∂A contains a smooth subset. For locally finite $\mathcal{X} \subset \mathbb{R}^d$, $x \in \mathcal{X}$, define the volume scores

$$
\begin{aligned}
&\nu^{\pm}(x, \mathcal{X}, \partial A) \\
&\quad := \begin{cases} \operatorname{Vol}(C(x, \mathcal{X}) \cap A^c), & \text{if } C(x, \mathcal{X}) \cap \partial A \neq \varnothing, x \in A, \\ \pm\operatorname{Vol}(C(x, \mathcal{X}) \cap A), & \text{if } C(x, \mathcal{X}) \cap \partial A \neq \varnothing, x \in A^c, \\ 0, & \text{if } C(x, \mathcal{X}) \cap \partial A = \varnothing. \end{cases}
\end{aligned} \tag{2.2}
$$

In view of limits such as (1.16), we need to define scores on hyperplanes $\mathbb{R}^{d-1}$. We thus put

$$
\begin{aligned}
&\nu^{\pm}(x, \mathcal{X}, \mathbb{R}^{d-1}) \\
&\quad := \begin{cases} \operatorname{Vol}(C(x, \mathcal{X}) \cap \mathbb{R}_+^{d-1}), & \text{if } C(x, \mathcal{X}) \cap \mathbb{R}^{d-1} \neq \varnothing, x \in \mathbb{R}_-^{d-1}, \\ \pm\operatorname{Vol}(C(x, \mathcal{X}) \cap \mathbb{R}_-^{d-1}), & \text{if } C(x, \mathcal{X}) \cap \mathbb{R}^{d-1} \neq \varnothing, x \in \mathbb{R}_+^{d-1}, \\ 0, & \text{if } C(x, \mathcal{X}) \cap \mathbb{R}^{d-1} = \varnothing, \end{cases}
\end{aligned} \tag{2.3}
$$

where $\mathbb{R}_+^{d-1} := \mathbb{R}^{d-1} \times [0, \infty)$ and $\mathbb{R}_-^{d-1} := \mathbb{R}^{d-1} \times (-\infty, 0]$. Define $\sigma^2(\nu^-, \partial A)$ by putting ξ and $\mathcal{M}$ to be ν^- and ∂A, respectively, in (1.16). Similarly, define $\sigma^2(\nu^+, \partial A)$. When $\kappa \equiv 1$, these expressions further simplify as at (1.21).

THEOREM 2.2. *If $\kappa \in \mathcal{C}(\partial A)$ and if ∂A contains a C^1 open subset, then*

$$
\operatorname{Var}\operatorname{Vol}(A_\lambda) = \Omega(\lambda^{-(d+1)/d}) \quad \textit{and} \quad \operatorname{Var}\operatorname{Vol}(A \Delta A_\lambda) = \Omega(\lambda^{-(d+1)/d}).
$$

Additionally, if $\partial A \in \mathbb{M}_2(d)$, then

$$
\lim_{\lambda \to \infty} \lambda^{(d+1)/d} \operatorname{Var}\operatorname{Vol}(A_\lambda) = \sigma^2(\nu^-, \partial A) \quad \textit{and}
$$

$$
\lim_{\lambda \to \infty} \lambda^{(d+1)/d} \operatorname{Var}\operatorname{Vol}(A \Delta A_\lambda) = \sigma^2(\nu^+, \partial A).
$$

Combining the above results gives the following central limit theorem for $\operatorname{Vol}(A_\lambda) - \operatorname{Vol}(A)$; identical results hold for $\operatorname{Vol}(A \Delta A_\lambda) - \mathbb{E}\operatorname{Vol}(A \Delta A_\lambda)$.

COROLLARY 2.1. *If $\kappa \in \mathcal{C}(\partial A)$ and if either ∂A contains a C^1 open subset or A is compact and convex, then*

$$
\sup_{t \in \mathbb{R}} \left| P\left[\frac{\operatorname{Vol}(A_\lambda) - \mathbb{E}\operatorname{Vol}(A_\lambda)}{\sqrt{\operatorname{Var}\operatorname{Vol}(A_\lambda)}} \leq t\right] - \Phi(t) \right| = O\big((\log \lambda)^{3d+1} \lambda^{-(d-1)/2d}\big).
$$

Additionally, if $\partial A \in \mathbb{M}_2(d)$, then as $\lambda \to \infty$

$$
\lambda^{(d+1)/2d}(\operatorname{Vol}(A_\lambda) - \mathbb{E}\operatorname{Vol}(A_\lambda)) \xrightarrow{\mathcal{D}} N(0, \sigma^2(\nu^-, \partial A)).
$$

Recall X_i, $i \geq 1$, are i.i.d. with density κ; $\mathcal{X}_n := \{X_i\}_{i=1}^n$. The binomial-Voronoi approximation of A is $A_n := \bigcup_{X_i \in A} C(X_i, \mathcal{X}_n)$. The above theorems extend to binomial input as follows.

THEOREM 2.3. *If $\kappa \in \mathcal{C}(\partial A)$ and if either ∂A contains a C^1 open subset or A is compact and convex, then*

$$\operatorname{Var}\operatorname{Vol}(A_n) = \Omega\big(n^{-(d+1)/d}\big) \quad \textit{and} \quad \operatorname{Var}\operatorname{Vol}(A \Delta A_n) = \Omega\big(n^{-(d+1)/d}\big).$$

Additionally, if $\partial A \in \mathbb{M}_2(d)$, then

$$\lim_{n\to\infty} n^{(d+1)/d} \operatorname{Var}\operatorname{Vol}(A_n) = \sigma^2\big(\nu^-, \partial A\big),$$

$$\lim_{n\to\infty} n^{(d+1)/d} \operatorname{Var}\operatorname{Vol}(A \Delta A_n) = \sigma^2\big(\nu^+, \partial A\big),$$

and as $n \to \infty$,

$$n^{(d+1)/2d}\big(\operatorname{Vol}(A_n) - \mathbb{E}\operatorname{Vol}(A_\lambda)\big) \xrightarrow{\mathcal{D}} N\big(0, \sigma^2\big(\nu^-, \partial A\big)\big).$$

REMARKS. (i) (*Theorem* 2.2.) When $\kappa \equiv 1$, Theorem 2.2 and (1.21) show that the limiting variance of $\operatorname{Vol}(A_\lambda)$ and $\operatorname{Vol}(A \Delta A_\lambda)$ involve multiples of $\mathcal{H}^{d-1}(\partial A)$, settling a conjecture implicit in Remark 2.2 of [26] when $\partial A \in \mathbb{M}_2(d)$. Up to now, it has been known that $\operatorname{Var}\operatorname{Vol}(A_\lambda) = \Theta(\lambda^{-(d+1)/d})$ for A compact and convex, where the upper and lower bounds follow from [12] and [27], respectively.

(ii) (*Corollary* 2.1.) When ∂A contains a C^1 open subset, Corollary 2.1 answers the first conjecture in Remark 2.2 of [12]; when A is convex it establishes a rate of normal convergence for $(\operatorname{Vol}(A_\lambda) - \mathbb{E}\operatorname{Vol}(A_\lambda))/\sqrt{\operatorname{Var}\operatorname{Vol}(A_\lambda)}$, extending the main result of [27] (Theorem 1.1).

(iii) (*The C^2 assumption.*) If $A \subset \mathbb{R}^d$ has finite perimeter, denoted $\operatorname{Per}(A)$, then [26] shows that $\lim_{\lambda\to\infty} \lambda^{1/d}\mathbb{E}\operatorname{Vol}(A \Delta A_\lambda) = c_d \operatorname{Per}(A)$, where c_d is an explicit constant depending only on dimension. This remarkable result, based on covariograms, holds with no other assumptions on A. Theorem 2.2 and Corollary 2.1 hold for ∂A not necessarily in $\mathbb{M}_2(d)$; see [29].

2.2. *Poisson–Voronoi surface integral estimators.* We show that the surface area of A_λ, when corrected by a factor independent of A, consistently estimates $\mathcal{H}^{d-1}(\partial A)$ and that it satisfies the limits in Theorems 1.1–1.3.

Given $\mathcal{X}$ locally finite and a Borel subset $A \subset \mathbb{R}^d$, define for $x \in \mathcal{X} \cap A$ the area score $\alpha(x, \mathcal{X}, \partial A)$ to be the $\mathcal{H}^{d-1}$ measure of the $(d-1)$-dimensional faces of $C(x, \mathcal{X})$ belonging to the boundary of $\bigcup_{w\in\mathcal{X}\cap A} C(w, \mathcal{X})$; if there are no such faces or if $x \notin \mathcal{X} \cap A$, then set $\alpha(x, \mathcal{X}, \partial A)$ to be zero. Similarly, for $x \in \mathcal{X} \cap \mathbb{R}_-^{d-1}$, put $\alpha(x, \mathcal{X}, \mathbb{R}^{d-1})$ to be the $\mathcal{H}^{d-1}$ measure of the $(d-1)$-dimensional faces of $C(x, \mathcal{X})$ belonging to the boundary of $\bigcup_{w\in\mathcal{X}\cap\mathbb{R}_-^{d-1}} C(w, \mathcal{X})$, otherwise $\alpha(x, \mathcal{X}, \mathbb{R}^{d-1})$ is zero.

The surface area of A_λ is then given by $\sum_{x\in\mathcal{P}_\lambda} \alpha(x, \mathcal{P}_\lambda, \partial A)$. We might expect that the statistic

$$\lambda^{-(d-1)/d} H^\alpha(\mathcal{P}_\lambda, \partial A) = \lambda^{-(d-1)/d} \sum_{x\in\mathcal{P}_\lambda} \alpha_\lambda(x, \mathcal{P}_\lambda, \partial A) \tag{2.4}$$

consistently estimates $\mathcal{H}^{d-1}(\partial A)$, $\lambda \to \infty$, and more generally, for $f \in B([0,1]^d)$ that

$$\lambda^{-(d-1)/d} \sum_{x \in \mathcal{P}_\lambda} \alpha_\lambda(x, \mathcal{P}_\lambda, \partial A) f(x)$$

consistently estimates the surface integral $\int_{\partial A} f(x)\mathcal{H}^{d-1}(dx)$. Provided that one introduces a *universal correction factor which is independent of the target* A, this turns out to be the case, as seen in the next theorem. Define $\mu(\alpha, d)$ and $\nu(\alpha, d)$ by putting ξ to be α in (1.17) and (1.19), respectively.

THEOREM 2.4. *If* $\kappa \equiv 1$ *and* $\partial A \in \mathbb{M}_2(d)$, *then*

$$\lim_{\lambda \to \infty} \big(\mu(\alpha, d-1)\big)^{-1} \mathcal{H}^{d-1}(\partial A_\lambda) = \mathcal{H}^{d-1}(\partial A) \qquad \text{in } L^2 \tag{2.5}$$

and

$$\begin{aligned} &\lim_{\lambda \to \infty} \lambda^{(d-1)/d} \operatorname{Var}\big[\mathcal{H}^{d-1}(\partial A_\lambda)\big] \\ &\qquad = \big[\mu\big(\alpha^2, d-1\big) + \nu(\alpha, d-1)\big]\mathcal{H}^{d-1}(\partial A). \end{aligned} \tag{2.6}$$

Further, for $f \in B([0,1]^d)$

$$\begin{aligned} &\lim_{\lambda \to \infty} \big(\mu(\alpha, d-1)\big)^{-1} \lambda^{-(d-1)/d} \sum_{x \in \mathcal{P}_\lambda} \alpha_\lambda(x, \mathcal{P}_\lambda, \partial A) f(x) \\ &\qquad = \int_{\partial A} f(x) \mathcal{H}^{d-1}(dx) \qquad \text{in } L^2. \end{aligned} \tag{2.7}$$

REMARKS. (i) (*Extensions.*) Assuming only $\partial A \in \mathbb{M}(d)$, it follows from Theorem 1.3 and the upcoming proof of Theorem 2.4 that $(\operatorname{Var} \mathcal{H}^{d-1}(\partial A_\lambda))^{-1/2} \times (\mathcal{H}^{d-1}(\partial A_\lambda) - \mathbb{E}\mathcal{H}^{d-1}(\partial A_\lambda))$ is asymptotically normal. When $\partial A \in \mathbb{M}_2(d)$ it follows by (1.22) that as $\lambda \to \infty$

$$\lambda^{-(d-1)/2d}\big(\mathcal{H}^{d-1}(\partial A_\lambda) - \mathbb{E}\mathcal{H}^{d-1}(\partial A_\lambda)\big) \xrightarrow{\mathcal{D}} N\big(0, \sigma^2\big),$$

with $\sigma^2 := [\mu(\alpha^2, d-1) + \nu(\alpha, d-1)]\mathcal{H}^{d-1}(\partial A)$. Analogs of (2.5)–(2.7) hold if $\mathcal{P}_\lambda$ is replaced by $\mathcal{X}_n := \{X_i\}_{i=1}^n$, A_λ is replaced by $A_n := \bigcup_{X_i \in A} C(X_i, \mathcal{X}_n)$, and $n \to \infty$.

(ii) (*Related work.*) Using the Delaunay triangulation of $\mathcal{P}_\lambda$, [15] introduces an a.s. consistent estimator of surface integrals of possibly nonsmooth boundaries. The limit theory for the Poisson–Voronoi estimator $H^\alpha(\mathcal{P}_\lambda, \partial A)$ extends to nonsmooth ∂A as in [29].

