Variance Asymptotics and Scaling Limits for Gaussian Polytopes

Pierre Calka*, J. E. Yukich**

December 18, 2013

Abstract

Let K_n be the convex hull of i.i.d. random variables distributed according to the standard normal distribution on $\mathbb{R}^d$. We establish variance asymptotics for the re-scaled volume and k-face functional of K_n, $k \in \{0, 1, ..., d-1\}$, resolving an open problem. Asymptotic variances and the scaling limit of the boundary of K_n are given in terms of functionals of germ-grain models having parabolic grains with apices at a Poisson point process on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity $e^h dhdv$.

1 Main results

For all $\lambda \in (0, \infty)$, let $\mathcal{P}_\lambda$ denote a Poisson point process of intensity $\lambda\phi(x)dx$, where

$$\phi(x) := (2\pi)^{-d/2} \exp(-\frac{|x|^2}{2})$$

is the standard normal density on $\mathbb{R}^d, d \geq 2$. Let $\mathcal{X}_n := \{X_1, ..., X_n\}$, where X_i are i.i.d. with density $\phi(\cdot)$. We put K_λ and K_n to be the Gaussian polytopes defined by the convex hull of $\mathcal{P}_\lambda$ and $\mathcal{X}_n$, respectively. The number of k-faces of K_λ and K_n are denoted by $f_k(K_\lambda)$ and $f_k(K_n)$, respectively.

American Mathematical Society 2010 subject classifications. Primary 60F05, 52A20; Secondary 60D05, 52A23

Key words and phrases. Random polytopes, parabolic germ-grain models, convex hulls of i.i.d. Gaussian samples

* Research partially supported by French ANR grant PRESAGE.

** Research supported in part by NSF grant DMS-1106619

In $d = 2$, Rényi and Sulanke [23] determined $\mathbb{E}\, f_1(K_n)$ and later Raynaud [21] determined $\mathbb{E}\, f_{d-1}(K_n)$ for all dimensions. Subsequently, work of Affentranger and Schneider [2] and Baryshnikov and Vitale [8] yielded the general formula

$$\mathbb{E}\, f_k(K_n) = \frac{2^d}{\sqrt{d}}\binom{d}{k+1}\beta_{k,d-1}(\pi \log n)^{(d-1)/2}(1+o(1)),$$

with $k \in \{0, ..., d-1\}$ and where $\beta_{k,d-1}$ is the internal angle of a regular $(d-1)$-simplex at one of its k-dimensional faces. Concerning the volume functional, Affentranger [1] showed that its expectation asymptotics satisfy

$$\mathbb{E}\,\mathrm{Vol}(K_n) = \kappa_d(2\log n)^{d/2}(1+o(1)),$$

where κ_d denotes the volume of the d-dimensional unit ball.

In a remarkable paper, Bárány and Vu [8] use dependency graph methods to establish a rate of normal convergence for $f_k(K_n)$ and $\mathrm{Vol}(K_n)$, $k \in \{0, ..., d-1\}$. A key part of their work involves obtaining sharp lower bounds for $\mathrm{Var} f_k(K_n)$ and $\mathrm{Var}\mathrm{Vol}(K_n)$. Their results stop short of determining precise variance asymptotics for $f_k(K_n)$ and $\mathrm{Vol}(K_n)$, an open problem going back to the 1993 survey of Weil and Wieacker (p. 1431 of [28]). We resolve this problem in Theorems 1.3 and 1.4, expressing the variance asymptotics in terms of scaling limit functionals of parabolic germ-grain models.

Let $\mathcal{P}$ be the Poisson point process on $\mathbb{R}^{d-1} \times \mathbb{R}$ with intensity

$$d\mathcal{P}((v,h)) := e^h dh dv, \quad \text{with} \quad (v,h) \in \mathbb{R}^{d-1} \times \mathbb{R}. \tag{1.1}$$

Let $\Pi^{\downarrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R}, h \leq -|v|^2/2\}$ and put $\Pi^{\downarrow}(w) := w \oplus \Pi^{\downarrow}$, where $\oplus$ denotes Minkowski addition. Consider the maximal union $\Phi(\mathcal{P})$ of parabolic grains $\Pi^{\downarrow}(w), w \in \mathbb{R}^d$, having the property that $\Pi^{\downarrow}(w)$ belongs to $\Phi(\mathcal{P})$ if its interior contains no point of $\mathcal{P}$. Thus

$$\Phi(\mathcal{P}) := \bigcup_{\begin{cases} w \in \mathbb{R}^{d-1}\times\mathbb{R} \\ \mathcal{P}\cap \mathrm{int}(\Pi^{\downarrow}(\mathrm{w})) = \emptyset \end{cases}} \Pi^{\downarrow}(w).$$

Remove all points of $\mathcal{P}$ not belonging to $\partial\Phi(\mathcal{P})$ and call the resulting thinned point set $\mathrm{Ext}(\mathcal{P})$. Notice that $\partial\Phi(\mathcal{P})$ is a union of inverted parabolic surfaces.

We show that the re-scaled random point configuration of extreme points in $\mathcal{P}_\lambda$ (and in $\mathcal{X}_n$) converges to the limit measure $\mathrm{Ext}(\mathcal{P})$ and that the scaling limit ∂K_λ as $\lambda \to \infty$ (and of ∂K_n and $n \to \infty$) coincides with $\partial\Phi(\mathcal{P})$. Curiously, this boundary features in the geometric construction of the zero-viscosity solution of Burgers' equation [10]. We consequently obtain a closed form expression for expectation and variance asymptotics for the number of shocks in the solution of the inviscid Burgers' equation adding to [7].

Fix $u_0 := (0, 0, ..., 1) \in \mathbb{R}^d$ and let $T_{u_0} := T_{u_0}\mathbb{S}^{d-1}$ denote the tangent space to the unit sphere $\mathbb{S}^{d-1}$ at u_0. The *exponential map* $\exp := \exp_{d-1} := \exp_{u_0} : T_{u_0}\mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ maps a vector v of the tangent space T_{u_0} to the point $u \in \mathbb{S}^{d-1}$ such that u lies at the end of the geodesic of length $|v|$ starting at u_0 and having direction v.

For all $\lambda \in [1, \infty)$ put

$$R_\lambda := \sqrt{2 \log \lambda - \log(2 \cdot (2\pi)^d \cdot \log \lambda)}. \tag{1.2}$$

Choose λ_0 so that for $\lambda \in [\lambda_0, \infty)$ we have $R_\lambda \in [0, \infty)$. Define the scaling transform $T^{(\lambda)} : \mathbb{R}^d \to \mathbb{R}^{d-1} \times \mathbb{R}$ by

$$T^{(\lambda)}(x) := \left(R_\lambda \exp_{d-1}^{-1} \frac{x}{|x|},\ R_\lambda^2 (1 - \frac{|x|}{R_\lambda}) \right), \quad x \in \mathbb{R}^d. \tag{1.3}$$

Theorem 1.1 *Under the transformations $T^{(\lambda)}$ and $T^{(n)}$, the extreme points of the respective Gaussian samples $\mathcal{P}_\lambda$ and $\mathcal{X}_n$ converge in distribution to the thinned process* $\operatorname{Ext}(\mathcal{P})$ *as $\lambda \to \infty$ (respectively, as $n \to \infty$).*

Let $B_d(v, r)$ be the d-dimensional Euclidean ball centered at $v \in \mathbb{R}^d$ and with radius r. $\mathcal{C}(B_d(v, r))$ is the space of continuous functions on $B_d(v, r)$ equipped with the supremum norm.

Theorem 1.2 *Fix $L \in (0, \infty)$. As $\lambda \to \infty$, the re-scaled boundary $T^{(\lambda)}(\partial K_\lambda)$ converges in probability to $\partial(\Phi(\mathcal{P}))$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$.*

In a companion paper we shall show that $\partial(\Phi(\mathcal{P}))$ is also the scaling limit of the boundary of the convex hull of i.i.d. points in polytopes. In $d = 2$, the reflection of $\partial(\Phi(\mathcal{P}))$ about the x-axis describes a festoon of parabolic arcs featuring in the geometric construction of the zero viscosity solution($\mu = 0$) to Burgers' equation

$$\frac{\partial v}{\partial t} + (v, \nabla)v = \mu \Delta v, \quad v = v(t, x),\ t > 0, \quad (x, t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+, \tag{1.4}$$

subject to Gaussian initial conditions [19]; see Remark (i) below. Given its prominence in the asymptotics of Burgers' equation and its role in scaling limits of boundaries of random polytopes, we shall henceforth refer to $\partial(\Phi(\mathcal{P}))$ as *the Burgers' festoon*.

The transformation $T^{(\lambda)}$ induces scaling limit k-face and volume functionals governing the large λ behavior of convex hull functionals, as seen in the next results. These scaling limit functionals are used in the description of the variance asymptotics for the k-face and volume functionals, $k \in \{0, 1, ..., d-1\}$.

Theorem 1.3 *For all $k \in \{0, 1, ..., d-1\}$, there exists a constant $F_{k,d} \in (0, \infty)$, defined in terms of averages of covariances of a scaling limit k-face functional on $\mathcal{P}$, such that*

$$\lim_{\lambda \to \infty} (2 \log \lambda)^{-(d-1)/2} \mathrm{Var} f_k(K_\lambda) = F_{k,d} \tag{1.5}$$

and

$$\lim_{n \to \infty} (2 \log n)^{-(d-1)/2} \mathrm{Var} f_k(K_n) = F_{k,d}. \tag{1.6}$$

Theorem 1.4 *There exists a constant $V_d \in (0, \infty)$, defined in terms of averages of covariances of a scaling limit volume functional on $\mathcal{P}$, such that*

$$\lim_{\lambda \to \infty} (2 \log \lambda)^{-(d-3)/2} \mathrm{Var}\mathrm{Vol}(K_\lambda) = V_d \tag{1.7}$$

and

$$\lim_{n \to \infty} (2 \log n)^{-(d-3)/2} \mathrm{Var}\mathrm{Vol}(K_n) = V_d. \tag{1.8}$$

We also have

$$\kappa_d^{-1} (2 \log \lambda)^{-d/2} \mathbb{E}\, \mathrm{Vol}(K_\lambda) = 1 + O((\log \lambda)^{-1}). \tag{1.9}$$

The thinned point set $\mathrm{Ext}(\mathcal{P})$ features in the description of asymptotic solutions to Burgers' equation (cf. Remark (i) below) and we next consider its limit theory with respect to a sequence of increasing windows in $\mathbb{R}^d$. Let $Q_\lambda \uparrow \mathbb{R}^{d-1}$ as $\lambda \to \infty$ and put $\tilde{Q}_\lambda := Q_\lambda \times \mathbb{R}$. The next result, a by-product of our general methods, yields a closed form expression for the limits appearing in variance and expectation asymptotics for the number of points in $\mathcal{P}_0$ over growing windows, adding to [7].