2.3. *Maximal points*. Let $K \subset \mathbb{R}^d$ be a cone with nonempty interior and apex at the origin of $\mathbb{R}^d$. Given $\mathcal{X} \subset \mathbb{R}^d$ locally finite, $x \in \mathcal{X}$ is called K-maximal, or simply maximal if $(K \oplus x) \cap \mathcal{X} = x$. Here, $K \oplus x$ is Minkowski addition, namely $K \oplus x := \{z + x : z \in K\}$. In the case $K = (\mathbb{R}^+)^d$, a point $x = (x_1, \dots, x_d) \in \mathcal{X}$ is maximal if there is no other point $(z_1, \dots, z_d) \in \mathcal{X}$ with $z_i \geq x_i$ for all $1 \leq i \leq d$. The maximal layer $m_K(\mathcal{X})$ is the collection of maximal points in $\mathcal{X}$. Let $M_K(\mathcal{X}) := \mathrm{card}(m_K(\mathcal{X}))$.

Maximal points feature in various disciplines. They are of broad interest in computational geometry; see books by Preparata and Shamos [25], Chen et al. [8]. Maximal points appear in pattern classification, multicriteria decision analysis, networks, data mining, analysis of linear programming and statistical decision theory; see Ehrgott [10] and Pomerol and Barba-Romero [24]. In economics, when $K = (\mathbb{R}^+)^d$, the maximal layer and K are termed the Pareto set and Pareto cone, respectively; see Sholomov [28] for a survey on Pareto optimality.

Next, let κ be a density having support

$$A := \big\{(v, w) : v \in D, 0 \leq w \leq F(v)\big\},$$

where $F : D \to \mathbb{R}$ has continuous partials F_i, $1 \leq i \leq d-1$, which are bounded away from zero and negative infinity; $D \subset [0, 1]^{d-1}$, and $|F| \leq 1$. Let $\mathcal{P}_\lambda := \mathcal{P}_{\lambda\kappa}$ and $\mathcal{X}_n := \{X_i\}_{i=1}^n$ as above.

Using Theorems 1.1–1.3, we deduce laws of large numbers, variance asymptotics, and central limit theorems for $M_K(\mathcal{P}_\lambda)$ and $M_K(\mathcal{X}_n)$, as $\lambda \to \infty$ and $n \to \infty$, respectively. Put $\partial A := \{(v, F(v)) : v \in D\}$ and let

$$\zeta(x, \mathcal{X}, \partial A) := \begin{cases} 1, & \text{if } \big((K \oplus x) \cap A\big) \cap \mathcal{X} = x, \\ 0, & \text{otherwise.} \end{cases}$$

When $x = (y, t)$, $y \in \partial A$, we write

$$\zeta(x, \mathcal{X}, \mathbb{H}_y) := \begin{cases} 1, & \text{if } \big((K \oplus x) \cap \mathbb{H}_+(y, \partial A)\big) \cap \mathcal{X} = x, \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

where $\mathbb{H}_+(y, \partial A)$ is the half-space containing $\mathbf{0}$ and with hyperplane $\mathbb{H}(y, \partial A)$.

To simplify the presentation, we take $K = (\mathbb{R}^+)^d$, but the results extend to general cones. Recalling definitions (1.11) and (1.12), we have the following results.

THEOREM 2.5. *If $\kappa \in \mathcal{C}(\partial A)$ and if κ is bounded away from 0 on A, then*

$$\begin{aligned} &\lim_{\lambda \to \infty} \lambda^{-(d-1)/d} M_K(\mathcal{P}_\lambda) \\ &= \mu(\zeta, \partial A) \qquad (2.9) \\ &= (d!)^{1/d} d^{-1} \Gamma(d^{-1}) \int_D \left| \prod_{i=1}^{d-1} F_i(v) \right|^{1/d} \kappa\big(v, F(v)\big)^{(d-1)/d}\, dv \qquad \textit{in } L^2 \end{aligned}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} \operatorname{Var}[M_K(\mathcal{P}_\lambda)] = \sigma^2(\zeta, \partial A) \in (0,\infty). \tag{2.10}$$

Moreover, as $\lambda \to \infty$, *we have*

$$\lambda^{-(d-1)/2d}\big(M_K(\mathcal{P}_\lambda) - \mathbb{E}M_K(\mathcal{P}_\lambda)\big) \xrightarrow{\mathcal{D}} N\big(0, \sigma^2(\zeta, \partial A)\big).$$

Identical limits hold with $M_K(\mathcal{P}_\lambda)$ *replaced by* $M_K(\mathcal{X}_n)$, $n \to \infty$. *We also have*

$$\sup_{t\in\mathbb{R}}\left| P\left[\frac{M_K(\mathcal{P}_\lambda) - \mathbb{E}M_K(\mathcal{P}_\lambda)}{\sqrt{\operatorname{Var}[M_K(\mathcal{P}_\lambda)]}} \le t\right] - \Phi(t)\right| \le c(\log\lambda)^{3q+1}\lambda^{(d-1)/2d}. \tag{2.11}$$

REMARKS. (i) (*Related expectation and variance asymptotics*.) Formula (2.10) is new for all dimensions d, whereas formula (2.9) is new for $d > 2$. For $d = 2$, (2.9) extends work of Devroye [9], who treats the case $\kappa \equiv 1$. Barbour and Xia [3, 4] establish growth rates for $\operatorname{Var}[M_K(\mathcal{P}_\lambda)]$ but do not determine limiting means or variances for $d > 2$. Hwang and Tsai [14] determine $\mathbb{E}M_K(\mathcal{X}_n)$ and $\operatorname{Var} M_K(\mathcal{X}_n)$ when $A := \{(x_1,\dots,x_d) : x_i \ge 0, \sum_{i=1}^d x_i \le 1\}$, that is, ∂A is a subset of the plane $\sum_{i=1}^d x_i = 1$.

(ii) (*Related central limit theorems*.) Using Stein's method, Barbour and Xia [3, 4] show for $d = 2$, κ uniform and $K = (\mathbb{R}^+)^2$ that $(M_K(\mathcal{X}_n) - \mathbb{E}M_K(\mathcal{X}_n))/\sqrt{\operatorname{Var} M_K(\mathcal{X}_n)}$ tends to a standard normal. Assuming differentiability conditions on F, they find rates of normal convergence of $M_K(\mathcal{X}_n)$ and $M_K(\mathcal{P}_\lambda)$ with respect to the bounded Wasserstein distance [3] and the Kolmogorov distance [4], respectively. Their work adds to Bai et al. [2], which for $K = (\mathbb{R}^+)^2$ establishes variance asymptotics and central limit theorems when κ is uniform on a convex polygonal region, and Baryshnikov [5], who proves a central limit theorem under general conditions on ∂A, still in the setting of homogeneous point sets.

(iii) (*Related results*.) Parametrizing points in $\mathbb{R}^d$ with respect to a fixed $(d-1)$-dimensional plane $\mathbb{H}_0$, the preprint [7] obtains expectation and variance asymptotics for $M_K(\mathcal{P}_\lambda)$ and $M_K(\mathcal{X}_n)$, with limits depending on an integral over the projection of ∂A onto $\mathbb{H}_0$. By comparison, the limits in Theorem 2.5 follow straightforwardly from the general limit theorems and exhibit an explicit dependence on the graph of F, that is, ∂A. Preprint [7] uses cumulants to show asymptotic normality without delivering the rate of convergence offered by Theorem 1.3.

(iv) (*Extensions*.) Separate analysis is needed to extend Theorem 2.5 to spherical boundaries $\mathbb{S}^{d-1} \cap [0,\infty)^d$, that is to say quarter circles in $d = 2$.

2.4. *Navigation in Poisson–Voronoi tessellations.* Put $\kappa \equiv 1$. Let $\mathcal{X} \subset \mathbb{R}^2$ be locally finite and let $r(t)$, $0 \le t \le 1$, be a C^1 curve $\mathcal{C}$ in $[0,1]^2$. Let $\mathcal{V}_\mathcal{C} := \mathcal{V}_\mathcal{C}(\mathcal{X})$ be the union of the Voronoi cells $C(x, \mathcal{X})$ meeting $\mathcal{C}$. Order the constituent cells of

$\mathcal{V}_C$ according to the "time" at which $r(t)$ first meets the cells. Enumerate the cells as

$$C(x_1, \mathcal{X}, \mathcal{C}), \dots, C(x_N, \mathcal{X}, \mathcal{C}); \qquad N \text{ random}.$$

The piecewise linear path joining the nodes $x_1, \dots, x_N$ is a path $\mathcal{C}(\mathcal{X})$ whose length $|\mathcal{C}(\mathcal{X})|$ approximates the length of $\mathcal{C}$. The random path $\mathcal{C}(\mathcal{P}_\lambda)$ has been studied by Bacelli et al. [1], which restricts to linear $\mathcal{C}$. For all $x \in \mathcal{X}$ define the score

$$\rho(x, \mathcal{X}, \mathcal{C}) := \begin{cases} \text{one half the sum of lengths of edges incident to } x \text{ in} \\ \qquad \mathcal{C}(\mathcal{X}) \text{ if } x \in \mathcal{C}(\mathcal{X}), \\ 0, \qquad \text{otherwise}. \end{cases}$$

Then the path length $|\mathcal{C}(\mathcal{P}_\lambda)|$ satisfies

$$\left|\mathcal{C}(\mathcal{P}_\lambda)\right| = \sum_{x \in \mathcal{P}_\lambda} \rho(x, \mathcal{P}_\lambda, \mathcal{C}) = \lambda^{-1/2} H^\rho(\mathcal{P}_\lambda, \mathcal{C}).$$

We claim that the score ρ satisfies the conditions of Theorems 1.1–1.3 and that therefore the limit theory of $|\mathcal{C}(\mathcal{P}_\lambda)|$ may be deduced from these general theorems, adding to [1]. Likewise, using the Delaunay triangulation of $\mathcal{P}_\lambda$, one can find a unique random path $\tilde{\mathcal{C}}_\lambda(\mathcal{P}_\lambda)$ whose edges meet $\mathcal{C}$ and belong to the triangulation of $\mathcal{P}_\lambda$, with length

$$\left|\tilde{\mathcal{C}}_\lambda(\mathcal{P}_\lambda)\right| = \sum_{x \in \mathcal{P}_\lambda} \tilde{\rho}(x, \mathcal{P}_\lambda, \mathcal{C}) = \lambda^{-1/2} H^{\tilde{\rho}}(\mathcal{P}_\lambda, \mathcal{C}),$$

where

$$\tilde{\rho}(x, \mathcal{P}_\lambda, \mathcal{C}) := \begin{cases} \text{one half the sum of lengths of edges incident to } x \text{ if } x \in \tilde{\mathcal{C}}_\lambda(\mathcal{P}_\lambda), \\ 0, \qquad \text{otherwise}. \end{cases}$$

Theorems 1.1–1.3 provide the limit theory for $|\tilde{\mathcal{C}}_\lambda(\mathcal{P}_\lambda)|$.

3. Auxiliary results.

We give three lemmas pertaining to the rescaled scores $\xi_\lambda, \lambda > 0$, defined at (1.4).

LEMMA 3.1. *Fix $\mathcal{M} \in \mathbb{M}_2(d)$. Assume that ξ is homogeneously stabilizing, satisfies the moment condition (1.8) for $p > 1$ and is well-approximated by $\mathcal{P}_\lambda$ input on half-spaces (1.10). Then for almost all $y \in \mathcal{M}$, all $u \in \mathbb{R}$, and all $x \in \mathbb{R}^d \cup \varnothing$ we have*

$$\lim_{\lambda \to \infty} \mathbb{E}\xi_\lambda\big((y, \lambda^{-1/d}u) + \lambda^{-1/d}x, \mathcal{P}_\lambda, \mathcal{M}\big) = \mathbb{E}\xi\big((\mathbf{0}_y, u) + x, \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big). \tag{3.1}$$

PROOF. Fix $\mathcal{M} \in \mathbb{M}_2(d)$. We first show for almost all $y \in \mathcal{M}$ that there exist coupled realizations $\mathcal{P}'_\lambda$ and $\mathcal{H}'_{\kappa(y)}$ of $\mathcal{P}_\lambda$ and $\mathcal{H}'_{\kappa(y)}$, respectively, such that for $u \in \mathbb{R}$ and $x \in \mathbb{R}^d$, we have as $\lambda \to \infty$

$$\xi_\lambda\big((y, \lambda^{-1/d}u) + \lambda^{-1/d}x, \mathcal{P}'_\lambda, \mathcal{M}\big) \xrightarrow{\mathcal{D}} \xi\big((\mathbf{0}_y, u) + x, \mathcal{H}'_{\kappa(y)}, \mathbb{H}_y\big). \tag{3.2}$$

By translation invariance of ξ, we have

$$\begin{aligned}\xi_\lambda\big((y,\lambda^{-1/d}u)+\lambda^{-1/d}x,\mathcal{P}_\lambda,\mathcal{M}\big) &= \xi_\lambda\big((\mathbf{0}_y,\lambda^{-1/d}u)+\lambda^{-1/d}x,\mathcal{P}_\lambda-y,\mathcal{M}-y\big)\\ &= \xi\big((\mathbf{0}_y,u)+x,\lambda^{1/d}(\mathcal{P}_\lambda-y),\lambda^{1/d}(\mathcal{M}-y)\big).\end{aligned}$$

By the half-space approximation assumption (1.10), we need only show for almost all $y\in\mathcal{M}$ that there exist coupled realizations $\mathcal{P}'_\lambda$ and $\mathcal{H}'_{\kappa(y)}$ of $\mathcal{P}_\lambda$ and $\mathcal{H}_{\kappa(y)}$, respectively, such that as $\lambda\to\infty$

$$\xi\big((\mathbf{0}_y,u)+x,\lambda^{1/d}(\mathcal{P}'_\lambda-y),\mathbb{H}_y\big)\xrightarrow{\mathcal{D}}\xi\big((\mathbf{0}_y,u)+x,\mathcal{H}'_{\kappa(y)},\mathbb{H}_y\big). \tag{3.3}$$

This, however, follows from the homogeneous stabilization of ξ and the continuous mapping theorem; see Lemmas 3.2 and 3.2 of [18], which proves this assertion for the more involved case of binomial input. Thus, (3.2) holds and Lemma 3.1 follows from uniform integrability of $\xi_\lambda((y,\lambda^{-1/d}u)+\lambda^{-1/d}x,\mathcal{P}'_\lambda,\mathcal{M})$, which follows from the moment condition (1.8). $\square$

LEMMA 3.2. *Fix $\mathcal{M}\in\mathbb{M}_2(d)$. Assume that ξ is homogeneously stabilizing, satisfies the moment condition (1.8) for $p>2$, and is well-approximated by $\mathcal{P}_\lambda$ input on half-spaces (1.10). Given $y\in\mathcal{M}$, $x\in\mathbb{R}^d$ and $u\in\mathbb{R}$, put*

$$\begin{aligned}X_\lambda &:= \xi_\lambda\big((y,\lambda^{-1/d}u),\mathcal{P}_\lambda\cup\big((y,\lambda^{-1/d}u)+\lambda^{-1/d}x\big),\mathcal{M}\big),\\ Y_\lambda &:= \xi_\lambda\big((y,\lambda^{-1/d}u)+\lambda^{-1/d}x,\mathcal{P}_\lambda\cup\big(y,\lambda^{-1/d}u\big),\mathcal{M}\big),\\ X &:= \xi\big((\mathbf{0}_y,u),\mathcal{H}_{\kappa(y)}\cup\big((\mathbf{0}_y,u)+x\big),\mathbb{H}_y\big) \quad \textit{and}\\ Y &:= \xi\big((\mathbf{0}_y,u)+x,\mathcal{H}_{\kappa(y)}\cup(\mathbf{0}_y,u),\mathbb{H}_y\big).\end{aligned}$$

Then for almost all $y\in\mathcal{M}$ we have $\lim_{\lambda\to\infty}\mathbb{E}X_\lambda Y_\lambda=\mathbb{E}XY$.

PROOF. By the moment condition (1.8), the sequence X_λ^2, $\lambda\ge1$, is uniformly integrable and hence the convergence in distribution $X_\lambda\xrightarrow{\mathcal{D}}X$ extends to L^2 convergence and likewise for $Y_\lambda\xrightarrow{\mathcal{D}}Y$. The triangle inequality and the Cauchy–Schwarz inequality give

$$\|X_\lambda Y_\lambda-XY\|_1\le\|Y_\lambda\|_2\|X_\lambda-X\|_2+\|X\|_2\|Y_\lambda-Y\|_2.$$

Lemma 3.2 follows since $\sup_{\lambda>0}\|Y_\lambda\|_2<\infty$ and $\|X\|_2<\infty$. $\square$

The next result quantifies the exponential decay of correlations between scores on re-scaled input separated by Euclidean distance $\|x\|$.

LEMMA 3.3. *Fix $\mathcal{M}\in\mathbb{M}(d)$. Let ξ be exponentially stabilizing (1.7) and assume the moment condition (1.8) holds for some $p>2$. Then there is a $c_0\in(0,\infty)$*

such that for all $w, x \in \mathbb{R}^d$ *and* $\lambda \in (0, \infty)$, *we have*

$$\begin{aligned}&\big|\mathbb{E}\xi_\lambda(w, \mathcal{P}_\lambda \cup (w + \lambda^{-1/d} x), \mathcal{M})\xi_\lambda(w + \lambda^{-1/d} x, \mathcal{P}_\lambda \cup w, \mathcal{M}) \\ &\qquad\qquad - \mathbb{E}\xi_\lambda(w, \mathcal{P}_\lambda, \mathcal{M})\mathbb{E}\xi_\lambda(w + \lambda^{-1/d} x, \mathcal{P}_\lambda, \mathcal{M})\big| \\ &\quad \le c_0 \exp\big(-c_0^{-1}\|x\|\big).\end{aligned}$$

PROOF. See the proof of Lemma 4.2 of [19] or Lemma 4.1 of [6]. □

4. Proofs of Theorems 1.1–1.2. Roughly speaking, putting $x = \varnothing$ in (3.1) and integrating (3.1) over $y \in \mathcal{M}$ and $u \in \mathbb{R}$, we obtain expectation convergence of $\lambda^{-(d-1)/d} H^\xi(\mathcal{P}_\lambda, \mathcal{M})$ in Theorem 1.1. We then upgrade this to L^1 and L^2 convergence. Regarding Theorem 1.2, Lemmas 3.1 and 3.2 similarly yield convergence of the covariance of scores ξ_λ at points $(y, \lambda^{-1/d} u)$ and $(y, \lambda^{-1/d} u) + \lambda^{-1/d} x$ and Lemma 3.3, together with dominated convergence, imply convergence of integrated covariances over $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$, as they appear in the iterated integral formula for $\lambda^{-(d-1)/d} \operatorname{Var} H^\xi(\mathcal{P}_\lambda, \mathcal{M})$. The details go as follows.

PROOF OF THEOREM 1.1. We first prove L^2 convergence. Recall the definitions of $H^\xi(\mathcal{P}_\lambda, \mathcal{M})$ and $\mu(\xi, \mathcal{M})$ at (1.2) and (1.11), respectively. In view of the identity

$$\begin{aligned}&\mathbb{E}\big(\lambda^{-(d-1)/d} H^\xi(\mathcal{P}_\lambda, \mathcal{M}) - \mu(\xi, \mathcal{M})\big)^2 \\ &\quad = \lambda^{-2(d-1)/d}\mathbb{E}H^\xi(\mathcal{P}_\lambda, \mathcal{M})^2 - 2\mu(\xi, \mathcal{M})\lambda^{-(d-1)/d}\mathbb{E}H^\xi(\mathcal{P}_\lambda, \mathcal{M}) \\ &\qquad + \mu(\xi, \mathcal{M})^2,\end{aligned}$$

it suffices to show

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\mathbb{E}H^\xi(\mathcal{P}_\lambda, \mathcal{M}) = \mu(\xi, \mathcal{M}) \tag{4.1}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-2(d-1)/d}\mathbb{E}H^\xi(\mathcal{P}_\lambda, \mathcal{M})^2 = \mu(\xi, \mathcal{M})^2. \tag{4.2}$$

To show (4.1), we first write

$$\lambda^{-(d-1)/d}\mathbb{E}H^\xi(\mathcal{P}_\lambda, \mathcal{M}) = \lambda^{1/d} \int_{[0,1]^d} \mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})\kappa(x)\, dx.$$

Given $\mathcal{M} \in \mathbb{M}_2(d)$ and $x \in [0, 1]^d$, recall from (1.3) the parameterization $x = y + t\mathbf{u}_y$, with $\mathbf{u}_y$ the unit outward normal to $\mathcal{M}$ at y. The Jacobian of the map $h : x \mapsto (y + t\mathbf{u}_y)$ at (y, t) is $J_h((y, t)) := \prod_{i=1}^{d-1}(1 + tC_{y,i})$, where $C_{y,i}$, $1 \le i \le d - 1$, are the principal curvatures of $\mathcal{M}$ at y. Surfaces in $\mathbb{M}_2(d)$ have bounded curvature, implying $\|J_h\|_\infty := \sup_{(y,t)\in[0,1]^d} |J_h((y, t))| < \infty$.

Given $y \in \mathcal{M}$, let N_y be the set of points in $[0,1]^d$ with parameterization (y,t) for some $t \in \mathbb{R}$. Define $T_y := \{t \in \mathbb{R} : (y,t) \in N_y\}$. This gives

$$\lambda^{-(d-1)/d}\mathbb{E}H^{\xi}(\mathcal{P}_\lambda, \mathcal{M}) = \lambda^{1/d} \int_{y\in\mathcal{M}} \int_{t\in T_y} \mathbb{E}\xi_\lambda\big((y,t), \mathcal{P}_\lambda, \mathcal{M}\big)\big|J_h\big((y,t)\big)\big|\kappa\big((y,t)\big)\,dt\,dy.$$

Let $t = \lambda^{-1/d}u$ to obtain

$$\begin{aligned}\lambda^{-(d-1)/d}&\mathbb{E}H^{\xi}(\mathcal{P}_\lambda, \mathcal{M}) \\ &= \int_{y\in\mathcal{M}} \int_{u\in\lambda^{1/d}T_y} \mathbb{E}\xi_\lambda\big((y,\lambda^{-1/d}u), \mathcal{P}_\lambda, \mathcal{M}\big)\big|J_h\big((y,\lambda^{-1/d}u)\big)\big| \\ &\qquad \times \kappa\big((y,\lambda^{-1/d}u)\big)\,du\,dy.\end{aligned} \tag{4.3}$$

By Lemma 3.1, for almost all $y \in \mathcal{M}$ and $u \in \mathbb{R}$, we have

$$\lim_{\lambda\to\infty} \mathbb{E}\xi_\lambda\big((y,\lambda^{-1/d}u), \mathcal{P}_\lambda, \mathcal{M}\big) = \mathbb{E}\xi\big((\mathbf{0}_y,u), \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big). \tag{4.4}$$

By (1.8), for $y \in \mathcal{M}$, $u \in \mathbb{R}$, and $\lambda \in (0,\infty)$, the integrand in (4.3) is bounded by $G^{\xi,1}(|u|)\|J_h\|_\infty\|\kappa\|_\infty$, which is integrable with respect to the measure $du\,dy$. Therefore, by the dominated convergence theorem, the limit $\lambda^{1/d}T_y \uparrow \mathbb{R}$, the continuity of κ, and (4.4), we obtain (4.1), namely

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d}\mathbb{E}H^{\xi}(\mathcal{P}_\lambda, \mathcal{M}) = \int_{y\in\mathcal{M}} \int_{-\infty}^{\infty} \mathbb{E}\big[\xi\big((\mathbf{0}_y,u), \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big)\big]\,du\,\kappa(y)\,dy.$$

To show (4.2), we note

$$\begin{aligned}\lambda^{-2(d-1)/d}&\mathbb{E}H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})^2 \\ &= \lambda^{-2(d-1)/d} \\ &\quad \times \Big[\lambda \int_{[0,1]^d} \mathbb{E}[\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})^2]\kappa(x)\,dx \\ &\qquad + \lambda^2 \int_{[0,1]^d}\int_{[0,1]^d} \mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})\xi_\lambda(w, \mathcal{P}_\lambda, \mathcal{M})\kappa(x)\kappa(w)\,dx\,dw\Big].\end{aligned}$$

The first integral goes to zero, since $\sup_{\lambda>0} \lambda^{1/d} \int_{[0,1]^d} \mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})^2\kappa(x)\,dx$ is bounded. The second integral simplifies to

$$\lambda^{2/d} \int_{[0,1]^d}\int_{[0,1]^d} \mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})\xi_\lambda(w, \mathcal{P}_\lambda, \mathcal{M})\kappa(x)\kappa(w)\,dx\,dw.$$

As $\lambda \to \infty$, this tends to $\mu(\xi, \mathcal{M})^2$ by independence, proving the asserted L^2 convergence of Theorem 1.1.