Corollary 1.1 *There exist constants E_d and $N_d \in (0, \infty)$ such that*

$$\lim_{\lambda \to \infty} (\mathrm{vol} Q_\lambda)^{-1} \mathbb{E}\, [\mathrm{card}(\mathrm{Ext}(\mathcal{P} \cap \tilde{Q}_\lambda))] = E_d \tag{1.10}$$

and

$$\lim_{\lambda \to \infty} (\mathrm{vol} Q_\lambda)^{-1} \mathrm{Var}[\mathrm{card}(\mathrm{Ext}(\mathcal{P} \cap \tilde{Q}_\lambda))] = N_d. \tag{1.11}$$

In particular, $N_d = F_{0,d}$.

The reader may wonder about the genesis of $T^{(\lambda)}$ and characteristic scaling by R_λ, which satisfies $R_\lambda^{-(d-1)} \lambda \int_{|x| > R_\lambda} \phi(x) dx \to c \in (0, \infty)$. Roughly speaking, the effect of $T^{(\lambda)}$ is to first re-scale the Gaussian sample by R_λ^{-1} so that ∂K_λ is close to $\mathbb{S}^{d-1}$. By considering the distribution of $\max_{i \leq n} |X_i|$ we see that $(1 - |x|/R_\lambda)$ is small when $x \in \partial K_\lambda$; cf [17]. Re-scale again according to the twin desiderata: (i) unit volume image subsets near the hyperplane $\mathbb{R}^{d-1}$ should host $\Theta(1)$ re-scaled points (ii) radial components of points should scale as the square of angular components $\exp_{d-1}^{-1} x/|x|$. Desideratum (ii) preserves the parabolic nature of the defect support function for $R_\lambda^{-1}K_\lambda$, namely the function $1-h_{R_\lambda^{-1}K_\lambda}(u), u \in \mathbb{S}^{d-1}$, where h_K is the support function of $K \subset \mathbb{R}^d$. Extreme value theory [25] for $|X_i|,\ i \geq 1$,suggests (i) is achieved via radial scaling by R_λ^2, whence by (ii) we obtain angular scaling of R_λ, and (1.3) follows. These heuristics are justified in Section 3 (especially Lemma 3.2).

Remarks.

(i) *Burgers' equation.* Let $\mathrm{Ext}(\mathcal{P})'$ be the reflection of $\mathrm{Ext}(\mathcal{P})$ about the hyperplane $\mathbb{R}^{d-1}$. The point process $\mathrm{Ext}(\mathcal{P})'$ features in the solution to Burgers' equation (1.4) for $\mu \in (0,\infty)$ as well as for $\mu = 0$ (inviscid limit). In the latter case, when $d = 2$ and when the initial conditions are specified by a stationary Gaussian process η having covariance $r(x) = \mathbb{E}\,\eta(\mathbf{0})\eta(x) = o(1/\log x), x \to \infty$, the re-scaled local maximum of the solutions converge in distribution to $\mathrm{Ext}(\mathcal{P})'$ [19]. The abscissas of points in $\mathrm{Ext}(\mathcal{P})'$ correspond to zeroes of the limit velocity process $v(L^2t, L^2x)$, as $L \to \infty$. See Figure 1 in [19] as well as Figure 13 in the seminal work of Burgers [10]. The shocks in the limit velocity process coincide with the local minima of the festoon $\partial(\Phi(\tilde{\mathcal{P}}))$.

By (1.3), the typical angular difference between consecutive extreme points of K_λ, after scaling by R_λ, converges in probability to the typical distance between abscissas of points in $\mathrm{Ext}(\mathcal{P})'$. *Thus the re-scaled angular increments between consecutive extreme points in K_λ behaves like the spacings between zeroes of the zero-viscosity solution to* (1.4).

In the case $\mu \in (0,\infty)$, the point set $\mathrm{Ext}(\mathcal{P})'$ is shown to be the scaling limit as $t \to \infty$ of centered and re-scaled local maxima of the solutions to Burgers' equation (1.4) when the initial conditions are specified by degenerate shot noise with Poissonian spatial locations; see Theorem 9 and Remark 3 of [3]. Correlation functions for $\mathrm{Ext}(\mathcal{P})'$ are given section 5 of [3].

(ii) *Theorems 1.1 and 1.2 - related work.* In 1961, Geffroy [17] states that the Hausdorff distance between K_n and $B_d(\mathbf{0}, \sqrt{2\log n})$ converges almost surely to zero. From [8] we also know that the extreme points of the polytope K_λ concentrate around the sphere $R_\lambda\mathbb{S}^{d-1}$ with high probability. Theorems 1.1-1.2 add to these results, showing convergence of the measure induced by the re-scaled extreme points as well as convergence of the re-scaled boundary.

(iii) *Theorem 1.3- - related work.* As mentioned, Bárány and Vu [8] show that $(\mathrm{Var} f_k(K_n))^{-1/2}(f_k(K_n) - \mathbb{E}\, f_k(K_n))$ converges to a normal random variable as $n \to \infty$. They also show (Theorem 6.3 of [8]) that $\mathrm{Var} f_k(K_n) = \Omega((\log n)^{(d-1)/2})$. These bounds are sharp, as Hug and Reitzner [14] had previously showed that $\mathrm{Var} f_k(K_n) = O((\log n)^{(d-1)/2})$. Aside from these variance bounds and work of Hueter [18], asserting that $\mathrm{Var} f_0(K_n) = c(\log n)^{(d-1)/2} + o(1)$, the second order issues raised by Weil and Weiacker [28] have largely remained unsettled in the case of Gaussian input. In particular the question of showing

$$\mathrm{Var} f_k(K_n) = c(\log n)^{(d-1)/2}(1 + o(1))$$

for $k \in \{1, ..., d-1\}$ has remained open. On page 298 of [14], Hug and Reitzner, commenting on the likelihood of progress, remarked that 'Most probably it is difficult to establish such a precise limit relation...'. Theorem 1.3 addresses these issues.

(iv) *Theorem 1.4- -related work.* Hug and Reitzner [14] show $\mathrm{Var}\mathrm{Vol}(K_n) = O((\log n)^{(d-3)/2})$ and later Bárány and Vu [8] show that $\mathrm{Var}\mathrm{Vol}(K_n) = \Theta((\log n)^{(d-3)/2})$, together with a central limit theorem for $\mathrm{Vol}(K_n)$ and $\mathrm{Vol}(K_\lambda)$. The expectation limit (1.9) improves upon Affentranger [1], who shows $\kappa_d^{-1}(2\log n)^{-d/2}\mathbb{E}\,\mathrm{Vol}(K_n) = 1 + o(1)$.

(v) *Corollary 1.1- -related work.* Baryshnikov [7] establishes the asymptotic normality of $\mathrm{card}(\mathrm{Ext}(\mathcal{P}) \cap \tilde{Q}_\lambda)$, obtaining expectation and variance asymptotics in Theorem 1.9.2 of [7]. Notice that $\mathrm{Ext}(\mathcal{P}) \cap \tilde{Q}_\lambda$ is the restriction to $\tilde{Q}_\lambda$ of the extreme points in $\mathcal{P}$, whereas $\mathrm{Ext}(\mathcal{P} \cap \tilde{Q}_\lambda)$ are the extreme points in $\mathcal{P} \cap \tilde{Q}_\lambda$, which in general is not the same set, by boundary effects. Baryshnikov left open the question of obtaining explicit limits, remarking that 'the question of constants is quite tricky'; see p. 180 of ibid.

In general, if a point process $\mathcal{P}_\infty$ is a scaling limit to the solution of (1.4), then $\mathrm{card}(\mathcal{P}_\infty \cap \tilde{Q}_\lambda)$ coincides with the number of Voronoi cells generated by the abcissas of points in $\mathcal{P}_\infty \cap \tilde{Q}_\lambda$; under conditions on the viscosity and initial input, such cells model the matterless voids in the Universe [3, 7, 19]

2 Parabolic germ-grain models and a general result

In this section we define scaling limit functionals of germ-grain models and we use their second order correlations to precisely define the limit constants $F_{k,d}$ and V_d in (1.5) and (1.7), respectively. We use the scaling limit functionals to establish variance asymptotics for the empirical measures induced by the k-face and volume functionals, thereby extending Theorems 1.3 and 1.4. Denote points in $\mathbb{R}^{d-1} \times \mathbb{R}$ by $w := (v, h)$ or $w' := (v', h')$.

2.1. Parabolic germ-grain models. Let

$$\Pi^{\uparrow} := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}^+, h \geq \frac{|v|^2}{2}\}.$$

Let $\Pi^{\uparrow}(w) := w \oplus \Pi^{\uparrow}$. The process $\mathcal{P}$ generates a germ-grain model of paraboloids

$$\Psi(\mathcal{P}) := \bigcup_{w \in \mathcal{P}} \Pi^{\uparrow}(w).$$

A point $w_0 \in \mathcal{P}$ is *extreme* with respect to $\Psi(\mathcal{P})$ if the grain $\Pi^{\uparrow}(w_0)$ is not a subset of the union of the grains $\Pi^{\uparrow}(w), w \in \mathcal{P} \setminus w_0$. It may be verified that the extreme points from this construction coincides with $\mathrm{Ext}(\mathcal{P})$.

2.2. Scaling limit k-face and volume functionals. A set of $(k+1)$ extreme points $\{x_1, ..., x_k\} \subset \mathrm{Ext}(\mathcal{P})$, generates a k-dimensional *parabolic face* of the Burgers festoon $\partial(\Phi(\mathcal{P}))$ if there exists a translate $\tilde{\Pi}^{\downarrow}$ of $\Pi^{\downarrow}$ such that $\{x_1, \cdots, x_{k+1}\} = \tilde{\Pi}^{\downarrow} \cap \mathrm{Ext}(\mathcal{P})$; cf. Definition 3.3 of [12]. When $k = d-1$ the parabolic face is a hyperface.

Definition 2.1 *Define the scaling limit k-face functional $\xi_k^{(\infty)}(x, \mathcal{P})$, $k \in \{0, 1, ..., d-1\}$, to be the product of $(k+1)^{-1}$ and the number of k-dimensional parabolic faces of the Burgers festoon $\partial(\Phi(\mathcal{P}))$ which contain x, if $x \in \mathrm{Ext}(\mathcal{P})$ and zero otherwise.*

Definition 2.2 *Define the scaling limit defect volume functional $\xi_V^{(\infty)}(x, \mathcal{P}), x \in \mathrm{Ext}(\mathcal{P})$, to be*

$$\xi_V^{(\infty)}(x, \mathcal{P}) := d^{-1} \int_{\mathrm{Cyl}(x)} \partial\Phi(v) dv,$$

where $\mathrm{Cyl}(x)$ denotes the projection onto $\mathbb{R}^{d-1}$ of the parabolic hyperfaces containing x. Otherwise, when $x \notin \mathrm{Ext}(\mathcal{P})$ we put $\xi_V^{(\infty)}(x, \mathcal{P}) = 0$.

One of the main features of our approach is that $\xi_k^{(\infty)}, k \in \{0, 1, ..., d-1\}$, are scaling limits of re-scaled k-face functionals, as defined in Section 3.3. A similar statement holds for $\xi_V^{(\infty)}$. Lemma 4.5 makes these assertions precise. Let $\Xi^{(\infty)}$ denote the collection of scaling limits $\xi_k^{(\infty)}, k \in \{0, 1, ..., d-1\}$, together with $\xi_V^{(\infty)}$.

2.3. Empirical k-face measures, empirical volume measures. Given a finite point set $\mathcal{X} \subset \mathbb{R}^d$, let $\mathrm{co}(\mathcal{X})$ be its convex hull.

Definition 2.3 *Given $k \in \{0, 1, ..., d-1\}$ and x a vertex of $\mathrm{co}(\mathcal{X})$, define the k-face functional $\xi_k(x, \mathcal{X})$ to be the product of $(k+1)^{-1}$ and the number of k-faces of $\mathrm{co}(\mathcal{X})$ which contain x. Otherwise we put $\xi_k(x, \mathcal{X}) = 0$. The empirical k-face measure is*

$$\mu_\lambda^{\xi_k} := \sum_{x \in \mathcal{P}_\lambda} \xi_k(x, \mathcal{P}_\lambda) \delta_x, \tag{2.1}$$

where δ_x is the unit point mass at x.

Thus the total number of k-faces in $\text{co}(\mathcal{X})$ is $\sum_{x \in \mathcal{X}} \xi_k(x, \mathcal{X})$.

Let $\mathcal{F}(x, \mathcal{P}_\lambda)$ be the collection of $(d-1)$-dimensional faces in K_λ which contain x and let $\mathcal{C}(x, \mathcal{P}_\lambda) := \{ry, r > 0, y \in \mathcal{F}(x, \mathcal{P}_\lambda)\}$ be the cone generated by $\mathcal{F}(x, \mathcal{P}_\lambda)$.

Definition 2.4 *Given x a vertex of $\text{co}(\mathcal{P}_\lambda)$, define the defect volume functional*

$$\xi_V(x, \mathcal{P}_\lambda) := d^{-1} \left[\text{Vol}(\mathcal{C}(x, \mathcal{P}_\lambda) \cap B_d(\mathbf{0}, R_\lambda)) - \text{Vol}(\mathcal{C}(x, \mathcal{P}_\lambda) \cap K_\lambda)\right].$$

When x is not a vertex of $\text{co}(\mathcal{P}_\lambda)$, we put $\xi_V(x, \mathcal{P}_\lambda) = 0$. The empirical defect volume measure is

$$\mu_\lambda^{\xi_V} := \sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda) \delta_x. \tag{2.2}$$

Thus the total defect volume of K_λ with respect to the ball $B_d(\mathbf{0}, R_\lambda)$ is given by $\sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda)$.

2.4. Limit theory for empirical k-face and volume measures. Define the following second order correlation functions for $\xi^{(\infty)} \in \Xi^{(\infty)}$.

Definition 2.5 *For all $w_1, w_2 \in \mathbb{R}^d$ and $\xi^{(\infty)} \in \Xi^{(\infty)}$ put*

$$c^{\xi^{(\infty)}}(w_1, w_2) := c^{\xi^{(\infty)}}(w_1, w_2, \mathcal{P}) := \tag{2.3}$$
$$\mathbb{E}\, \xi^{(\infty)}(w_1, \mathcal{P} \cup \{w_2\}) \xi^{(\infty)}(w_2, \mathcal{P} \cup \{w_1\}) - \mathbb{E}\, \xi^{(\infty)}(w_1, \mathcal{P}) \mathbb{E}\, \xi^{(\infty)}(w_2, \mathcal{P}).$$

and

$$\sigma^2(\xi^{(\infty)}) := \int_{-\infty}^{\infty} \mathbb{E}\, \xi^{(\infty)}((\mathbf{0}, h), \mathcal{P})^2 e^h dh + \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^{\xi^{(\infty)}}((\mathbf{0}, h), (v', h')) e^{h'+h} dh' dv' dh. \tag{2.4}$$

Theorem 1.3 is a special case of the following general result expressing the asymptotic behavior of the empirical k-face measures in terms of the scaling limit functional $\xi_k^{(\infty)}$ of parabolic germ grain models. Let $\mathcal{C}(\mathbb{S}^{d-1})$ be the class of bounded functions on $\mathbb{R}^d$ whose set of continuity points includes $\mathbb{S}^{d-1}$. Given $g \in \mathcal{C}(\mathbb{S}^{d-1})$, let $g_r(x) := g(x/r)$ and let $\langle g, \mu_\lambda^\xi \rangle$ denote the integral of g with respect to μ_λ^ξ.

Theorem 2.1 *For all $k \in \{0, 1, ..., d-1\}$, and $g \in \mathcal{C}(\mathbb{S}^{d-1})$ we have*

$$\lim_{\lambda \to \infty} (2 \log \lambda)^{-(d-1)/2} \mathbb{E}\, [\langle g_{R_\lambda}, \mu_\lambda^{\xi_k} \rangle] = \int_{-\infty}^{\infty} \mathbb{E}\, \xi_k^{(\infty)}((\mathbf{0}, h), \mathcal{P}) e^h dh \int_{\mathbb{S}^{d-1}} g(u) du \tag{2.5}$$

and

$$\lim_{\lambda \to \infty} (2 \log \lambda)^{-(d-1)/2} \text{Var}[\langle g_{R_\lambda}, \mu_\lambda^{\xi_k} \rangle] = \sigma^2(\xi_k^{(\infty)}) \int_{\mathbb{S}^{d-1}} g(u)^2 du \in (0, \infty). \tag{2.6}$$

Likewise Theorem 1.4 is a special case of the following general result for the empirical volume measure.

Theorem 2.2 *For all $g \in \mathcal{C}(\mathbb{S}^{d-1})$ we have*

$$\lim_{\lambda \to \infty} (2 \log \lambda)^{-(d-2)/2} \mathbb{E}\, [\langle g_{R_\lambda}, \mu_\lambda^{\xi_V} \rangle] = \int_{-\infty}^{\infty} \mathbb{E}\, \xi_V^{(\infty)}((\mathbf{0}, h), \mathcal{P}) e^h dh \int_{\mathbb{S}^{d-1}} g(u) du \tag{2.7}$$

and

$$\lim_{\lambda \to \infty} (2 \log \lambda)^{-(d-3)/2} \mathrm{Var}[\langle g_{R_\lambda}, \mu_\lambda^{\xi_V} \rangle] = \sigma^2(\xi_V^{(\infty)}) \int_{\mathbb{S}^{d-1}} g(u)^2 du \in (0, \infty). \tag{2.8}$$

Remarks.
(i) *Deducing Theorems 1.3 and 1.4 from Theorems 2.1 and 2.2 .* The convergence (1.5) is implied by (2.6) with $F_{k,d} = \sigma^2(\xi_k^{(\infty)})$. Indeed, applying (2.6) to $g \equiv 1$, we have

$$\langle 1, \mu_\lambda^{\xi_k} \rangle = \sum_{x \in \mathcal{P}_\lambda} \xi_k(x, \mathcal{P}_\lambda) = f_k(K_\lambda).$$

Likewise, putting $g \equiv 1$ in (2.8) we get the convergence (1.7), with $V_d = \sigma^2(\xi_V^{(\infty)})$.

To obtain (1.9), we put $g \equiv 1$ in (2.7) to get $(2 \log \lambda)^{-d/2+1} \mathbb{E}\, [\mathrm{Vol}(B_d(\mathbf{0}, R_\lambda)) - \mathrm{Vol}(K_\lambda)] = O(1)$. It follows that (1.9) holds since

$$\begin{aligned} \kappa_d^{-1} (2 \log \lambda)^{-d/2} \mathbb{E}\, \mathrm{Vol}(K_\lambda) &= \kappa_d^{-1} (2 \log \lambda)^{-d/2} \mathrm{Vol}(B_d(\mathbf{0}, R_\lambda)) \\ &\quad + \kappa_d^{-1} (2 \log \lambda)^{-d/2} \mathbb{E}\, [\mathrm{Vol}(B_d(\mathbf{0}, R_\lambda)) - \mathrm{Vol}(K_\lambda)] \\ &= 1 + O((\log \lambda)^{-1}). \end{aligned}$$

We are unable to show that the right side of (2.7) is non-zero, that is to say we are unable to show $(2 \log \lambda)^{-d/2} (\mathbb{E}\, [\mathrm{Vol}(B_d(\mathbf{0}, R_\lambda))] - \mathrm{Vol}(K_\lambda)) \neq o((\log \lambda)^{-1})$.

The de-Poissonized limit (1.8) follows from the coupling of binomial and Poisson points used in Bárány and Vu [8], in particular Lemma 8.1 of [8]. The limit (1.6) similarly follows from (1.5) and the same coupling, as described in Section 13.2 of [8].

(ii) *Central limit theorems.* Combining (2.6) with the results of [8] shows the following central limit theorem, as $\lambda \to \infty$:

$$(2 \log \lambda)^{-(d-1)/2} (\langle g_{R_\lambda}, \mu_\lambda^{\xi_k} \rangle - \mathbb{E}\, \langle g_{R_\lambda}, \mu_\lambda^{\xi_k} \rangle) \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

where $N(0, \sigma^2)$ denotes a mean zero normal random variable with variance $\sigma^2 := \sigma^2(\xi_k^{(\infty)}) \int_{\mathbb{S}^{d-1}} g(u)^2 du$.