To prove L^1 convergence we follow a truncation argument similar to that for the proof of Proposition 3.2 in [21]. Given $K > 0$, we put

$$\xi^K(x, \mathcal{X}, \mathcal{M}) := \min\big(\xi(x, \mathcal{X}, \mathcal{M}), K\big).$$

Then ξ^K is homogenously stabilizing and uniformly bounded and, therefore, by the first part of this proof we get

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/d} H^{\xi^K}(\mathcal{P}_\lambda, \mathcal{M}) = \mu(\xi^K, \mathcal{M}) \qquad \text{in } L^2. \tag{4.5}$$

Also, following the arguments around (4.3), we have

$$\begin{aligned}
&\big|\lambda^{-(d-1)/d}\big(\mathbb{E}H^{\xi}(\mathcal{P}_\lambda, \mathcal{M}) - \mathbb{E}H^{\xi^K}(\mathcal{P}_\lambda, \mathcal{M})\big)\big| \\
&\quad \le \int_{y\in\mathcal{M}} \int_{u\in\lambda^{1/d}T_y} \mathbb{E}[\cdots]\big|J_h\big((y, \lambda^{-1/d}u)\big)\big|\kappa\big((y, \lambda^{-1/d}u)\big)\, du\, dy,
\end{aligned}$$

where $\mathbb{E}[\cdots] := \mathbb{E}[|\xi_\lambda((y, \lambda^{-1/d}u), \mathcal{P}_\lambda, \mathcal{M}) - \xi_\lambda^K((y, \lambda^{-1/d}u), \mathcal{P}_\lambda, \mathcal{M})|]$. This expected difference tends to zero as $K \to \infty$, because the moments condition (1.8) with $p > 1$ implies that $|\xi_\lambda((y, \lambda^{-1/d}u), \mathcal{P}_\lambda, \mathcal{M}) - \xi_\lambda^K((y, \lambda^{-1/d}u), \mathcal{P}_\lambda, \mathcal{M})|$ is uniformly integrable. By monotone convergence, $\mu(\xi^K, \mathcal{M}) \to \mu(\xi, \mathcal{M})$ as $K \to \infty$. Thus, letting $K \to \infty$ in (4.5) we get the desired L^1 convergence. $\square$

PROOF OF THEOREM 1.2. We have

$$\begin{aligned}
\lambda^{-(d-1)/d} \operatorname{Var} H^{\xi}(\mathcal{P}_\lambda, \mathcal{M}) &= \lambda^{1/d} \int_{[0,1]^d} \mathbb{E}\xi_\lambda^2(x, \mathcal{P}_\lambda, \mathcal{M})\kappa(x)\mathcal{H}^d(dx) \\
&\quad + \lambda^{1+1/d} \int_{x\in[0,1]^d} \int_{w\in[0,1]^d} \{\cdots\}\kappa(x)\kappa(w)\, dx\, dw,
\end{aligned}$$

where

$$\{\cdots\} := \mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda \cup w, \mathcal{M})\xi_\lambda(w, \mathcal{P}_\lambda \cup x, \mathcal{M}) - \mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})\mathbb{E}\xi_\lambda(w, \mathcal{P}_\lambda, \mathcal{M}).$$

For a fixed $(y, t) \in \mathcal{M} \times \mathbb{R}$, parameterize points $x \in [0, 1]^d$ by $x_y := (z_y, s_y)$, where $z_y \in \mathbb{H}_y$ and $s_y \in \mathbb{R}$. Given $(y, t) \in [0, 1]^d$ and $z_y \in \mathbb{H}_y$, let $S_{z_y} := S_{z_y,t} := \{s_y \in \mathbb{R} : (y, t) + (z_y, s_y) \in [0, 1]^d\}$ and let $Z_y := [0, 1]^d \cap \mathbb{H}_y$. We have

$$\begin{aligned}
&\lambda^{-(d-1)/d} \operatorname{Var}\big[H^{\xi}(\mathcal{P}_\lambda, \mathcal{M})\big] \\
&\quad = \lambda^{1/d} \int_{[0,1]^d} \mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})^2\kappa(x)\, dx \\
&\qquad + \lambda^{1+1/d} \int_{y\in\mathcal{M}} \int_{T_y} \int_{Z_y} \int_{S_{z_y}} \{\cdots\}\big|J_h\big((y, t)\big)\big| \\
&\qquad\qquad \times \kappa\big((y, t)\big)\kappa\big((y, t) + (z_y, s_y)\big)\, ds_y\, dz_y\, dt\, dy,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
\{\cdots\} &:= \mathbb{E}\xi_\lambda\big((y, t), \mathcal{P}_\lambda \cup (y, t) + (z_y, s_y), \mathcal{M}\big)\xi_\lambda\big((y, t) + (z_y, s_y), \mathcal{P}_\lambda \cup (y, t), \mathcal{M}\big) \\
&\quad - \mathbb{E}\xi_\lambda\big((y, t), \mathcal{P}_\lambda, \mathcal{M}\big)\mathbb{E}\xi_\lambda\big((y, t) + (z_y, s_y), \mathcal{P}_\lambda, \mathcal{M}\big).
\end{aligned}$$

As in the proof of Theorem 1.1, the first integral in (4.6) converges to

$$\int_{\mathcal{M}} \int_{-\infty}^{\infty} \mathbb{E}\xi^2\big((\mathbf{0}_y, u), \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big)\, du\, \kappa(y)\, dy. \tag{4.7}$$

In the second integral in (4.6), we let $t = \lambda^{-1/d}u$, $s_y = \lambda^{-1/d}s$, $z_y = \lambda^{-1/d}z$ so that $dz = \lambda^{(d-1)/d}\, dz_y$. These substitutions transform the multiplicative factor

$$\big|J_h\big((y,t)\big)\big|\kappa\big((y,t)\big)\kappa\big((y,t) + (z_y, s_y)\big)$$

into

$$\big|J_h\big((y, \lambda^{-1/d}u)\big)\big|\kappa\big((y, \lambda^{-1/d}u)\big)\kappa\big((y, \lambda^{-1/d}u) + \big(\lambda^{-1/d}z, \lambda^{-1/d}s\big)\big), \tag{4.8}$$

they transform the differential $\lambda^{1+1/d}\, ds_y\, dz_y\, dt\, dy$ into $ds\, dz\, du\, dy$, and, lastly, they transform [recalling $x_y = (z_y, s_y)$] the covariance term $\{\cdots\}$ into

$$\begin{aligned} \{\cdots\}' := {}& \mathbb{E}\xi_\lambda\big((y, \lambda^{-1/d}u), \mathcal{P}_\lambda \cup \big(y, \lambda^{-1/d}u\big) + \lambda^{-1/d}x_y, \mathcal{M}\big) \\ & \times \xi_\lambda\big((y, \lambda^{-1/d}u) + \lambda^{-1/d}x_y, \mathcal{P}_\lambda \cup \big(y, \lambda^{-1/d}u\big), \mathcal{M}\big) \\ & - \mathbb{E}\xi_\lambda\big((y, \lambda^{-1/d}u), \mathcal{P}_\lambda, \mathcal{M}\big)\mathbb{E}\xi_\lambda\big((y, \lambda^{-1/d}u) + \lambda^{-1/d}x_y, \mathcal{P}_\lambda, \mathcal{M}\big). \end{aligned} \tag{4.9}$$

The factor at (4.8) is bounded by $\|J_h\|_\infty \|\kappa\|_\infty^2$ and converges to $\kappa(y)^2$, as $\lambda \to \infty$. By Lemma 3.2, for almost all $y \in \mathcal{M}$, the covariance term $\{\cdots\}'$ at (4.9) converges to

$$c^\xi\big((\mathbf{0}_y, u), (\mathbf{0}_y, u) + (z, s), \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big).$$

By Lemma 3.3 as well as (1.8), the factor $\{\cdots\}'$ is dominated by an integrable function of $(y, u, x_y) \in \mathcal{M} \times \mathbb{R} \times \mathbb{R}^d$. By dominated convergence, together with the set limits $\lambda^{1/d}Z_y \uparrow \mathbb{R}^{d-1}$, $\lambda^{1/d}S_{z_y} \uparrow \mathbb{R}$, and $\lambda^{1/d}T_y \uparrow \mathbb{R}$ the second integral converges to

$$\begin{aligned} \int_{\mathcal{M}} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} & c^\xi\big((\mathbf{0}_y, u), (\mathbf{0}_y, u) + (z, s); \mathcal{H}_{\kappa(y)}, \mathbb{H}_y\big) \\ & \times \kappa(y)^2\, du\, ds\, dz\, dy, \end{aligned} \tag{4.10}$$

which is finite. Combining (4.7) and (4.10), we obtain Theorem 1.2. □

5. Proof of Theorem 1.3. Put $T_\lambda := H^\xi(\mathcal{P}_\lambda, \mathcal{M})$, $\mathcal{M} \in \mathbb{M}(d)$. We shall first prove that Theorem 1.3 holds when T_λ is replaced by a version T'_λ on input concentrated near $\mathcal{M}$. To show asymptotic normality of T'_λ, we follow the set-up of [22], which makes use of dependency graphs, allowing applicability of Stein's method. We show that T'_λ is close to T_λ, thus yielding Theorem 1.3. This goes as follows.

Put $\rho_\lambda := \beta \log \lambda$, $s_\lambda := \rho_\lambda \lambda^{-1/d} = \beta \log \lambda \cdot \lambda^{-1/d}$, $\beta \in (0, \infty)$ a constant to be determined. Consider the collection of cubes Q of the form $\prod_{i=1}^d [j_i s_\lambda, (j_i + 1)s_\lambda)$, with all $j_i \in \mathbb{Z}$, such that $\int_Q \kappa(x)\, dx > 0$. Further, consider only cubes Q such that $d(Q, \mathcal{M}) < 2s_\lambda$, where for Borel subsets A and B of $\mathbb{R}^d$, we put $d(A, B) := \inf\{|x - y| : x \in A, y \in B\}$. Relabeling if necessary, write the union of the cubes as $\mathcal{Q} := \bigcup_{i=1}^W Q_i$, where $W := W(\lambda) = \Theta((s_\lambda^{-1})^{d-1})$, because $\mathcal{H}^{d-1}(\mathcal{M}) < \infty$.

We have $\mathrm{card}(Q_i \cap \mathcal{P}_\lambda) := N_i := N(\nu_i)$, where N_i is an independent Poisson random variable with parameter

$$\nu_i := \lambda \int_{Q_i} \kappa(x)\, dx \le \|\kappa\|_\infty \rho_\lambda^d.$$

We may thus write $\mathcal{P}_\lambda \cap \bigcup_{i=1}^W Q_i = \bigcup_{i=1}^W \{X_{ij}\}_{j=1}^{N_i}$, where for $1 \le i \le W$, we have X_{ij} are i.i.d. on Q_i with density

$$\kappa_i(\cdot) := \frac{\kappa(\cdot)}{\int_{Q_i} \kappa(x)\, dx} \mathbf{1}(Q_i).$$

Define

$$\tilde{T}_\lambda := \sum_{x \in \mathcal{P}_\lambda \cap \mathcal{Q}} \xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M}).$$

Then by definition of W, N_i and X_{ij}, we may write

$$\tilde{T}_\lambda = \sum_{i=1}^W \sum_{j=1}^{N_i} \xi_\lambda(X_{ij}, \mathcal{P}_\lambda, \mathcal{M}).$$

As in [22], it is useful to consider a version T'_λ of $\tilde{T}_\lambda$ which has more independence between summands. This goes as follows. For all $1 \le i \le W$ and all $j = 1, 2, \dots$, recalling the definition (1.7), let $R_{ij} := R^\xi(X_{ij}, \mathcal{P}_\lambda, \mathcal{M})$ denote the radius of stabilization of ξ at X_{ij} if $1 \le j \le N_i$ and otherwise let R_{ij} be zero. Put $E_{ij} := \{R_{ij} \le \rho_\lambda\}$, let

$$E_\lambda := \bigcap_{i=1}^W \bigcap_{j=1}^\infty E_{ij} \tag{5.1}$$

and define

$$T'_\lambda := \sum_{i=1}^W \sum_{j=1}^{N_i} \xi_\lambda(X_{ij}, \mathcal{P}_\lambda, \mathcal{M}) \mathbf{1}(E_{ij}).$$

For all $1 \le i \le W$, define

$$S_i := S_{Q_i} := (\mathrm{Var}\, T'_\lambda)^{-1/2} \sum_{j=1}^{N_i} \xi_\lambda(X_{ij}, \mathcal{P}_\lambda, \mathcal{M}) \mathbf{1}(E_{ij}).$$

Note that S_i and S_j are independent if $d(Q_i, Q_j) > 2\lambda^{-1/d}\rho_\lambda$. Put

$$S_\lambda := (\mathrm{Var}\, T'_\lambda)^{-1/2}(T'_\lambda - \mathbb{E}T'_\lambda) = \sum_{i=1}^W (S_i - \mathbb{E}S_i).$$

We aim to show that T'_λ closely approximates T_λ, but first we show that $\tilde{T}_\lambda$ closely approximates T_λ.

LEMMA 5.1. *Given $\mathcal{M} \in \mathbb{M}(d)$, let $G^{\xi,2} := G^{\xi,2,\mathcal{M}}$ satisfy* (1.8) *and* (1.9). *Choose $\beta \in (0, \infty)$ so that*

$$\beta \limsup_{|u|\to\infty} |u|^{-1} \log G^{\xi,2}(|u|) < -8. \tag{5.2}$$

Then

$$\|\tilde{T}_\lambda - T_\lambda\|_2 = O(\lambda^{-3}) \tag{5.3}$$

and

$$|\operatorname{Var} \tilde{T}_\lambda - \operatorname{Var} T_\lambda| = O(\lambda^{-2}). \tag{5.4}$$

PROOF. Writing $\tilde{T}_\lambda = T_\lambda + (\tilde{T}_\lambda - T_\lambda)$ gives

$$\operatorname{Var} \tilde{T}_\lambda = \operatorname{Var} T_\lambda + \operatorname{Var}[\tilde{T}_\lambda - T_\lambda] + 2 \operatorname{Cov}(T_\lambda, \tilde{T}_\lambda - T_\lambda).$$

Now

$$\begin{aligned}
&\operatorname{Var}[\tilde{T}_\lambda - T_\lambda] \\
&\quad \le \|\tilde{T}_\lambda - T_\lambda\|_2^2 = \mathbb{E}\biggl(\sum_{x\in\mathcal{P}_\lambda\setminus\mathcal{Q}} \xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})\biggr)^2 \\
&\quad = \lambda^2 \int_{[0,1]^d\setminus\mathcal{Q}} \int_{[0,1]^d\setminus\mathcal{Q}} \mathbb{E}[\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})\xi_\lambda(y, \mathcal{P}_\lambda, \mathcal{M})]\kappa(x)\kappa(y)\, dx\, dy.
\end{aligned}$$