2.5. Further extensions

(i) Brownian limits. By following the scaling methods of this paper and by appealing to the methods of section 8 of [12] we may deduce that the process given as the integrated version of the defect volume converges to a Brownian sheet process. This goes as follows. For $\mathcal{X} \subset \mathbb{R}^d$ and $u \in \mathbb{S}^{d-1}$ we put

$$r(u, \mathcal{X}) := R_\lambda - \sup\{\rho > 0 : \rho u \in \mathrm{co}(\mathcal{X})\}$$

and for all $\lambda \in [\lambda_0, \infty)$ let $r_\lambda(u) := r(u, \mathcal{P}_\lambda)$ be the defect radius-vector function. Let σ_{d-1} be the $(d-1)$-dimensional surface measure on $\mathbb{S}^{d-1}$ and let $\mathbb{B}_{d-1}(\pi)$ be the closure of the injectivity region of $\exp_{d-1}$, i.e. $\exp(\mathbb{B}_{d-1}(\pi)) = \mathbb{S}^{d-1}$. Define for $v \in \mathbb{B}_{d-1}(\pi)$ the *defect volume process*

$$V_\lambda(v) := \int_{\exp([\mathbf{0},v])} r_\lambda(u) d\sigma_{d-1}(u).$$

Here the 'segment' $[\mathbf{0}, v]$ for $v \in \mathbb{R}^{d-1}$ is the rectangular solid in $\mathbb{R}^{d-1}$ with vertices $\mathbf{0}$ and v, that is to say $[\mathbf{0}, v] := \prod_{i=1}^{d-1}[\min(0, v^{(i)}), \max(0, v^{(i)})]$, with $v^{(i)}$ standing for the ith coordinate of v. Define $\hat{V}_\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ by

$$\hat{V}_\lambda(v) := (2\log\lambda)^{-(d+3)/4}(V_\lambda(v) - \mathbb{E}\, V_\lambda(v)),\ v \in \mathbb{R}^{d-1}.$$

For any $\sigma^2 > 0$ let B^{σ^2} be the Brownian sheet of variance coefficient σ^2 on the injectivity region $\mathbb{B}_{d-1}(\pi)$. Extend the index set of B^{σ^2} to all of $\mathbb{R}^{d-1}$ by putting $B^{\sigma^2}(v) = B^{\sigma^2}(w)$ whenever $[\mathbf{0}, v] \cap \mathbb{B}_{d-1}(\pi) = [\mathbf{0}, w] \cap \mathbb{B}_{d-1}(\pi)$. In other words B^{σ^2} is the mean zero continuous path Gaussian process indexed by $\mathbb{R}^{d-1}$ with

$$\mathrm{Cov}(B^{\sigma^2}(v), B^{\sigma^2}(w)) = \sigma^2 \cdot \sigma_{d-1}(\exp([\mathbf{0}, v] \cap [\mathbf{0}, w])).$$

Put $\sigma_V^2 := \sigma^2(\xi_V^{(\infty)})$. The following shows that $\hat{V}_\lambda, \lambda \geq 1$, converges to a Brownian sheet.

Theorem 2.3 *As $\lambda \to \infty$, the random functions $\hat{V}_\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ converge in law to $B^{\sigma_V^2}$ in $\mathcal{C}(\mathbb{R}^{d-1})$.*

We shall not prove this result, as it follows closely the proof of Theorem 8.1 of [12].

(ii) Intrinsic volumes. For $k \in \{1, \cdots, d-1\}$, we denote by $V_k(K_\lambda)$ the k-th intrinsic volume of K_λ. In [14], Hug and Reitzner show the expectation asymptotics for $V_k(K_\lambda)$ as well as an upper-bound for its variance. We assert that a result similar to Theorem 1.4 can be obtained for the asymptotic expectation and variance of $V_k(K_\lambda)$. In other words, we have the following theorem.

Theorem 2.4 *There exists a constant $v_k \in [0, \infty)$, defined in terms of averages of covariances of a scaling limit volume functional on $\mathcal{P}$, such that*

$$\lim_{\lambda \to \infty} (2 \log \lambda)^{-k+(d+3)/2} \mathrm{Var} V_k(K_\lambda) = v_k.$$

Moreover, when λ goes to ∞, we have

$$\frac{\kappa_{d-k}}{\binom{d}{k} \kappa_d} (2 \log \lambda)^{-k/2} \mathbb{E}\, V_k(K_\lambda) = 1 + O((\log \lambda)^{-1}).$$

It is a notable improvement from Theorem 1.2 in [14] which claims that $(\log \lambda)^{-\frac{k-3}{2}} \mathrm{Var} V_k(K_\lambda)$ is bounded. In [22], Reitzner says already that 'it seems that these upper bounds are not best possible'. Hopefully the bounds that we get here are optimal but unfortunately we were not able to show that the limits v_k, $1 \le k \le (d-1)$, are different from zero. In particular, $\mathrm{Var} V_k(K_\lambda)$ goes to infinity for $k > (d+3)/2$ as soon as $v_k \neq 0$. We postpone the proof of Theorem 2.4 to Section 5.

(iii) Binomial input. By coupling binomial and Poisson points as in [8], one may deduce the binomial analogs of Theorems 2.1 and 2.2 for the measures $\sum_{i=1}^n \xi_k(X_i, \mathcal{X}_n) \delta_{X_i}, k \in \{0, 1, ..., d-1\}$ and $\sum_{i=1}^n \xi_V(X_i, \mathcal{X}_n) \delta_{X_i}$, where we recall that X_i are i.i.d. with density ϕ, $\mathcal{X}_n := \{X_j\}_{j=1}^n$.

(iv) Random polytopes on general Poisson input. We expect that our main results extend to random polytopes generated by Poisson points having an isotropic intensity density. As shown by Carnal [13] and others, there are qualitative differences in the behavior of $\mathbb{E}\, f_k(K_n)$ according to whether the input $\mathcal{X}_n$ has algebraic tails modulated by a slowly varying function or whether the input has an exponentially decaying tail. The choice for the critical radius R_α and the scaling transform $T^{(\lambda)}$ would thus need to reflect such behavior. For example, if $|X_i|$ have an exponential intensity density on $\mathbb{R}^d$ and $R_\lambda = \log \lambda - \log \log \lambda$, then $T^{(\lambda)}(\mathcal{P}_\lambda) \xrightarrow{\mathcal{D}} \mathcal{H}_1$, where $\mathcal{H}_1$ is a rate one homogenous Poisson point process on $\mathbb{R}^d$.

3 Scaling transformations

For all $\lambda \in [\lambda_0, \infty)$, the scaling transform $T^{(\lambda)}$ defined at (1.3) maps $\mathbb{R}^d$ onto the rectangular solid $W_\lambda \subset \mathbb{R}^{d-1} \times \mathbb{R}$ given by

$$W_\lambda := (R_\lambda \cdot \mathbb{B}_{d-1}(\pi)) \times (-\infty, R_\lambda^2],$$

where we recall $\mathbb{B}_{d-1}(\pi)$ is the closure of the injectivity region of $\exp_{d-1}$. Let (v, h) be the coordinates in W_λ, that is

$$v = R_\lambda \exp_{d-1}^{-1} \frac{x}{|x|}, \quad h = R_\lambda^2 (1 - \frac{|x|}{R_\lambda}). \tag{3.1}$$

Note that $\mathbb{S}^{d-1}$ is geodesically complete in that the exponential map $\exp_{u_0}$ is well defined on the whole tangent space $\mathbb{R}^{d-1} \simeq T_{u_0}\mathbb{S}^{d-1}$, although it is injective only on $\{v \in T_{u_0}\mathbb{S}^{d-1},\ |v| < \pi\}$. In this and in the following section, our aim is to show:

(i) $T^{(\lambda)}$ defines a $1-1$ correspondence between boundaries of convex hulls of point sets $\mathcal{X} \subset \mathbb{R}^d$ and a subset of piecewise smooth functions on W_λ,

(ii) $T^{(\lambda)}$ carries $\mathcal{P}_\lambda$ into a point process on W_λ converging in distribution to $\mathcal{P}$ defined at (1.1), and

(iii) $T^{(\lambda)}$ defines re-scaled k-face and volume functionals on W_λ; when the input is $\mathcal{P}_\lambda$ then the means and covariances converge to the respective means and covariances of the corresponding functionals in $\Xi^{(\infty)}$.

3.1. The re-scaled boundary of the convex hull under $T^{(\lambda)}$. We examine the image under $T^{(\lambda)}$ of the boundary of $\mathrm{co}(\mathcal{X})$. For all $\lambda \in [\lambda_0, \infty)$ and $(v_1, h_1) \in W_\lambda$, consider the grain given by

$$[\Pi^{\uparrow}(v_1, h_1)]^{(\lambda)} := \{(v, h) \in W_\lambda,\ h \geq h_1 + R_\lambda^2(1 - \cos[e_\lambda(v, v_1)]) - h_1(1 - \cos[e_\lambda(v, v_1)])\}, \tag{3.2}$$

with

$$e_\lambda(v, v_1) := d_{\mathbb{S}^{d-1}}(\exp_{d-1}(R_\lambda^{-1} v), \exp_{d-1}(R_\lambda^{-1} v_1)), \tag{3.3}$$

where $d_{\mathbb{S}^{d-1}}$ stands for the geodesic distance in $\mathbb{S}^{d-1}$.

For $\lambda \in [\lambda_0, \infty)$, every locally finite $\mathcal{X} \subset W_\lambda$ generates the germ-grain model

$$\Psi^{(\lambda)}(\mathcal{X}) := \bigcup_{w \in \mathcal{X}} [\Pi^{\uparrow}(w)]^{(\lambda)}. \tag{3.4}$$

Considering the defect support function of $\mathrm{co}(\mathcal{X})$, it may be seen (see e.g. section 4 of [26], sections 2 and 4 of [12]) that $x_0 \in \mathcal{X}$ is a vertex of $\mathrm{co}(\mathcal{X})$ if and only if for each $\lambda \in [\lambda_0, \infty)$ it is the case that $[\Pi^{\uparrow}(T^{(\lambda)}(x_0))]^{(\lambda)}$ is not covered by the union $\Psi^{(\lambda)}(T^{(\lambda)}(\mathcal{X} \setminus x_0))$. In this case we call $T^{(\lambda)}(x_0)$ a vertex of $T^{(\lambda)}(\mathcal{X})$.

For $x_0 \in \mathbb{R}^d$ consider the half-space

$$H(x_0) := \{x \in \mathbb{R}^d :\ \langle x, \frac{x_0}{|x_0|}\rangle \geq |x_0|\}.$$

Letting $\theta := d_{S^{d-1}}(\frac{x}{|x|}, \frac{x_0}{|x_0|})$, we rewrite $H(x_0)$ as

$$H(x_0) := \{x \in \mathbb{R}^d :\ |x_0| \leq |x| \cos\theta\}.$$

Recalling the change of variable at (3.1), let $T^{\lambda}(x_0) := (v_0, h_0)$, so that $h_0 = R_\lambda^2(1 - |x_0|/R_\lambda)$. We may rewrite $H(x_0)$ as

$$H(x_0) := \{x \in \mathbb{R}^d : \ R_\lambda^2(1 - \frac{|x_0|}{R_\lambda \cos\theta}) \geq R_\lambda^2(1 - \frac{|x|}{R_\lambda})\}.$$

Thus $T^{(\lambda)}$ transforms $H(x_0)$ into

$$T^{(\lambda)}(H(x_0)) := [\Pi^{\downarrow}(v_0, h_0)]^{(\lambda)} := \{(v, h) \in W_\lambda, \ h \leq R_\lambda^2 - \frac{R_\lambda^2 - h_0}{\cos[e_\lambda(v, v_0)]}\}. \tag{3.5}$$

Noting that $\mathbb{R}^d \setminus \mathrm{co}(\mathcal{X})$ is the union of half-spaces not containing points in $\mathcal{X}$, it follows that $T^{(\lambda)}$ transforms $\mathbb{R}^d \setminus \mathrm{co}(\mathcal{X})$ into the subset of W_λ given by

$$\Phi^{(\lambda)}(T^{(\lambda)}(\mathcal{X})) := \bigcup_{\left\{\substack{w \in W_\lambda \\ [\Pi^{\downarrow}(w)]^{(\lambda)} \cap T^{(\lambda)}(\mathcal{X}) = \emptyset}\right.} [\Pi^{\downarrow}(w)]^{(\lambda)}.$$

Thus $T^{(\lambda)}$ sends the boundary of $\mathrm{co}(\mathcal{X})$ to the continuous function on W_λ whose graph coincides with the upper boundary of $\Phi^{(\lambda)}(T^{(\lambda)}(\mathcal{X}))$. There is thus a $1{-}1$ correspondence between convex hull boundaries and a subset of the continuous functions on $\mathbb{R}^{d-1} \times \mathbb{R}$. This contrasts with Eddy [15], who mapped *support functions* of convex hulls into a subset of the continuous functions on $\mathbb{R}^{d-1} \times \mathbb{R}$.

The random sets $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ and $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ link the geometry of the convex hull K_λ with that of the limit paraboloid germ-grain models $\Psi(\mathcal{P})$ and $\Phi(\mathcal{P})$. Theorem 1.2 and the upcoming Proposition 5.1 show that the boundaries $\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ and $\partial\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ respectively converge in probability to $\partial(\Psi(\mathcal{P}))$ and to $\partial(\Phi(\mathcal{P}))$ as $\lambda \to \infty$.

The next lemma is suggestive of this convergence and shows for fixed $w \in W_\lambda$ that $[\Pi^{\uparrow}(w)]^{(\lambda)}$ and $[\Pi^{\downarrow}(w)]^{(\lambda)}$ locally approximate the paraboloids $[\Pi^{\uparrow}(w)]^{(\infty)} := \Pi^{\uparrow}(w)$ and $[\Pi^{\downarrow}(w)]^{(\infty)} := \Pi^{\downarrow}(w)$, respectively. We may henceforth refer to $[\Pi^{\uparrow}(w)]^{(\lambda)}$ and $[\Pi^{\downarrow}(w)]^{(\lambda)}$ as *quasi-paraboloids*. Recalling that $B_{d-1}(v, r)$ is the $(d-1)$ dimensional ball centered at $v \in \mathbb{R}^{d-1}$ with radius r, define the cylinder $C(v, r) \subset \mathbb{R}^{d-1} \times \mathbb{R}$ by

$$C(v, r) := C_{d-1}(v, r) := B_{d-1}(v, r) \times \mathbb{R}. \tag{3.6}$$

Lemma 3.1 *For all $w := (v, h) \in W_\lambda$, $L \in (0, \infty)$, and all $\lambda \in [\lambda_0, \infty)$, we have*

$$||\partial([\Pi^{\uparrow}(w)]^{(\lambda)} \cap C(v, L)) - \partial([\Pi^{\uparrow}(w)]^{(\infty)} \cap C(v, L))||_\infty \leq cL^3 R_\lambda^{-1} + chL^2 R_\lambda^{-2} \tag{3.7}$$

and

$$||\partial([\Pi^{\downarrow}(w)]^{(\lambda)} \cap C(v, L)) - \partial([\Pi^{\downarrow}(w)]^{(\infty)} \cap C(v, L))||_\infty \leq cL^3 R_\lambda^{-1} + chL^2 R_\lambda^{-2}. \tag{3.8}$$

Proof. We first prove (3.7). For $w_1 := (v_1, h_1) \in W_\lambda$ we recall from (3.2) that

$$\partial([\Pi^\uparrow(w_1)]^{(\lambda)}) := \{(v,h) \in W_\lambda,\ h = h_1 + R_\lambda^2(1-\cos[e_\lambda(v,v_1)]) - h_1(1-\cos[e_\lambda(v,v_1)])\}. \tag{3.9}$$

For $v \in B_{d-1}(v_1, L)$, notice that

$$e_\lambda(v,v_1) = |R_\lambda^{-1}v - R_\lambda^{-1}v_1| + O(|R_\lambda^{-1}v - R_\lambda^{-1}v_1|^2) \tag{3.10}$$

and thus

$$1-\cos(e_\lambda(v,v_1)) = \frac{|R_\lambda^{-1}v - R_\lambda^{-1}v_1|^2}{2} + O(L^3R_\lambda^{-3}).$$

It follows that

$$R_\lambda^2(1-\cos(e_\lambda(v,v_1))) = \frac{(v-v_1)^2}{2} + O(L^3R_\lambda^{-1})$$

and

$$|h_1(1-\cos(e_\lambda(v,v_1)))| = O(h_1L^2R_\lambda^{-2}).$$

Thus the boundary of $[\Pi^\uparrow(w_1)]^{(\lambda)} \cap C(v_1, L)$ is within $cL^3R_\lambda^{-1} + ch_1L^2R_\lambda^{-2}$ of the graph of

$$v \mapsto h_1 + \frac{|v-v_1|^2}{2},$$

which establishes (3.7). The proof of (3.8) is similar, and goes as follows. For $w_1 := (v_1, h_1) \in W_\lambda$ we get from (3.5) that

$$\partial([\Pi^\downarrow(w_1)]^{(\lambda)}) := \{(v,h) \in W_\lambda,\ h = R_\lambda^2 - \frac{R_\lambda^2 - h_1}{\cos[e_\lambda(v,v_1)]}\}. \tag{3.11}$$

Using (3.10), Taylor expanding $\cos\theta$ up to second order, and writing $1/(1-r) = 1 + r + r^2 + ...$ gives

$$\partial([\Pi^\downarrow(w_1)]^{(\lambda)}) := \{(v,h) \in W_\lambda,\ h = h_1 - \frac{|v-v_1|^2}{2} + O(R_\lambda|v-v_1|^3) + O(h_1R_\lambda^{-2}|v-v_1|^2)\}, \tag{3.12}$$

and (3.8) follows. □

3.2. The weak limit of $T^{(\lambda)}(\mathcal{P}_\lambda)$. Put

$$\mathcal{P}^{(\lambda)} := T^{(\lambda)}(\mathcal{P}_\lambda).$$

Part (a) of the next result is the analog of Lemma 3.1 of [11] and the discussion around (2.14) of [12]. Let Vol_d denote d-dimensional volume measure on $\mathbb{R}^d$ and recall the definition of $\mathcal{P}$ at (1.1).

Lemma 3.2 *As $\lambda \to \infty$, we have*
(a) $\mathcal{P}^{(\lambda)} \xrightarrow{\mathcal{D}} \mathcal{P}$, and
(b) $T^{(\lambda)}(R_\lambda \cdot \mathrm{Vol}_d) \xrightarrow{\mathcal{D}} \mathrm{Vol}_d$.
The convergence is in the sense of total variation convergence on compact sets.

Remarks. (i) It is likewise the case that the image of the binomial point process $\sum_{x \in \mathcal{X}_n} \delta_x$ under $T^{(n)}$ converges in distribution to $\mathcal{P}$ as $n \to \infty$.
(ii) The transformation $T^{(\lambda)}$ carries $\mathcal{P}_\lambda$ into a point process on $\mathbb{R}^{d-1} \times \mathbb{R}$ which in the large λ limit is stationary in the spatial coordinate. This contrast with the transformation of Eddy [15] (and generalized in Eddy and Gale [16]) which carries $\sum_{x \in \mathcal{X}_n} \delta_x$ into a point process (T_k, Z_k) on $\mathbb{R} \times \mathbb{R}^{d-1}$ where $T_k, k \geq 1$, are points of a Poisson point process on $\mathbb{R}$ with intensity $e^{-h} dh$ and $Z_k, k \geq 1$, are i.i.d. standard Gaussian on $\mathbb{R}^{d-1}$.

Proof. Representing $x \in \mathbb{R}^d$ by $x = ur, u \in \mathbb{S}^{d-1}, r \in [0, \infty)$, we find the image by $T^{(\lambda)}$ of the Poisson measure on $\mathbb{R}^d$ with intensity

$$\lambda \phi(x) dx = \lambda \phi(ur) r^{d-1} dr d\sigma_{d-1}(u). \tag{3.13}$$

Make the change of variables

$$v := R_\lambda \exp_{d-1}^{-1}(u) = R_\lambda v_u, \quad h := R_\lambda^2 (1 - \frac{r}{R_\lambda}),$$

The exponential map $\exp_{d-1} : T_{u_0} \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ has the following expression:

$$\exp_{d-1}(v) = \cos(|v|)(0, \cdots, 0, 1) + \sin(|v|)(\frac{v}{|v|}, 0), \quad v \in \mathbb{R}^{d-1} \setminus \{\mathbf{0}\}. \tag{3.14}$$

Therefore, since $v_u := \exp_{d-1}^{-1}(u)$ we have

$$d\sigma_{d-1}(u) = \sin^{d-2}(|v_u|) d(|v_u|) d\sigma_{d-2}(\frac{v_u}{|v_u|}) = \frac{\sin^{d-2}(|v_u|) dv_u}{|v_u|^{d-2}}.$$

Since $v_u = R_\lambda^{-1} v$, this gives

$$d\sigma_{d-1}(u) = \frac{\sin^{d-2}(R_\lambda^{-1} |v_u|)}{|R_\lambda^{-1} v|^{d-2}} (R_\lambda^{-1})^{d-1} dv. \tag{3.15}$$

We also have

$$r^{d-1} dr = |R_\lambda (1 - \frac{h}{R_\lambda^2})|^{d-1} R_\lambda^{-1} dh \tag{3.16}$$

as well as

$$\lambda \phi(x) = \lambda \phi(u R_\lambda (1 - \frac{h}{R_\lambda^2})) = \frac{\sqrt{2 \cdot (2\pi)^d \cdot \log \lambda}}{(2\pi)^{d/2}} \exp\left(h - \frac{h^2}{2R_\lambda^2}\right). \tag{3.17}$$

Combining (3.13) and (3.15)-(3.17), we get that $\mathcal{P}^{(\lambda)}$ has intensity density

$$\frac{d\mathcal{P}^{(\lambda)}}{dvdh}((v,h)) = \frac{\sqrt{2\log\lambda}}{R_\lambda}\frac{\sin^{d-2}(R_\lambda^{-1}|v|)}{|R_\lambda^{-1}v|^{d-2}}|(1-\frac{h}{R_\lambda^2})|^{d-1}e^{h-\frac{h^2}{2R_\lambda^2}},\ (v,h)\in W_\lambda. \tag{3.18}$$

Given a fixed compact subset D of W_λ, this intensity converges in the $L^1(D)$ sense to the intensity of $\mathcal{P}$, completing the proof of part (a).