If $x \in [0, 1]^d \setminus \mathcal{Q}$, then $d(x, \mathcal{M}) \ge \beta \log \lambda \cdot \lambda^{-1/d}$. Thus, by (1.8) and (1.9), for large λ we have $\mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})^2 \le G^{\xi,2}(\beta \log \lambda) \le \exp(-8 \log \lambda) = \lambda^{-8}$. Applying the Cauchy–Schwarz inequality to $\mathbb{E}\xi_\lambda(x, \mathcal{P}_\lambda, \mathcal{M})\xi_\lambda(y, \mathcal{P}_\lambda, \mathcal{M})$ with $x, y \in [0, 1]^d \setminus Q$, we obtain

$$\|\tilde{T}_\lambda - T_\lambda\|_2^2 = O(\lambda^{-6}) \tag{5.5}$$

which gives (5.3). Also, since $\|T_\lambda\|_2 = O(\lambda)$ and $\|\tilde{T}_\lambda - T_\lambda\|_2 = O(\lambda^{-3})$, another application of the Cauchy–Schwarz inequality gives

$$\operatorname{Cov}(T_\lambda, \tilde{T}_\lambda - T_\lambda) \le \|T_\lambda\|_2 \|\tilde{T}_\lambda - T_\lambda\|_2 = O(\lambda^{-2}). \tag{5.6}$$

Combining (5.5) and (5.6) gives (5.4). $\square$

LEMMA 5.2. *Assume that ξ satisfies the moment conditions* (1.8) *and* (1.9) *for some $p > q$, $q \in (2, 3]$. For β large, we have*

$$\|T_\lambda - T'_\lambda\|_2 = O(\lambda^{-3}) \tag{5.7}$$

and

$$|\operatorname{Var} T_\lambda - \operatorname{Var} T'_\lambda| = O(\lambda^{-2}). \tag{5.8}$$

PROOF. We have $\|T_\lambda - T'_\lambda\|_2 \le \|T_\lambda - \tilde{T}_\lambda\|_2 + \|\tilde{T}_\lambda - T'_\lambda\|_2 = O(\lambda^{-3}) + \|\tilde{T}_\lambda - T'_\lambda\|_2$, by Lemma 5.1. Note that $|\tilde{T}_\lambda - T'_\lambda| = 0$ on E_λ, with E_λ defined at (5.1). Choosing β large enough, we have $P[E^c_\lambda] = O(\lambda^{-D})$ for any $D > 0$. By the analog of Lemma 4.3 of [22], and using condition (1.8), we get for $q \in (2, 3]$ that $\|\tilde{T}_\lambda - T'_\lambda\|_q = O(\lambda)$. This, together with the Hölder inequality, gives $\|(\tilde{T}_\lambda - T'_\lambda)\mathbf{1}(E^c_\lambda)\|_2 = O(\lambda^{-3})$, whence (5.7).

To show (5.8), we note that by (5.4) and the triangle inequality, it is enough to show $|\operatorname{Var}\tilde{T}_\lambda - \operatorname{Var}T'_\lambda| = O(\lambda^{-2})$. However, this follows by writing

$$\operatorname{Var}\tilde{T}_\lambda = \operatorname{Var}T'_\lambda + \operatorname{Var}[\tilde{T}_\lambda - T'_\lambda] + 2\operatorname{Cov}(T'_\lambda, \tilde{T}_\lambda - T'_\lambda),$$

noting $\operatorname{Var}[\tilde{T}_\lambda - T'_\lambda] \le \|\tilde{T}_\lambda - T'_\lambda\|_2 = O(\lambda^{-3})$, and then using $\|T'_\lambda\|_2 = O(\lambda)$ and the Cauchy–Schwarz inequality to bound $\operatorname{Cov}(T'_\lambda, \tilde{T}_\lambda - T'_\lambda)$ by $O(\lambda^{-2})$. □

Now we are ready to prove Theorem 1.3. Since (1.14) trivially holds for large enough λ when $\operatorname{Var}T_\lambda < 1$, we may without loss of generality assume $\operatorname{Var}T_\lambda \ge 1$.

As in [22], we define a dependency graph $G_\lambda := (\mathcal{V}_\lambda, \mathcal{E}_\lambda)$ for $\{S_i\}_{i=1}^V$. The set $\mathcal{V}_\lambda$ consists of the cubes $Q_1, \dots, Q_V$ and edges (Q_i, Q_j) belong to $\mathcal{E}_\lambda$ iff $d(Q_i, Q_j) < 2\lambda^{-1/d}\rho_\lambda$. Using Stein's method in the context of dependency graphs, we adapt the proof in [22] to show the asymptotic normality of S_λ, $\lambda \to \infty$, and then use this to show the asymptotic normality of $T_\lambda, \lambda \to \infty$. In [22], we essentially replace the term $V = \Theta(\lambda/(\log\lambda)^d)$ by the smaller term $W = \Theta(\lambda^{(d-1)/d}/(\log\lambda)^{d-1})$, and instead of (4.16) and (4.17) of [22], we use (5.7) and (5.8). Note that for $p > q, q \in (2, 3]$, we have $\|S_i\|_q = O((\operatorname{Var}[T'_\lambda])^{-1/2}\rho_\lambda^{d(p+1)/p})$. We sketch the argument as follows.

Let c denote a generic constant whose value may change at each occurrence. Following Section 4.3 of [22] verbatim up to (4.18) gives, via Lemma 4.1 of [22], with $p > q, q \in (2, 3]$ and $\theta := c(\operatorname{Var}[T'_\lambda])^{-1/2}\rho_\lambda^{d(p+1)/p}$:

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\big|P[S_\lambda \le t] - \Phi(t)\big| \\
&\qquad \le cW\theta^q \le c\lambda^{(d-1)/d}\rho_\lambda^{-(d-1)}\big(\operatorname{Var}T'_\lambda\big)^{-q/2}\rho_\lambda^{d(p+1)q/p} \\
&\qquad \le c\lambda^{(d-1)/d}\big(\operatorname{Var}[T_\lambda]\big)^{-q/2}\rho_\lambda^{dq+1},
\end{aligned}\tag{5.9}$$

where we use $\operatorname{Var}[T'_\lambda] \ge \operatorname{Var}[T_\lambda)]/2$, which follows (for λ large) from (5.8).

Follow verbatim the discussion between (4.18)–(4.20) of [22], with $V(\lambda)$ there replaced by W. Recall that $q \in (2, 3]$ with $p > q$. Making use of (5.7), this gives the analog of (4.20) of [22]. In other words, this gives a constant c depending on d, ξ, p, and q such that for all $\lambda \ge 2$ the inequality (5.9) becomes

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\big|P\big[\big(\operatorname{Var}T'_\lambda\big)^{-1/2}(T_\lambda - \mathbb{E}T_\lambda) \le t\big] - \Phi(t)\big| \\
&\qquad \le c\lambda^{(d-1)/d}(\operatorname{Var}T_\lambda)^{-q/2}\rho_\lambda^{dq+1} + c\lambda^{-2}.
\end{aligned}\tag{5.10}$$

By [6, 19], we have $\operatorname{Var} T_\lambda = O(\lambda)$ and so $c\lambda^{-2}$ is negligible with respect to the first term on the right-hand side of (5.10).

Finally we replace $\operatorname{Var} T'_\lambda$ by $\operatorname{Var} T_\lambda$ on the left-hand side of (5.10). As in [22], we have by the triangle inequality

$$\sup_{t\in\mathbb{R}} \left| P\left[(\operatorname{Var} T_\lambda)^{-1/2}(T_\lambda - \mathbb{E}T_\lambda) \le t\right] - \Phi(t)\right|$$

$$(5.11)\qquad \begin{aligned} &\le \sup_{t\in\mathbb{R}} \left| P\left[(\operatorname{Var} T'_\lambda)^{-1/2}(T_\lambda - \mathbb{E}T_\lambda) \le t \cdot \left(\frac{\operatorname{Var} T_\lambda}{\operatorname{Var} T'_\lambda}\right)^{1/2}\right] \right. \\ &\qquad\qquad \left. - \Phi\left(t\left(\frac{\operatorname{Var} T_\lambda}{\operatorname{Var} T'_\lambda}\right)^{1/2}\right)\right| \\ &\quad + \sup_{t\in\mathbb{R}} \left|\Phi\left(t\left(\frac{\operatorname{Var} T_\lambda}{\operatorname{Var} T'_\lambda}\right)^{1/2}\right) - \Phi(t)\right|. \end{aligned}$$

We have

$$\left|\sqrt{\frac{\operatorname{Var} T_\lambda}{\operatorname{Var} T'_\lambda}} - 1\right| \le \left|\frac{\operatorname{Var} T_\lambda}{\operatorname{Var} T'_\lambda} - 1\right| = O(\lambda^{-2}).$$

Let $\phi := \Phi'$ be the density of Φ. Following the analysis after (4.21) of [22], we get

$$\sup_{t\in\mathbb{R}} \left|\Phi\left(t\sqrt{\frac{\operatorname{Var} T_\lambda}{\operatorname{Var} T'_\lambda}}\right) - \Phi(t)\right| \le c \sup_{t\in\mathbb{R}} \left(\left(\frac{|t|}{\lambda^2}\right)\left(\sup_{u\in[t-tc/\lambda^2, t+tc/\lambda^2]} \phi(u)\right)\right) = O(\lambda^{-2}).$$

This gives (1.14) as desired.

6. Proofs of Theorems 2.1–2.5.

We first give a general result useful in proving versions of Theorems 1.1–1.3 for binomial input. Say that ξ is *binomially exponentially stabilizing* with respect to the pair $(\mathcal{X}_n, \mathcal{M})$ if for all $x \in \mathbb{R}^d$ there is a radius of stabilization $R := R^\xi(x, \mathcal{X}_n, \mathcal{M}) \in (0, \infty)$ a.s. such that

$$(6.1)\qquad \xi_n(x, \mathcal{X}_n \cap B_{n^{-1/d}R}(x), \mathcal{M}) = \xi_n(x, (\mathcal{X}_n \cap B_{n^{-1/d}R}(x)) \cup \mathcal{A}, \mathcal{M})$$

for all locally finite $\mathcal{A} \subset \mathbb{R}^d \setminus B_{n^{-1/d}R}(x)$, and moreover, the tail probability $\tilde{\tau}(t) := \tilde{\tau}(t, \mathcal{M}) := \sup_{n\ge 1, x\in\mathbb{R}^d} P[R(x, \mathcal{X}_n, \mathcal{M}) > t]$ satisfies $\limsup_{t\to\infty} t^{-1}\log\tilde{\tau}(t) < 0$.

LEMMA 6.1. *Let $\mathcal{M} \in \mathbb{M}(d)$. Let ξ be exponentially stabilizing* (1.7), *binomially exponentially stabilizing* (6.1), *and assume the moment conditions* (1.8) *and* (1.9) *hold for some $p > 2$. If there is constant $c_1 \in (0, \infty)$ such that*

$$(6.2)\qquad P\left[|\xi_n(X_1, \mathcal{X}_n, \mathcal{M})| \ge c_1 \log n\right] = O(n^{-1-2/(1-1/p)}),$$

and if $N(n)$ is an independent Poisson random variable with parameter n, then

$$(6.3)\qquad |\operatorname{Var} H^\xi(\mathcal{X}_n, \mathcal{M}) - \operatorname{Var} H^\xi(\mathcal{X}_{N(n)}, \mathcal{M})| = o(n^{(d-1)/d}).$$

PROOF. Let $D := 2/(1 - 1/p)$. By (6.2), there is an event $F_{n,1}$, with $P[F_{n,1}^c] = O(n^{-D})$ such that on $F_{n,1}$ we have

$$\max_{1 \le i \le n+1} \left|\xi_n(X_i, \mathcal{X}_n, \mathcal{M})\right| \le c_1 \log n. \tag{6.4}$$

As in the proof of Theorem 1.3, put $s_n := \beta \log n / n^{1/d}$, $\mathcal{Q} := \mathcal{Q}(n) := \bigcup_{i=1}^W Q_i$, where $d(Q_i, \mathcal{M}) < 2s_n$, $W := W(n) = O((s_n^{-1})^{d-1})$, and β is a constant to be determined. Consider the event $F_{n,2}$ such that for all $1 \le i \le n+1$, we have $\xi_n(X_i, \mathcal{X}_n, \mathcal{M}) = \xi_n(X_i, \mathcal{X}_n \cap B_{s(n)}(X_i), \mathcal{M})$. By binomial exponential stabilization (6.1) and for β large enough, we have $P[F_{n,2}^c] = O(n^{-D})$. Define for all $n = 1, 2, \dots$

$$\tilde{T}_n := \sum_{X_i \in \mathcal{X}_n \cap \mathcal{Q}_n} \xi_n(X_i, \mathcal{X}_n \cap \mathcal{Q}_n, \mathcal{M}).$$

As in Lemma 5.1, for β large we have the generous bounds

$$\left|\operatorname{Var} H^\xi(\mathcal{X}_n, \mathcal{M}) - \operatorname{Var} \tilde{T}_n\right| = o\big(n^{(d-1)/d}\big)$$

and

$$\left|\operatorname{Var} H^\xi(\mathcal{X}_{N(n)}, \mathcal{M}) - \operatorname{Var} \tilde{T}_{N(n)}\right| = o\big(n^{(d-1)/d}\big).$$

Therefore, to show (6.3), it is enough to show

$$\left|\operatorname{Var} \tilde{T}_n - \operatorname{Var} \tilde{T}_{N(n)}\right| = o\big(n^{(d-1)/d}\big). \tag{6.5}$$

Write $\xi_n(X_i, \mathcal{X}_n)$ for $\xi_n(X_i, \mathcal{X}_n, \mathcal{M})$. If $X_i \in B_{s_n}^c(X_{n+1})$, $1 \le i \le n$, then on $F_{n,2}$ we have $\xi_n(X_i, \mathcal{X}_n) = \xi_n(X_i, \mathcal{X}_{n+1})$. On $F_{n,2}$, we thus have

$$|\tilde{T}_n - \tilde{T}_{n+1}| \le \xi_n(X_{n+1}, \mathcal{X}_{n+1}) + \sum_{1 \le i \le n :\, X_i \in B_{s_n}(X_{n+1})} \left|\xi_n(X_i, \mathcal{X}_n) - \xi_n(X_i, \mathcal{X}_{n+1})\right|.$$