Part (b) follows by replacing the intensity $\lambda\phi dx$ with dx in the above computations, which gives the intensity density

$$\frac{d\mathrm{Vol}_d}{dvdh}((v,h)) = \frac{\sin^{d-2}(R_\lambda^{-1}|v|)}{|R_\lambda^{-1}v|^{d-2}}|(1-\frac{h}{R_\lambda^2})|^{d-1}R_\lambda^{-1},\ (v,h)\in W_\lambda. \tag{3.19}$$

The product of this intensity density with R_λ converges pointwise to 1 as $\lambda\to\infty$, showing part (b). $\square$

3.3. Re-scaled k-face and volume functionals $\xi^{(\lambda)}$. Fix $\lambda\in[\lambda_0,\infty)$. *Let* $\xi_k, k\in\{0,1,...,d-1\}$, *be a generic k-face functional,* as in Definition 2.3. Given $\mathcal{X}\subset W_\lambda$, we say that $w\in\mathcal{X}$ is extreme with respect to $[T^{(\lambda)}]^{-1}$ if $[T^{(\lambda)}]^{-1}(x)$ is extreme in $[T^{(\lambda)}]^{-1}(\mathcal{X})$. Write $\mathrm{Ext}^{(\lambda)}(\mathcal{X})$ for the set of points in $\mathcal{X}$ which are extreme with respect to $[T^{(\lambda)}]^{-1}$. The inverse transformation $[T^{(\lambda)}]^{-1}$ defines generic re-scaled k-face functionals $\xi^{(\lambda)}$ defined for $w\in W_\lambda$ and $\mathcal{X}\subset W_\lambda$ by

$$\xi^{(\lambda)}(w,\mathcal{X}) := \xi_k^{(\lambda)}(w,\mathcal{X}) := \xi_k([T^{(\lambda)}]^{-1}(w),[T^{(\lambda)}]^{-1}(\mathcal{X})). \tag{3.20}$$

It follows for all $k\in\{0,1,...,d-1\}$, $\lambda\in[\lambda_0,\infty)$, that $\xi_k(x,\mathcal{P}_\lambda) := \xi_k^{(\lambda)}(T^{(\lambda)}(x),\mathcal{P}^{(\lambda)})$. Note that for all $\lambda\in[\lambda_0,\infty)$, $k\in\{0,1,...,d-1\}$, $w\in W_\lambda$, and $\mathcal{X}\subset W_\lambda$, that $\xi_k^{(\lambda)}(w,\mathcal{X})$ is the product of $(k+1)^{-1}$ and the number of quasi-parabolic k-dimensional faces of $\Phi^{(\lambda)}(\mathcal{X}) := \partial(\bigcup_{x\in\mathcal{X}}[\Pi^{\downarrow}(x)]^{(\lambda)})$ which contain w, $w\in\mathrm{Ext}^{(\lambda)}(\mathcal{X})$, otherwise $\xi_k^{(\lambda)}(w,\mathcal{X})=0$.

Define the re-scaled volume functional $\xi_V^{(\lambda)}$ similarly, that is to say for $w\in W_\lambda$ and $\mathcal{X}\subset W_\lambda$ we have

$$\xi_V^{(\lambda)}(w,\mathcal{X}) := \xi_V([T^{(\lambda)}]^{-1}(w),[T^{(\lambda)}]^{-1}(\mathcal{X})). \tag{3.21}$$

Let $\mathrm{Vol}_d^{(\lambda)}$ be the image of $\mathrm{Vol_d}$ under $T^{(\lambda)}$. For $w\in\mathrm{Ext}^{(\lambda)}(\mathcal{X})$ we have

$$\begin{aligned}\xi_V^{(\lambda)}(w,\mathcal{X}) = \mathrm{Vol}_d^{(\lambda)}(\{(v,h):0\le h\le\partial\Phi^{(\lambda)}(\mathcal{X})(v), v\in\mathrm{Cyl}^{(\lambda)}(\mathrm{w}), \Phi^{(\lambda)}(\mathcal{X})(\mathrm{v})\ge 0\})-\\ -\mathrm{Vol}_d^{(\lambda)}(\{(v,h):\Phi^{(\lambda)}(\mathcal{X})(v)\le h\le 0, v\in\mathrm{Cyl}^{(\lambda)}(\mathrm{w}), \Phi^{(\lambda)}(\mathcal{X})(\mathrm{v})<0\}).\end{aligned} \tag{3.22}$$

where $\mathrm{Cyl}^{(\lambda)}(w)$ denotes the projection onto $\mathbb{R}^{d-1}$ of the quasi-parabolic faces of $\Phi^{(\lambda)}(\mathcal{X})$ containing w. When $w\notin\mathrm{Ext}^{(\lambda)}(\mathcal{X})$ we put $\xi_V^{(\lambda)}(w,\mathcal{X})=0$.

Let $\Xi^{(\lambda)}$ denote the collection of re-scaled functionals $\xi_k^{(\lambda)}, k \in \{0, 1, ..., d-1\}$, together with $\xi_V^{(\lambda)}$. Write Ξ for $\Xi^{(1)}$. Our main goal in the next section is to show that, given a generic $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \in [\lambda_0, \infty)$, the means and covariances of $\xi^{(\lambda)}(\cdot, \mathcal{P}^{(\lambda)})$ converge as $\lambda \to \infty$ to the respective means and covariances of $\xi^{(\infty)}(\cdot, \mathcal{P})$, with $\xi^{(\infty)} \in \Xi^{(\infty)}$.

4 Properties of re-scaled k-face and volume functionals

To establish convergence of the re-scaled functionals $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \in [\lambda_0, \infty)$, to their respective counterparts $\xi^{(\infty)} \in \Xi^{(\infty)}$, we first need to show that $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \in [\lambda_0, \infty]$ satisfy a localization in the spatial and time coordinates v and h, respectively. These localization results are the analogs of Lemmas 7.2 and 7.3 of [12] and they hold for all k-face functionals $\xi_k, k \in \{0, 1, ..., d-1\}$ as well as the volume functional ξ_V. *In the following the point process $\mathcal{P}^{(\lambda)}$, $\lambda = \infty$, is taken to be $\mathcal{P}$ and the set W_λ, $\lambda = \infty$, is taken to be $\mathbb{R}^d$.*

4.1. Localization of $\xi^{(\lambda)}$. We establish localization properties of the functionals in $\Xi^{(\lambda)}, \lambda \in [\lambda_0, \infty]$, in both the space and time domains. Recall the definition of the cylinder $C(v, r) := C_{d-1}(v, r) := B_{d-1}(v, r) \times \mathbb{R}$ at (3.6). Given a generic functional $\xi^{(\lambda)}, \lambda \in [\lambda_0, \infty]$, and $w := (v, h) \in W_\lambda$, we shall write

$$\xi_{[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)} \cap C_{d-1}(v, r)). \tag{4.1}$$

Given $\xi^{(\lambda)}, \lambda \in [\lambda_0, \infty]$, recall from [12, 26] that a random variable $R := R^{\xi^{(\lambda)}}[w] := R^{\xi^{(\lambda)}}[w, \mathcal{P}^{(\lambda)}] \in \mathbb{N}$ is a *spatial localization radius* for $\xi^{(\lambda)}$ at w with respect to $\mathcal{P}^{(\lambda)}$ iff a.s.

$$\xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \xi_{[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \text{ for all } r \geq R. \tag{4.2}$$

There are in general more than one R satisfying (4.2) and we shall henceforth assume R is the smallest integer satisfying (4.2). As seen in [20], R is measurable. The functionals $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \in [\lambda_0, \infty]$, admit spatial localization radii with tails decaying super-exponentially fast.

Lemma 4.1 *For each $\xi \in \Xi$, there are constants $c \in (0, \infty)$ and $\lambda_1 \in [\lambda_0, \infty)$ such that the localization radius $R^{\xi^{(\lambda)}}[w]$ for all $\lambda \in [\lambda_0, \infty]$ satisfies for every $w := (v, h) \in W_\lambda$, and $t \geq (-h \vee 4)$*

$$P[R^{\xi^{(\lambda)}}[w] > t] \leq c \exp(-\frac{t^2}{c}). \tag{4.3}$$

Proof. We first prove (4.3) for $\lambda = \infty$. Since $\mathcal{P} := \mathcal{P}^{(\infty)}$ is stationary with respect to the spatial coordinate, it suffices to consider $R^{\xi^{(\infty)}}[w_0]$, with $w_0 := (\mathbf{0}, h)$. For $t \geq (-h \vee 4)$ we have

$$\{R^{\xi^{(\infty)}}[w_0] > t\} \subset E_1 \cup E_2,$$

where

$$E_1 := \left\{R^{\xi^{(\infty)}}[w_0] > t,\ w_0 \notin \mathrm{Ext}(\mathcal{P})\right\} \text{ and } E_2 := \left\{R^{\xi^{(\infty)}}[w_0] > t,\ w_0 \in \mathrm{Ext}(\mathcal{P})\right\}.$$

We may rewrite E_1 as

$$E_1 = \{w_0 \notin \mathrm{Ext}(\mathcal{P}),\ w_0 \in \mathrm{Ext}(\mathcal{P} \cap C(\mathbf{0}, t))\}.$$

If E_1 occurs then there is

$$w_1 := (v_1, h_1) \in \partial \left(\Pi^{\uparrow}(w_0)\right) \cap C(\mathbf{0}, t)$$

which belongs to some $\Pi^{\uparrow}(y),\ y \in \mathcal{P} \cap C^c(\mathbf{0}, t)$, but $w_1 \notin \bigcup_{w \in \mathcal{P} \cap C(\mathbf{0},t)} \Pi^{\uparrow}(w)$. In other words, w_1 is covered by paraboloids with apices in $\mathcal{P}$, but not by paraboloids with apices restricted to $\mathcal{P} \cap C(\mathbf{0}, t)$. This means that the down paraboloid $\Pi^{\downarrow}(w_1)$ must contain a point of $\mathcal{P}$ in $C(\mathbf{0}, t)^c$. In other words, if

$$\begin{aligned} F_1 :=& \{\exists w_1 := (v_1, h_1) \in \partial(\Pi^{\uparrow}(w_0)) \cap C(\mathbf{0}, t) :\ w_0 \in \partial \Pi^{\downarrow}(w_1),\ h_1 \in (-\infty, t), \\ & \Pi^{\downarrow}(w_1) \cap C(\mathbf{0}, t) \cap \mathcal{P} = \emptyset\ , \Pi^{\downarrow}(w_1) \cap C(\mathbf{0}, t)^c \cap \mathcal{P} \neq \emptyset\} \end{aligned}$$

and if

$$\begin{aligned} F_2 :=& \{\exists w_1 := (v_1, h_1) \in \partial(\Pi^{\uparrow}(w_0)) \cap C(\mathbf{0}, t) :\ w_0 \in \partial \Pi^{\downarrow}(w_1),\ h_1 \in [t, \infty), \\ & \Pi^{\downarrow}(w) \cap C(\mathbf{0}, t) \cap \mathcal{P} = \emptyset\}, \end{aligned}$$

then we have $E_1 \subset F_1 \cup F_2$.

If E_2 happens then there is $w_1 := (v_1, h_1) \in C(\mathbf{0}, t)^c \cap \Pi^{\uparrow}(w_0)$ which is not covered by paraboloids with apices in $\mathcal{P}$ and w_0 belongs to a hyperface of $\Pi^{\downarrow}(w_1)$. Also, h_1 satisfies $h_1 \geq t^2/2 + h \geq t$ whenever $t \geq (-h \vee 4)$. Thus we have

$$E_2 \subset \tilde{E}_2 := \left\{w_0 \in \mathrm{Ext}(\mathcal{P}) \text{ belongs to a hyperface of } \Pi^{\downarrow}(w_1),\, h_1 \in [t, \infty)\right\}. \tag{4.4}$$

On $\tilde{E}_2$, $\Pi^{\downarrow}(w_1)$ does not contain points in $\mathcal{P}$ and therefore it doesn't contain points of $\mathcal{P} \cap C(\mathbf{0}, t)$. Thus $\tilde{E}_2 \subset F_2$ and we have

$$E_1 \cup E_2 \subset E_1 \cup \tilde{E}_2 \subset F_1 \cup F_2.$$

We bound separately the probability of events F_1 and F_2.

Upper-bound for $P[F_1]$. Let us consider a fixed $w_1 \in \partial\Pi^\uparrow(w_0)$ with $h_1 = h + \frac{1}{2}v_1^2 \leq t$. The probability that $\Pi^\downarrow(w_1) \cap C(\mathbf{0},t)^c \cap \mathcal{P} \neq \emptyset$ is bounded by the $d\mathcal{P}$ measure of $\Pi^\downarrow(w_1) \cap C(\mathbf{0},t)^c$. A small calculation shows that the maximal height of a point in $\Pi^\downarrow(w_1) \cap C(\mathbf{0},t)^c$ is $h_1 - \frac{1}{2}(t - \sqrt{2(h_1-h)})^2$. Consequently, the $d\mathcal{P}$ measure of $\Pi^\downarrow(w_1) \cap C(\mathbf{0},t)^c$ is bounded by the $d\mathcal{P}$ measure of $\Pi^\downarrow(w_1) \cap \{(v,h') : h' \leq h_1 - \frac{1}{2}(t - \sqrt{2(h_1-h)})^2\}$. Recall that c is a constant which changes from line to line. Up to a multiplicative constant, the $d\mathcal{P}$ measure of $\Pi^\downarrow(w_1) \cap C(\mathbf{0},t)^c$ is bounded by

$$\begin{aligned}\int_{-\infty}^{h_1-\frac{1}{2}(t-\sqrt{2(h_1-h)})^2} e^{h'}(2(h_1-h'))^{\frac{d-1}{2}}dh' &= e^{h_1}\int_{\frac{1}{2}(t-\sqrt{2(h_1-h)})^2}^{+\infty} e^{-u}(2u)^{\frac{d-1}{2}}du \\ &\leq c\exp\left(h_1 - \frac{1}{c}(\frac{t^2}{2} + h_1 - h - t\sqrt{2(h_1-h)})\right),\end{aligned}$$

where the equality follows with $u := h_1 - h'$.

Consequently, discretizing $\partial\Pi^\uparrow(w_0) \cap (\mathbb{R}^{d-1} \times (-\infty, t])$, we get

$$\begin{aligned}P[F_1] &\leq ce^{-t^2/(2c)}\int_h^t (2(h_1-h))^{\frac{d-2}{2}} e^{(1-1/c)h_1 + h/c + t\sqrt{h_1-h}/c}dh_1 \\ &\leq ce^{-t^2/(2c)}\int_0^{t-h}(2h_1)^{\frac{d-2}{2}}e^{(1-1/c)h_1+h+t\sqrt{h_1}/c}dh_1 \\ &\leq ce^{-(1/c)(t^2-t^{3/2}-t)} \\ &\leq ce^{-t^2/c}.\end{aligned}$$

When λ is fixed, we proceed as follows. Let $w_0 := (v_0, h_0)$. Recall from (3.9) that

$$[\Pi^\uparrow(v_0,h_0)]^{(\lambda)} := \{(v,h) \in W_\lambda,\ h \geq h_0 + R_\lambda^2(1-\cos[e_\lambda(v,v_0)]) - h_0(1-\cos[e_\lambda(v,v_0)])\}.$$

We claim that for λ large and independent of t, that $[\Pi^\uparrow(v_0,h_0)]^{(\lambda)} \cap (\mathbb{R}^{d-1} \times (-\infty,t])$ has a spatial diameter (in the v coordinates) bounded by $c_1\sqrt{t}$. We see this as follows. Let $(v,h) \in [\Pi^\uparrow((v_0,h_0))]^{(\lambda)} \cap (-\infty,t]$. When $h \leq t$ and $|h_0| \leq t$ we get $R_\lambda^2(1-\cos[e_\lambda(v,v_0)]) \leq 3t$. Thus $1-\cos[e_\lambda(v,v_0)] \leq 3tR_\lambda^{-2}$. We may without loss of generality assume $t \leq R_\lambda$, since the stabilization radius never exceeds the spatial diameter of W_λ. It follows that $1-\cos[e_\lambda(v,v_0)]$ is small when λ exceeds some λ_1 (independent of t), that is to say there is c such that

$$ce_\lambda(v,v_0)^2 \leq 1-\cos[e_\lambda(v,v_0)] \leq 3tR_\lambda^{-2}$$

which gives $|v-v_0| \leq c_1\sqrt{t}$, since $e_\lambda(v,v_0)^2 \geq (v-v_0)^2R_\lambda^{-2}$ (geodesic distance exceeds Euclidean distance). Let

$$w_1 := (v_1,h_1) \in \partial[(\Pi^\uparrow(w_0)]^{(\lambda)} \cap C(v_0,t).$$

We now estimate the maximal height of a point on $[(\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap C(v_0,t)^c$. If (v,h) belongs to the boundary of $[(\Pi^{\downarrow}(w_1)]^{(\lambda)}$ then we have from (3.11) that

$$h = R_\lambda^2 - \frac{R_\lambda^2 - h_1}{\cos[e_\lambda(v,v_1)]}$$

which gives

$$\cos[e_\lambda(v,v_1)] = 1 - \frac{h_1 - h}{R_\lambda^2 - h}.$$

Now $\cos x \le 1 - c_2 x^2$ for $x \in [0,\pi]$, giving

$$1 - \frac{h_1 - h}{R_\lambda^2 - h} \le 1 - c_2 e_\lambda(v,v_1)^2.$$

This gives

$$\frac{c_2(v_1 - v)^2}{R_\lambda^2} \le c_2 e_\lambda(v,v_1)^2 \le \frac{h_1 - h}{R_\lambda^2 - h} \le \frac{h_1 - h}{R_\lambda^2 - t} \le \frac{h_1 - h}{R_\lambda^2 - R_\lambda}$$

where we use that without loss of generality $t \le R_\lambda$. In other words, since $R_\lambda^2 - R_\lambda$ exceeds $R_\lambda^2/2$ we have

$$h \le h_1 - c_3(v - v_1)^2. \tag{4.5}$$

The maximal height of a point on $[(\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap C(v_0,t)^c$ is found by setting v equal to $v_0 + t$ in the above, giving that maximal height is at most

$$\le h_1 - c_3(v_0 + t - v_1)^2.$$

Now $|v_1 - v_0| \le c_1\sqrt{t}$ $(v_1 \le v_0 + c_1\sqrt{t})$ which shows that the maximal height is at most

$$\le h_1 - c_3(t - c_1\sqrt{t})^2 \le t - c_3(t - c_1\sqrt{t})^2.$$

Now we follow the proof for the case $\lambda = \infty$. The $d\mathcal{P}^{(\lambda)}$ measure of $[(\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap C(v_0,t)^c$ is bounded by the $d\mathcal{P}^{(\lambda)}$ measure of

$$[(\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap \{(v,h) : \ h \le t - c_3(t - c_1\sqrt{t})^2\}.$$

The intensity measure of $d\mathcal{P}^{(\lambda)}$ is upper bounded by

$$\le C|(1 - \frac{h}{R_\lambda^2})|^{d-1} e^{h/2}, \quad (v,h) \in W_\lambda.$$

The cross section of $[(\Pi^{\downarrow}(w_1)]^{(\lambda)}$ at height $h \in (-\infty, h_1]$ has radius which may be upper bounded by solving for $|v - v_1|$ in (4.5). This gives

$$|v - v_1| \le c_3^{-1}\sqrt{h_1 - h}.$$

The $d\mathcal{P}^{(\lambda)}$ measure of $[(\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap C(v_0,t)^c$ is thus bounded by

$$C\int_{-\infty}^{t-c_3(t-c_1\sqrt{t})^2} |(1-\frac{h}{R_\lambda^2})|^{d-1}e^{h/2}(h_1-h)^{(d-1)/2}dh.$$

Let $c_4 = c_3/2$. For t large the upper limit of integration is at most $-c_4t^2$, independent of the choice of h. There is a small positive constant c_5 such that $|(1-\frac{h}{R_\lambda^2})|^{d-1}e^{h/2} \leq e^{c_5h}$ holds for all $h \in (-\infty, 0]$. Also,

$$(h_1-h)^{(d-1)/2} \leq C(t^{(d-1)/2} + |h|^{(d-1)/2}).$$

Putting these estimates together shows that the $d\mathcal{P}^{(\lambda)}$ measure of $[(\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap C(v_0,t)^c$ is bounded by

$$C\int_{-\infty}^{-c_4t^2} e^{c_5h}(t^{(d-1)/2} + |h|^{(d-1)/2})dh.$$

This last integral is bounded by $\leq c_6\exp(-t^2/c_6)$.