Given a constant $c_2 \in (0, \infty)$, define

$$F_{n,3} := \left\{\operatorname{card}\left\{X_n \cap B_{s_n}(X_{n+1})\right\} \le c_2 \log n\right\}.$$

Choose c_2 large such that $P[F_{n,3}^c] = O(n^{-D})$. On $F_{n,1} \cap F_{n,2} \cap F_{n,3}$ we have by (6.4) $|\tilde{T}_n - \tilde{T}_{n+1}| = O((\log n)^2)$. We deduce there is a c_3 such that on $F_{n,1} \cap F_{n,2} \cap F_{n,3}$ and all integers $l \in \{1, \dots, n\}$

$$|\tilde{T}_n - \tilde{T}_{n+l}| \le c_3 l (\log n)^2. \tag{6.6}$$

To show (6.5), we shall show

$$|\operatorname{Var} \tilde{T}_n - \operatorname{Var} \tilde{T}_{N(n)}| = O\big((\log n)^4 n^{1-3/2d}\big). \tag{6.7}$$

To show (6.7), write

$$\operatorname{Var} \tilde{T}_n = \operatorname{Var} \tilde{T}_{N(n)} + (\operatorname{Var} \tilde{T}_n - \operatorname{Var} \tilde{T}_{N(n)}) + 2\operatorname{cov}(\tilde{T}_{N(n)}, \tilde{T}_n - \tilde{T}_{N(n)}). \tag{6.8}$$

The proof of Theorem 1.2 shows $\operatorname{Var} \tilde{T}_{N(n)} = O(n^{(d-1)/2d})$, yielding

$$\begin{aligned}\operatorname{cov}(\tilde{T}_{N(n)}, \tilde{T}_n - \tilde{T}_{N(n)}) &\leq \sqrt{\operatorname{Var} \tilde{T}_{N(n)}} \cdot \|(\tilde{T}_n - \tilde{T}_{N(n)})\|_2 \\ &= O\big(n^{(d-1)/2d} \|(\tilde{T}_n - \tilde{T}_{N(n)})\|_2\big).\end{aligned}$$

It is thus enough to show

$$\|(\tilde{T}_n - \tilde{T}_{N(n)})\|_2^2 = O\big((\log n)^8 n^{1-2/d}\big), \tag{6.9}$$

since the last two terms in (6.8) are then $O((\log n)^4 n^{1-3/2d})$. Relabel the X_i, $i \geq 1$, so that $\mathcal{X}_n \cap \mathcal{Q}_n = \{X_1, \dots, X_{B(n,s_n)}\}$, $\mathcal{X}_{N(n)} \cap \mathcal{Q}_n = \{X_1, \dots, X_{N(n\cdot s_n)}\}$.

Put $E_n := \{B(n, s_n) \neq N(n \cdot s_n)\}$. There is a coupling of $B(n, s_n)$ and $N(n \cdot s_n)$ such that $P[E_n] \leq s_n$. By definition of E_n,

$$\begin{aligned}&\|(\tilde{T}_n - \tilde{T}_{N(n)})\|_2^2 \\ &\quad = \int \Bigg| \sum_{X_i \in \mathcal{X}_n \cap \mathcal{Q}_n} \xi_n(X_i, \mathcal{X}_n \cap \mathcal{Q}_n) - \sum_{X_i \in \mathcal{X}_{N(n)} \cap \mathcal{Q}_n} \xi_n(X_i, \mathcal{X}_{N(n)} \cap \mathcal{Q}_n) \Bigg|^2 \\ &\qquad \times \mathbf{1}(E_n)\, dP.\end{aligned}$$

Now $|B(n, s_n) - N(n \cdot s_n)| \leq c_4 \log n \sqrt{n s_n}$ on an event $F_{n,4}$ with $P[F_{n,4}^c] = O(n^{-D})$. Let $F_n := \bigcap_{i=1}^4 F_{n,i}$ and note that $P[F_n^c] = O(n^{-D})$. By (6.6), we have

$$\begin{aligned}&\int \Bigg| \sum_{X_i \in \mathcal{X}_n \cap \mathcal{Q}_n} \xi_n(X_i, \mathcal{X}_n \cap \mathcal{Q}_n) - \sum_{X_i \in \mathcal{X}_{N(n)} \cap \mathcal{Q}_n} \xi_n(X_i, \mathcal{X}_{N(n)} \cap \mathcal{Q}_n) \Bigg|^2 \\ &\quad \times \mathbf{1}(E_n)\mathbf{1}(F_n)\, dP \\ &\quad \leq \big(c_3 c_4 \log n \sqrt{n s_n} (\log n)^2\big)^2.\end{aligned} \tag{6.10}$$

For random variables U and Y, we have $\|UY\|_2^2 \leq \|U\|_{2p}^2 \|Y\|_{2q}^2$, $p^{-1} + q^{-1} = 1$, giving

$$\begin{aligned}\|(\tilde{T}_n - \tilde{T}_{N(n)})\mathbf{1}(F_n^c)\|_2^2 &= \|\tilde{T}_n - \tilde{T}_{N(n)}\|_{2p}^2 \|\mathbf{1}(F_n^c)\|_{2q}^2 \\ &= O(n^2)\big(P[F_n^c]\big)^{1/q} = O(1).\end{aligned} \tag{6.11}$$

Combining (6.10)–(6.11) yields (6.9) as desired:

$$\begin{aligned}\|(\tilde{T}_n - \tilde{T}_{N(n)})\|_2^2 &= O\bigg((\log n)^6 n s_n \int \mathbf{1}(E_n)\mathbf{1}(F_n)\, dP\bigg) + O(1) \\ &= O\big((\log n)^6 n s_n P[E_n]\big) + O(1) \\ &= O\big((\log n)^6 n s_n^2\big) = O\big((\log n)^8 n^{1-2/d}\big). \qquad \square\end{aligned}$$

PROOF OF THEOREM 2.1. Recalling the definition of ν^- at (2.2), we have

$$\lambda\big(\mathrm{Vol}(A_\lambda) - \mathrm{Vol}(A)\big) = \sum_{x \in \mathcal{P}_\lambda} \nu_\lambda^-(x, \mathcal{P}_\lambda, \partial A) = H^{\nu^-}(\mathcal{P}_\lambda, \partial A), \tag{6.12}$$

where the last equality follows from (1.5). Therefore,

$$\lambda^{(d+1)/d}\,\mathrm{Var}\big[\mathrm{Vol}(A_\lambda) - \mathrm{Vol}(A)\big] = \lambda^{-(d-1)/d}\,\mathrm{Var}\big[H^{\nu^-}(\mathcal{P}_\lambda, \partial A)\big].$$

Likewise,

$$\lambda \mathrm{Vol}(A \triangle A_\lambda) = \sum_{x \in \mathcal{P}_\lambda} \nu_\lambda^+(x, \mathcal{P}_\lambda, \partial A) = H^{\nu^+}(\mathcal{P}_\lambda, \partial A).$$

It is therefore enough to show that ν^- and ν^+ satisfy the conditions of Theorem 1.3. We show this for ν^-; similar arguments apply for ν^+. Write ν for ν^- in all that follows.

As seen in Lemma 5.1 of [18], when κ is bounded away from 0 and infinity, the functional $\tilde{\nu}(x, \mathcal{X}) := \mathrm{Vol}(C(x, \mathcal{X}))$ is homogeneously stabilizing and exponentially stabilizing with respect to $\mathcal{P}_\lambda$. Identical arguments show that ν is homogeneously stabilizing and exponentially stabilizing with respect to $(\mathcal{P}_\lambda, \partial A)$. The arguments in [18] may be adapted to show that ν satisfies the p-moment condition (1.8), and we provide the details. For all $y \in \partial A$, $z \in \mathbb{R}^d$, $u \in \mathbb{R}$, we have

$$\begin{aligned} &\big|\nu_\lambda\big((y, \lambda^{-1/d}u), \mathcal{P}_\lambda \cup z, \partial A\big)\big| \\ &\quad \le \omega_d \,\mathrm{diam}\big[C\big((\lambda^{1/d}y, u), \lambda^{1/d}(\mathcal{P}_\lambda \cup z)\big)\big]^d \\ &\qquad \times \mathbf{1}\big(C\big((\lambda^{1/d}y, u), \lambda^{1/d}(\mathcal{P}_\lambda \cup z)\big) \cap \partial A \neq \varnothing\big), \end{aligned} \tag{6.13}$$

where $\omega_d := \pi^{d/2}[\Gamma(1 + d/2)]^{-1}$ is the volume of the d-dimensional unit ball. When κ is bounded away from zero, the factor $\mathrm{diam}[C((\lambda^{1/d}y, u), \lambda^{1/d}(\mathcal{P}_\lambda \cup z))]^d$ has finite moments of all orders, uniformly in y and z [17]. It may be seen that $\mathbb{E}[\mathbf{1}(C((\lambda^{1/d}y, u), \lambda^{1/d}(\mathcal{P}_\lambda \cup z)) \cap \partial A \neq \varnothing)]$ decays exponentially fast in u, uniformly in y and z (see, e.g., Lemma 2.2 of [17]), giving condition (1.8). The Cauchy–Schwarz inequality gives exponential decay (1.9) for ν.

Thus, $\nu := \nu^-$ satisfies all conditions of Theorem 1.3 and, therefore, recalling (6.12), the first part of Theorem 2.1 follows. The second part of Theorem 2.1 follows from identical arguments involving $\nu := \nu^+$. $\square$

PROOF OF THEOREM 2.2. As seen above, ν is homogeneously and exponentially stabilizing with respect to $(\mathcal{P}_\lambda, \partial A)$. It remains only to establish that ν is well-approximated by $\mathcal{P}_\lambda$ input on half-spaces (1.10) and we may then deduce the second part of Theorem 2.2 from Theorem 1.2. This goes as follows.

Fix $\partial A \in \mathbb{M}_2(d)$, $y \in \partial A$. Translating y to the origin, letting $\mathcal{P}_\lambda$ denote a Poisson point process on $[0, 1]^d - y$, letting ∂A denote $\partial A - y$, and using rotation invariance of ν, it is enough to show for all $w \in \mathbb{R}^d$ that

$$\lim_{\lambda \to \infty} \mathbb{E}\big|\nu\big(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A\big) - \nu\big(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1}\big)\big| = 0.$$

Without loss of generality, we assume, locally around the origin, that $\partial A \subset \mathbb{R}^{d-1}_-$.

Let $\tilde{C}(w, \lambda^{1/d}\mathcal{P}_\lambda)$ be the union of $C(w, \lambda^{1/d}\mathcal{P}_\lambda)$ and the Voronoi cells adjacent to $C(w, \lambda^{1/d}\mathcal{P}_\lambda)$ in the Voronoi tessellation of $\mathcal{P}_\lambda$. Consider the event

$$E(\lambda, w) := \{\mathrm{diam}[\tilde{C}(w, \lambda^{1/d}\mathcal{P}_\lambda)] \le \beta \log \lambda\}. \tag{6.14}$$

For β large, we have $P[E(\lambda, w)^c] = O(\lambda^{-2})$ (see, e.g., Lemma 2.2 of [17]). Note that $\nu(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A)$ and $\nu(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})$ have finite second moments, uniformly in $w \in \mathbb{R}^d$ and $\lambda \in (0, \infty)$. By the Cauchy–Schwarz inequality, for large $\beta \in (0, \infty)$, we have for all $w \in \mathbb{R}^d$,

$$\lim_{\lambda\to\infty} \mathbb{E}\big|\big(\nu(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \nu(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})\big)\mathbf{1}\big(E(\lambda, w)^c\big)\big| = 0.$$

It is therefore enough to show for all $w \in \mathbb{R}^d$ that

$$\lim_{\lambda\to\infty} \mathbb{E}\big|\big(\nu(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \nu(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})\big)\mathbf{1}\big(E(\lambda, w)\big)\big| = 0. \tag{6.15}$$

We first assume $w \in \mathbb{R}^{d-1}_-$; the arguments with $w \in \mathbb{R}^{d-1}_+$ are nearly identical. Moreover, we may assume $w \in \lambda^{1/d}A$ for λ large. Consider the (possibly degenerate) solid

$$\Delta_\lambda(w) := \Delta_\lambda(w, \beta) := \big(\mathbb{R}^{d-1}_- \setminus \lambda^{1/d}A\big) \cap B_{2\beta\log\lambda}(w). \tag{6.16}$$

Since ∂A is C^2, the solid $\Delta_\lambda(w)$ has maximal "height" $o((\|w\| + 2\beta\log\lambda)\lambda^{-1/d})$ with respect to the hyperplane $\mathbb{R}^{d-1}$. It follows that

$$\mathrm{Vol}\big(\Delta_\lambda(w)\big) = O\big((\|w\| + 2\beta\log\lambda)\lambda^{-1/d}(2\beta\log\lambda)^{d-1}\big) = O\big((\log\lambda)^d\lambda^{-1/d}\big).$$

On the event $E(\lambda, w)$, the difference of the volumes $C(w, \lambda^{1/d}\mathcal{P}_\lambda) \cap \lambda^{1/d}A^c$ and $C(w, \lambda^{1/d}\mathcal{P}_\lambda) \cap \mathbb{R}^{d-1}_+$ is at most $\mathrm{Vol}(\Delta_\lambda(w))$. Thus,

$$\begin{aligned}&\mathbb{E}\big|\big(\nu(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A) - \nu(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})\big)\mathbf{1}\big(E(\lambda, w)\big)\big| \\ &\quad \le \mathrm{Vol}\big(\Delta_\lambda(w)\big) = O\big((\log\lambda)^d\lambda^{-1/d}\big),\end{aligned}$$

which gives (6.15), and thus the variance asymptotics follow.

We next prove the first part of Theorem 2.2, namely $\mathrm{Var}\,\mathrm{Vol}(A_\lambda) = \Omega(\lambda^{-(d-1)/d})$. By assumption, there is a C^1 subset Γ of ∂A, with $\mathcal{H}^{d-1}(\Gamma) > 0$. Recalling $A \subset [0,1]^d$, subdivide $[0,1]^d$ into cubes of edge length $l(\lambda) := (\lfloor\lambda^{1/d}\rfloor)^{-1}$. The number $L(\lambda)$ of cubes having nonempty intersection with Γ satisfies $L(\lambda) = \Omega(\lambda^{(d-1)/d})$, as otherwise the cubes would partition Γ into $o(\lambda^{(d-1)/d})$ sets, each of $\mathcal{H}^{d-1}$ measure $O((\lambda^{-1/d})^{d-1})$, giving $\mathcal{H}^{d-1}(\Gamma) = o(1)$, a contradiction.

Find a subcollection $Q_1, \dots, Q_M$ of the $L(\lambda)$ cubes such that $d(Q_i, Q_j) \ge 2\sqrt{d}l(\lambda)$ for all $i, j \le M$, and $M = \Omega(\lambda^{(d-1)/d})$. Rotating and translating Q_i, $1 \le i \le M$, by a distance at most $(\sqrt{d}/2)l(\lambda)$, if necessary, we obtain a collection $\tilde{Q}_1, \dots, \tilde{Q}_M$ of disjoint cubes (with faces not necessarily parallel to a coordinate plane) such that:

- $d(\tilde{Q}_i, \tilde{Q}_j) \geq \sqrt{d} l(\lambda)$ for all $i, j \leq M$,
- Γ contains the center of each $\tilde{Q}_i$, here denoted x_i, $1 \leq i \leq M$.

By the C^1 property, Γ is well-approximated locally around each x_i by a hyperplane $\mathbb{H}_i$ tangent to Γ at x_i. Making a further rotation of Q_i, if necessary, we may assume that $\mathbb{H}_i$ partitions $\tilde{Q}_i$ into congruent rectangular solids.

Write ν for ν^-. We now exhibit a configuration of Poisson points $\mathcal{P}_\lambda$ which has strictly positive probability, for which $\lambda^{(d-1)/d}\mathrm{Vol}(A_\lambda)$ has variability bounded away from zero, uniform in λ. Let $\overrightarrow{\mathbf{0}n_i}$, $n_i \in \mathbb{R}^d$, be the unit normal to Γ at x_i. Let $\varepsilon := \varepsilon(\lambda) := l(\lambda)/8$ and subdivide each $\tilde{Q}_i$ into 8^d subcubes of edge length ε. Recall that $B_r(x)$ denotes the Euclidean ball centered at $x \in \mathbb{R}^d$ with radius r. Consider cubes $\tilde{Q}_i$, $1 \leq i \leq M$, having these properties:

(a) the subcubes of $\tilde{Q}_i$ having a face on $\partial \tilde{Q}_i$, called the "boundary subcubes," each contain at least one point from $\mathcal{P}_\lambda$,

(b) $\mathcal{P}_\lambda \cap [B_{\varepsilon/20}(x_i - \frac{\varepsilon}{10} n_i) \cup B_{\varepsilon/20}(x_i + \frac{\varepsilon}{10} n_i)]$ consists of a singleton, say w_i, and

(c) $\mathcal{P}_\lambda$ puts no other points in $\tilde{Q}_i$.

Relabeling if necessary, let $I := \{1, \dots, K\}$ be the indices of cubes $\tilde{Q}_i$ having properties (a)–(c). It is easily checked that the probability a given $\tilde{Q}_i$ satisfies property (a) is strictly positive, uniform in λ. This is also true for properties (b)–(c), showing that

$$\mathbb{E}K = \Omega\bigl(\lambda^{(d-1)/d}\bigr). \tag{6.17}$$

Without loss of generality, we may assume that A contains $B_{\varepsilon/20}(x_i - \frac{\varepsilon}{10} n_i)$ but that $A \cap B_{\varepsilon/20}(x_i + \frac{\varepsilon}{10} n_i) = \varnothing$. Abusing notation, let $\mathcal{Q} := \bigcup_{i=1}^K \tilde{Q}_i$ and put $\mathcal{Q}^c := [0,1]^d \setminus \mathcal{Q}$. Let $\mathcal{F}_\lambda$ be the sigma algebra determined by the random set I, the positions of points of $\mathcal{P}_\lambda$ in all boundary subcubes, and the positions of points $\mathcal{P}_\lambda$ in $\mathcal{Q}^c$. Given $\mathcal{F}_\lambda$, properties (a) and (c) imply that $\mathrm{Vol}(C(w_i, \mathcal{P}_\lambda)) = \Omega(\varepsilon^d)$. Simple geometry shows that when $w_i \in B_{\varepsilon/20}(x_i - \frac{\varepsilon}{10} n_i)$ we have $\mathrm{Vol}(C(w_i, \mathcal{P}_\lambda) \cap A^c) = \Omega(\varepsilon^d)$, that is the contribution to A_λ by the cell $C(w_i, \mathcal{P}_\lambda)$ is $\Omega(\varepsilon^d)$. On the other hand, when $w_i \in B_{\varepsilon/20}(x_i + \frac{\varepsilon}{10} n_i)$, then there is no contribution to A_λ. Moreover, in either case, the volume contribution to A_λ arising from points of $\mathcal{P}_\lambda$ in the boundary subcubes is modified by $o(\varepsilon^d)$ regardless of the position of w_i. Conditional on $\mathcal{F}_\lambda$, and using that w_i is equally likely to belong to either ball, it follows that $\mathrm{Vol}(A_\lambda \cap \tilde{Q}_i)$ has variability $\Omega(\varepsilon^{2d}) = \Omega(\lambda^{-2})$, uniformly in $i \in I$, that is,

$$\mathrm{Var}\bigl[\mathrm{Vol}(A_\lambda \cap \tilde{Q}_i) | \mathcal{F}_\lambda\bigr] = \Omega\bigl(\lambda^{-2}\bigr), \qquad i \in I. \tag{6.18}$$

By the conditional variance formula,

$$\begin{aligned}
\mathrm{Var}\bigl[\mathrm{Vol}(A_\lambda)\bigr] &= \mathrm{Var}\bigl[\mathbb{E}\bigl[\mathrm{Vol}(A_\lambda)|\mathcal{F}_\lambda\bigr]\bigr] + \mathbb{E}\bigl[\mathrm{Var}\bigl[\mathrm{Vol}(A_\lambda)|\mathcal{F}_\lambda\bigr]\bigr] \\
&\geq \mathbb{E}\bigl[\mathrm{Var}\bigl[\mathrm{Vol}(A_\lambda)|\mathcal{F}_\lambda\bigr]\bigr] \\
&= \mathbb{E}\bigl[\mathrm{Var}\bigl[\mathrm{Vol}(A_\lambda \cap \mathcal{Q}) + \mathrm{Vol}\bigl(A_\lambda \cap \mathcal{Q}^c\bigr)|\mathcal{F}_\lambda\bigr]\bigr].
\end{aligned}$$

Given $\mathcal{F}_\lambda$, the Poisson–Voronoi tessellation of $\mathcal{P}_\lambda$ admits variability only inside $\mathcal{Q}$, that is $\mathrm{Vol}(A_\lambda \cap \mathcal{Q}^c)$ is constant. Thus,

$$\begin{aligned}\mathrm{Var}[\mathrm{Vol}(A_\lambda)] &\ge \mathbb{E}[\mathrm{Var}[\mathrm{Vol}(A_\lambda \cap \mathcal{Q})|\mathcal{F}_\lambda]] \\ &= \mathbb{E}\Biggl[\mathrm{Var}\Biggl[\sum_{i\in I}\mathrm{Vol}(A_\lambda \cap \tilde{Q}_i)|\mathcal{F}_\lambda\Biggr]\Biggr] = \mathbb{E}\Biggl[\sum_{i\in I}\mathrm{Var}[\mathrm{Vol}(A_\lambda \cap \tilde{Q}_i)|\mathcal{F}_\lambda]\Biggr],\end{aligned}$$

since, given $\mathcal{F}_\lambda$, $\mathrm{Vol}(A_\lambda \cap \tilde{Q}_i), i \in I$, are independent. By (6.17) and (6.18), we have

$$\mathrm{Var}[\mathrm{Vol}(A_\lambda)] \ge c_5 \lambda^{-2}\mathbb{E}[K] = \Omega(\lambda^{-(d+1)/d}),$$

concluding the proof of Theorem 2.2 when ν is set to ν^-.

To show $\mathrm{Var}[\mathrm{Vol}(A \triangle A_\lambda)] = \Omega(\lambda^{-(d+1)/d})$, consider cubes $\tilde{Q}_i, 1 \le i \le M$, having these properties:

(a′) the "boundary subcubes," each contain at least one point from $\mathcal{P}_\lambda$,
(b′) $\mathcal{P}_\lambda \cap B_{\varepsilon/20}(x_i - \frac{\varepsilon}{10}n_i)$ consists of a singleton, say w_i, and
(c′) $\mathcal{P}_\lambda \cap [B_{\varepsilon/20}(x_i + \frac{\varepsilon}{10}n_i) \cup B_{\varepsilon/20}(x_i + \varepsilon n_i)]$ consists of a singleton, say z_i,
(d′) $\mathcal{P}_\lambda$ puts no other points in $\tilde{Q}_i$.

Let $I' := \{1, \dots, K'\}$ be the indices of cubes $\tilde{Q}_i$ having properties (a′)–(d′). Let $\mathcal{F}_\lambda$ be as above, with I replaced by I'. It suffices to notice that on $\mathcal{F}_\lambda$, we have

$$\begin{aligned}\mathrm{Vol}(A \triangle A_\lambda)\mathbf{1}(z_i \in B_{\varepsilon/20}(x_i + \varepsilon n_i)) &\ge 2\mathrm{Vol}(A \triangle A_\lambda)\mathbf{1}(z_i \in B_{\varepsilon/20}(x_i + \varepsilon/10 n_i)) \\ &= \Omega(\lambda^{-2}).\end{aligned}$$

From this, we may deduce the analog of (6.18), namely

$$\mathrm{Var}[\mathrm{Vol}((A \triangle A_\lambda) \cap \tilde{Q}_i)|\mathcal{F}_\lambda] = \Omega(\lambda^{-2}), \qquad i \in I,$$

and follow the above arguments nearly verbatim. This concludes the proof when ν is set to ν^+. $\square$

PROOF OF THEOREM 2.3. For any ∂A, we have $|\nu_n^{\pm}(X_i, \mathcal{X}_n, \partial A)| \le \mathrm{Vol}(C(X_i, \mathcal{X}_n)) \le \omega_d(\mathrm{diam}[C(n^{1/d}X_i, n^{1/d}\mathcal{X}_n)])^d$. Let $D = 2/(1 - 1/p)$. Modifications of Lemma 2.2 of [17] show that with probability at least $1 - n^{-D-1}$ we have $(\mathrm{diam}[C(n^{1/d}X_i, n^{1/d}\mathcal{X}_n)])^d = O(\log n)$, that is to say $\nu^{\pm}$ satisfies (6.2). The discussion in Section 6.3 of [19] shows that the functionals ν^+ and ν^- are binomially exponentially stabilizing as at (6.1). Theorem 2.3 follows from Lemma 6.1, Theorems 2.1–2.2, and Corollary 2.1. $\square$

PROOF OF THEOREM 2.4. It suffices to show that the functional α defining the statistics (2.4) satisfies the conditions of Theorems 1.1 and 1.2 and then apply (1.20) and (1.21) to the statistic (2.4) to obtain (2.5) and (2.6), respectively. To do this, we shall follow the proof that the volume functional ν defined at (2.2) satisfies these conditions. The proof that α is homogeneously stabilizing and satisfies the moment condition (1.8) follows nearly verbatim the proof that ν satisfies these conditions, where we only need to replace the factor $\omega_d \operatorname{diam}[C((\lambda^{1/d} y, u), \lambda^{1/d}(\mathcal{P}_\lambda \cup z))]^d$ in (6.13) by $\omega_{d-1} \operatorname{diam}[C((\lambda^{1/d} y, u), \lambda^{1/d}(\mathcal{P}_\lambda \cup z))]^{d-1}$.

To show that α is well-approximated by $\mathcal{P}_\lambda$ input on half-spaces (1.10), by moment bounds on α and the Cauchy–Schwarz inequality, it is enough to show the analog of (6.15), namely for all $w \in \mathbb{R}^d$ that

$$\lim_{\lambda \to \infty} \mathbb{E}\big|\big(\alpha\big(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A\big) - \alpha\big(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1}\big)\big)\mathbf{1}\big(E(\lambda, w)\big)\big| = 0, \tag{6.19}$$

where $E(\lambda, w)$ is at (6.14). Recalling the definition of $\Delta_\lambda(w)$ at (6.16), define

$$E_0(\lambda, w) := \{\lambda^{1/d}\mathcal{P}_\lambda \cap \Delta_\lambda(w) = \varnothing\}.$$

Since the intensity measure of $\lambda^{1/d}\mathcal{P}_\lambda$ is upper bounded by $\|\kappa\|_\infty$, we have

$$\begin{aligned} P\big[E_0(\lambda, w)^c\big] &= 1 - P\big[E_0(\lambda, w)\big] \le 1 - \exp\big(-\|\kappa\|_\infty \mathrm{Vol}\big(\Delta_\lambda(w)\big)\big) \\ &\le 1 - \exp\big(-c_6 (\log \lambda)^d \lambda^{-1/d}\big) = O\big((\log \lambda)^d \lambda^{-1/d}\big). \end{aligned} \tag{6.20}$$

On the event $E(\lambda, w) \cap E_0(\lambda, w)$, the scores $\alpha(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A)$ and $\alpha(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1})$ coincide. Indeed, on this event it follows that f is face of the boundary cell $C(w, \lambda^{1/d}\mathcal{P}_\lambda)$ of $\lambda^{1/d} A_\lambda$ iff f is a face of a boundary cell of the Poisson–Voronoi tessellation of $\mathbb{R}^{d-1}_-$. [If f is a face of the boundary cell $C(w, \lambda^{1/d}\mathcal{P}_\lambda)$, $w \in \lambda^{1/d} A$, then f is also a face of $C(z, \lambda^{1/d}\mathcal{P}_\lambda)$ for some $z \in \lambda^{1/d} A^c$. If $\Delta_\lambda(w) = \varnothing$, then z must belong to $\mathbb{R}^{d-1}_+$, showing that f is a face of a boundary cell of the Poisson–Voronoi tessellation of $\mathbb{R}^{d-1}_-$. The reverse implication is shown similarly.]

On the other hand, since

$$\big\|\big(\alpha\big(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A\big) - \alpha\big(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1}\big)\big)\mathbf{1}\big(E(\lambda, w)\big)\big\|_2 = O(1),$$

and since by (6.20) we have $P[E_0(\lambda, w)^c] = O((\log \lambda)^d \lambda^{-1/d})$, it follows by the Cauchy–Schwarz inequality that as $\lambda \to \infty$,

$$\begin{aligned} &\mathbb{E}\big|\big(\alpha\big(w, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A\big) - \alpha\big(w, \lambda^{1/d}\mathcal{P}_\lambda, \mathbb{R}^{d-1}\big)\big) \\ &\qquad \times \mathbf{1}\big(E(\lambda, w)\big)\mathbf{1}\big(E_0(\lambda, w)^c\big)\big| \to 0. \end{aligned} \tag{6.21}$$

Therefore, (6.19) holds and so α is well-approximated by $\mathcal{P}_\lambda$ input on half-spaces and α satisfies all conditions of Theorems 1.1 and 1.2. This proves statements (2.5)–(2.6). Note that (2.7) follows from (1.23), proving Theorem 2.4. To show these limits hold when Poisson input is replaced by binomial input $\mathcal{X}_n$ we shall show that α satisfies the conditions of Lemma 6.1. Notice that $|\alpha_n(X_1, \mathcal{X}_n, \partial A)| \le$

$\mathcal{H}^{d-1}(C(X_1, \mathcal{X}_n)) = O(\mathrm{diam}[C(n^{1/d}X_1, n^{1/d}\mathcal{X}_n)]^{d-1}) = O((\log n)^{(d-1)/d})$ with probability at least $1 - n^{-D-1}$, that is α satisfies condition (6.2), where $D = 2/(1-1/p)$. The arguments in Section 6.3 of [19] may be modified to show that α is binomially exponentially stabilizing as at (6.1) and, therefore, by Lemma 6.1, the limits (2.5)–(2.7) hold for input $\mathcal{X}_n$, as asserted in remark (i) following Theorem 2.4. $\square$

PROOF OF THEOREM 2.5. Orient ∂A so that points $(y, t) \in A$, have positive t coordinate. Notice that ζ satisfies the decay condition (1.9) for all $p \in [1, \infty)$. Indeed, for all $z \in \mathbb{R}^d \cup \varnothing$, $y \in \partial A$, $u \in (-\infty, \infty)$, and $\lambda \in (0, \infty)$, we have

$$\big|\zeta_\lambda\big((y, \lambda^{-1/d}u), \mathcal{P}_\lambda \cup z, \partial A\big)\big| \le \mathbf{1}\big(\big(K \oplus \big(y, \lambda^{-1/d}u\big)\big) \cap A \cap \mathcal{P}_\lambda = \varnothing\big).$$

Now

$$P\big[\big(K \oplus \big(y, \lambda^{-1/d}u\big)\big) \cap A \cap \mathcal{P}_\lambda = \varnothing\big] = \exp\big(-\lambda \mathrm{Vol}\big(\big(K \oplus \big(y, \lambda^{-1/d}u\big)\big) \cap A\big)\big)$$

decays exponentially fast in $|u| \in (0, \infty)$, uniformly in $y \in \partial A$ and $\lambda \in (0, \infty)$ and therefore (1.9) holds for all $p \in [1, \infty)$.

To see that ζ is homogeneously stabilizing as at (1.6), we argue as follows. Without loss of generality, let $\mathbf{0}$ belong to the half-space H with hyperplane $\mathbb{H}$, as otherwise $\zeta(\mathbf{0}, \mathcal{H}_\tau, \mathbb{H}) = 0$. Now $\zeta(\mathbf{0}, \mathcal{H}_\tau, \mathbb{H})$ is insensitive to point configurations outside $K \cap H$ and so $R^\zeta(\mathcal{H}_\tau, \mathbb{H}) := \mathrm{diam}(K \cap H)$ is a radius of stabilization for ζ.

To show exponential stabilization of ζ as at (1.7), we argue similarly. By definition of maximality, $\zeta_\lambda(x, \mathcal{P}_\lambda, \partial A)$ is insensitive to point configurations outside $(K \oplus x) \cap A$. In other words, $\zeta(\lambda^{1/d}x, \lambda^{1/d}\mathcal{P}_\lambda, \lambda^{1/d}\partial A)$ is unaffected by point configurations outside

$$K_\lambda(x) := \big(K \oplus \lambda^{1/d}x\big) \cap \lambda^{1/d}A.$$

Let $R(x) := R^\zeta(x, \mathcal{P}_\lambda, \partial A)$ be the distance between $\lambda^{1/d}x$ and the nearest point in $K_\lambda(x) \cap \lambda^{1/d}\mathcal{P}_\lambda$, if there is such a point; otherwise let $R(x, \mathcal{P}_\lambda, \partial A)$ be the maximal distance between $\lambda^{1/d}x$ and $K_\lambda(x) \cap \partial(\lambda^{1/d}A)$, denoted here by $D(\lambda^{1/d}x)$. By the smoothness assumptions on the boundary, it follows that $K_\lambda(x) \cap B_t(x)$ has volume at least $c_7 t^d$ for all $0 \le t \le D(\lambda^{1/d}x)$. It follows that uniformly in $x \in \partial A$ and $\lambda > 0$

$$P\big[R(x) > t\big] \le \exp\big(-c_7 t^d\big), \qquad 0 \le t \le D\big(\lambda^{1/d}x\big). \tag{6.22}$$

For $t \in [D(\lambda^{1/d}x), \infty)$, this inequality holds trivially. Moreover, we claim that $R(x)$ is a radius of stabilization for ζ at x. Indeed, if $R(x) \in (0, D(\lambda^{1/d}x))$, then x is not maximal, and so

$$\zeta\big(x, \lambda^{1/d}\mathcal{P}_\lambda \cap B_R(x), \lambda^{1/d}\partial A\big) = 0.$$

Point configurations outside $B_R(x)$ do not modify the score ζ. If $R(x) \in [D(\lambda^{1/d}x), \infty)$ then

$$\zeta\big(x, \lambda^{1/d}\mathcal{P}_\lambda \cap B_R(x), \lambda^{1/d}\partial A\big) = 1$$

and point configurations outside $B_R(x)$ do not modify ζ, since maximality of x is preserved. Thus, $R(x) := R^\zeta(x, \mathcal{P}_\lambda, \partial A)$ is a radius of stabilization for ζ at x, it decays exponentially fast by (6.22), and (1.7) holds.

It remains to show that ζ is well-approximated by $\mathcal{P}_\lambda$ input on half-spaces (1.10). As with the Poisson–Voronoi functional, it is enough to show the convergence (6.15), with ν replaced by ζ there. However, since ζ is either 0 or 1, we have that (6.15) is bounded by the probability of the event that $\lambda^{1/d}\mathcal{P}_\lambda$ puts points in the region $\Delta_\lambda(w)$ defined at (6.16). However, this probability tends to zero as $\lambda \to \infty$, since the complement probability satisfies

$$\lim_{\lambda\to\infty} P\big[\lambda^{1/d}\mathcal{P}_\lambda \cap \Delta_\lambda(w) = \varnothing\big] = \lim_{\lambda\to\infty} \exp\big(-\mathrm{Vol}\big(\Delta_\lambda(w)\big)\big) = 1.$$

This gives the required analog of (6.15) for ζ and so ζ_λ satisfies (1.10), which was to be shown. Thus, Theorem 2.5 holds for Poisson input $\mathcal{P}_\lambda$, where we note $\sigma^2(\zeta, \partial A) \in (0, \infty)$ by Theorem 4.3 of [3]. Straightforward modifications of the above arguments show that ζ is binomially exponentially stabilizing as at (6.1). Now $|\zeta| \le 1$, so ζ trivially satisfies (6.2). Therefore, by Lemma 6.1, Theorem 2.5 holds for binomial input $\mathcal{X}_n$.

This completes the proof of Theorem 2.5, save for showing (2.9). First notice that

$$\mu(\zeta, \partial A) = \int_{\partial A} \int_0^\infty \mathbb{E}\zeta\big((\mathbf{0}_y, u), \mathcal{H}_1, \mathbb{H}_y\big)\kappa(y)^{(d-1)/d}\, du\, dy, \tag{6.23}$$

which follows from (1.11) and $\mathbb{E}\zeta((\mathbf{0}_y, u), \mathcal{H}_\tau, \mathbb{H}_y) = \mathbb{E}\zeta((\mathbf{0}_y, u\tau^{1/d}), \mathcal{H}_1, \mathbb{H}_y)$.

The limit (6.23) further simplifies as follows. In $d = 2$, we have for $y = (v, F(v)) \in \partial A$ and all $u \in (0, \infty)$ that

$$\mathbb{E}\zeta\big((\mathbf{0}_y, u), \mathcal{H}_1, \mathbb{H}_y\big) = \exp\bigg(-\frac{u^2}{2}\frac{(1 + F'(v)^2)}{|F'(v)|}\bigg),$$

where we use that a right triangle with legs on the coordinate axes, hypotenuse distant u from the origin and having slope $m \in (0, -\infty)$ has area $u^2(1+m^2)/2|m|$. Put $b := (1 + F'(v)^2)/2|F'(v)|$ and $z = u^2 b$. Then

$$\begin{aligned}
\mu(\zeta, \partial A) &= \int_{\partial A} \int_0^\infty \mathbb{E}\zeta\big((\mathbf{0}_y, u), \mathcal{H}_1, \mathbb{H}_y\big)\, du\, \kappa(y)^{1/2}\, dy \\
&= \frac{1}{2} \int_{v\in[0,1]} \int_0^\infty \exp(-z)(bz)^{-1/2}\sqrt{1 + F'(v)^2}\kappa\big(v, F(v)\big)^{1/2}\, dz\, dv \\
&= \frac{1}{2}\Gamma\bigg(\frac{1}{2}\bigg) \int_{v\in[0,1]} b^{-1/2}\sqrt{1 + F'(v)^2}\kappa\big(v, F(v)\big)^{1/2}\, dv
\end{aligned}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\int_{v\in[0,1]} 2^{1/2}\left|F'(v)\right|^{1/2}\kappa\big(v, F(v)\big)^{1/2}\,dv$$

$$= \left(\frac{\pi}{2}\right)^{1/2}\int_0^1 \left|F'(v)\right|^{1/2}\kappa\big(v, F(v)\big)^{1/2}\,dv.$$

More generally, in $d > 2$, assume that F is continuously differentiable with partials which are negative and bounded away from 0 and $-\infty$. Let $y \in \partial A$ be given by $y = (v, F(v))$, $v \in D$, and put $F_i := \partial F/\partial v_i$. Then for $u \in (0, \infty)$ we have

$$\mathbb{E}\zeta\big((\mathbf{0}_y, u), \mathcal{H}_1, \mathbb{H}_y\big) = \exp\left(\frac{-u^d(1 + \sum_{i=1}^{d-1} F_i'(v)^2)^{d/2}}{d!|\prod_{i=1}^{d-1} F_i(v)|^{-1}}\right).$$

Let $z = u^d b$, where $b := \frac{1}{d!}(1 + \sum_{i=1}^{d-1} F_i'(v)^2)^{d/2}|\prod_{i=1}^{d-1} F_i(v)|^{-1}$. This yields

$$\mu(\zeta, \partial A) := \int_{\partial A}\int_0^\infty \mathbb{E}\zeta\big((\mathbf{0}_y, u), \mathcal{H}_1, \mathbb{H}_y\big)\,du\,\kappa(y)^{(d-1)/d}\,dy$$

$$= (d!)^{1/d} d^{-1}\Gamma\big(d^{-1}\big)\int_D \left|\prod_{i=1}^{d-1} F_i(v)\right|^{1/d} \kappa\big(v, F(v)\big)^{(d-1)/d}\,dv,$$

that is to say (2.9) holds. $\square$

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