Consequently, discretizing $\partial\Pi^{\uparrow}(w_0)^{(\lambda)} \cap (\mathbb{R}^{d-1}\times(-\infty,t])$, we get

$$\begin{aligned} P[F_1] &\leq c_6e^{-t^2/c_6}\int_h^t (c_1\sqrt{h_1-h})^{d-2}dh_1 \\ &\leq c_7e^{-t^2/c_7} \end{aligned}$$

Upper-bound for $P[F_2]$. Suppose $h_1 \in [t,\infty)$. Since $w_1 \notin \bigcup_{w\in\mathcal{P}\cap C(\mathbf{0},t)}\Pi^{\uparrow}(w)$, it follows that the downward paraboloid $\Pi^{\downarrow}(w_1)$ does not contain points in $\mathcal{P}\cap C(\mathbf{0},t)$. Now the $d\mathcal{P}$ measure of $\Pi^{\downarrow}(w_1)\cap C(\mathbf{0},t)$ is bounded below by its $d\mathcal{P}$ measure 'above' $\mathbb{R}^{d-1}$, which is at least as large as $ch_1^de^{h_1/2}$ which we generously bound below by $e^{h_1/2}$. Thus the probability that $\Pi^{\downarrow}(w_1)$ does not contain points in $\mathcal{P}\cap C(\mathbf{0},t)$ is bounded by $\exp(-e^{h_1/2})$.

Discretizing $(\mathbb{R}^{d-1}\times[t,\infty))\cap C(\mathbf{0},t)$ into unit cubes, we see that the probability that there is $w_1 := (v_1,h_1) \in \partial\left(\Pi^{\uparrow}(w_0)\right)\cap C(\mathbf{0},t)$ such that $\Pi^{\downarrow}(w_1)$ does not contain points in $\mathcal{P}\cap C(\mathbf{0},t)$ is bounded by

$$c\int_t^\infty t^{d-1}\exp(-e^{h_1/2})dh_1 \leq ct^{d-1}\exp(\frac{-e^{t/2}}{c}).$$

Thus there is a constant c such that $P[F_2] \leq c\exp(-t^2/c)$ for $t \geq (-h\vee 4)$.

When λ is fixed, we consider $w_1 = (v_1,h_1) \in \partial\left[\Pi^{\uparrow}(w_0)\right]^{(\lambda)}\cap C(\mathbf{0},t)$ such that $h_1 \geq t$ and $\left[\Pi^{\downarrow}(w_1)\right]^{(\lambda)}\cap C(\mathbf{0},t)\cap\mathcal{P} = \emptyset$. We use the same notations $x_0 = [T^{(\lambda)}]^{-1}((\mathbf{0},h))$ and $x_1 = [T^{(\lambda)}]^{-1}(w_1)$. We recall that x_1 is on the sphere of diameter $[0,x_0]$ and that the inverse image of $\left[\Pi^{\downarrow}(w_1)\right]^{(\lambda)}$ by $[T^{(\lambda)}]^{-1}$ is the half-space containing x_1, orthogonal to x_1 and not containing the origin. We denote by $C'_{x_1,t,\lambda}$ the intersection of that half-space with the cone $\{x \in \mathbb{R}^d : d_{\mathbb{S}^{d-1}}(x/|x|, u_0) \le t/R_\lambda\}$. Let us find a lower bound for the $d\mathcal{P}_\lambda$ measure of $C'_{x_1,t,\lambda}$: we recall that a point x is on the hyperplane containing x_1 and orthogonal to x_1 if $|x| = |x_1|/\cos(\theta)$ where $\theta = d_{\mathbb{S}^{d-1}}(\frac{x_1}{|x_1|}, \frac{x}{|x|})$. Consequently, we verify that there exists $c > 0$ such that when $\theta \le c\sqrt{h_1}R_\lambda$, we have $|x| \le R_\lambda(1 - \frac{h_1}{2R_\lambda^2})$. In particular, $C'_{x_1,t,\lambda}$ contains the set

$$\{x \in \mathbb{R}^d : |x| \ge R_\lambda(1 - \frac{h_1}{2R_\lambda^2})\} \cap \{x \in \mathbb{R}^d : d_{\mathbb{S}^{d-1}}(x/|x|, u_0) \le t/R_\lambda\}$$
$$\cap \{x \in \mathbb{R}^d : d_{\mathbb{S}^{d-1}}(x/|x|, x_1/|x_1|) \le c\sqrt{h_1}/R_\lambda\}.$$

The area measure of the intersection of the two cones $\{x \in \mathbb{R}^d : d_{\mathbb{S}^{d-1}}(x/|x|, u_0) \le t/R_\lambda\} \cap \{x \in \mathbb{R}^d : d_{\mathbb{S}^{d-1}}(x/|x|, x_1/|x_1|) \le c\sqrt{h_1}/R_\lambda\}$ with the unit-sphere is at least $c(\sqrt{h} \wedge t/R_\lambda)^{d-1}$, i.e. at least $cR_\lambda^{-(d-1)}$ if $t \ge 1$. Consequently, the $d\mathcal{P}_\lambda$ measure of $C'_{x_1,t,\lambda}$ satisfies

$$\begin{aligned}
d\mathcal{P}_\lambda(C'_{x_1,t,\lambda}) &\ge c\lambda R_\lambda^{-(d-1)} \int_{\rho=R_\lambda(1-\frac{h_1}{2R_\lambda^2})}^{\infty} e^{-\rho/2}\rho^{d-1}d\rho \\
&\ge c\lambda R_\lambda^{-(d-1)} R_\lambda^{d-2}(1 - \frac{h_1}{2R_\lambda^2})^{d-2} \exp\left(-R_\lambda^2(1 - \frac{h_1}{2R_\lambda^2})^2/2\right) \\
&\ge c\lambda R_\lambda^{-1} e^{-R_\lambda^2/2} e^{h_1/2} e^{-h_1^2/(8R_\lambda)^2} \\
&\ge cR_\lambda^{-1}\sqrt{\log(\lambda)} e^{h_1/2} \\
&\ge ce^{h_1/2} \ge ce^{t/2}.
\end{aligned}$$

Notice that we use that $h_1 \le R_\lambda^2$ to get $(1 - \frac{h_1}{2R_\lambda}) \ge 1/2$.

In particular, the probability that $C'_{x_1,t,\lambda}$ is empty is upper-bounded by $e^{-ce^{t/2}}$. We discretize the set which is the intersection of the sphere of diameter $[\mathbf{0}, x_0]$ with the cone $\{x \in \mathbb{R}^d : d_{\mathbb{S}^{d-1}}(x/|x|, u_0) \le t/R_\lambda\}$ and the ball centered at the origin and of radius $R_\lambda(1 - t/R_\lambda^2)$. That set has an area of order $\Theta(1)$. Consequently, we have $P[F_2] \le e^{-ce^{t/2}}$.

Therefore,

$$P[E_1] + P[E_2] \le P[F_1] + P[F_2] \le c\exp(-\frac{t^2}{c}),$$

showing (4.3) as desired. □

Whereas the previous lemma localizes k-face and volume functionals in the spatial domain, we now localize in the height/time domain. We show that the boundaries of the paraboloid germ-grain processes $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ and $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$, $\lambda \in [\lambda_0, \infty]$, are not too far from $\mathbb{R}^{d-1}$. Recall that the point process $\mathcal{P}^{(\lambda)}, \lambda = \infty$, is taken to be $\mathcal{P}$ and we also write $\Psi(\mathcal{P})$ for $\Psi^{(\infty)}(\mathcal{P}^{\infty})$. We also localize the height coordinate of faces containing extreme points. If $w \in \mathrm{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)})$ we put $H(w) := H(w, \mathcal{P}^{(\lambda)})$ to be the maximal height coordinate (with respect to $\mathbb{R}^{d-1}$) of a parabolic face in $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ containing w, otherwise we put $H(w) = 0$.

Lemma 4.2 *(a) There is a constant c such that for all $\lambda \in [\lambda_1, \infty]$, $w = (v_1, h_1) \in \mathcal{P}^{(\lambda)}$, and $t \geq (-h \vee 4)$ we have*

$$P[H(w, \mathcal{P}^{(\lambda)}) \geq t] \leq c \exp(-\frac{e^t}{c}). \tag{4.6}$$

(b) There is a constant c such that for all $L \in (0, \infty)$, $t \in (0, \infty)$, $\lambda \in [\lambda_1, \infty]$ and $(v_1, h_1) \in W_\lambda$ we have

$$P[d((\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(v_1, L)), B_{d-1}(v_1, L)) > t] \leq cL^{d-1}e^{-\frac{t}{c}}. \tag{4.7}$$

The bounds (4.6)-(4.7) *also hold for the dual processes $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$.*

Proof. The bound (4.6) is just a restatement of the probability bound $P[\tilde{E}_2]$, with $\tilde{E}_2$ defined at (4.5).

We now prove (4.7). Recall that we write $\mathcal{P}$ for $\mathcal{P}^{(\infty)}$. We bound the probability of the two events

$$E_3 := \{\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap \{(v, h) : |v - v_1| \leq L, h > t\} \neq \emptyset\}$$

and

$$E_4 := \{\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap \{(v, h) : |v - v_1| \leq L, h < -t\} \neq \emptyset\}.$$

When in E_3, there is a point $w := (v, h) \in \mathrm{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}), h \in [t, \infty), |v - v_1| \leq L$. By (4.6) and discretization of $\{(v, h) : \ |v - v_1| \leq L, h \in [t, \infty)\}$ into unit volume sub-cubes, we get

$$P[E_3] \leq cL^{d-1} \exp(-e^t/c).$$

On the event E_4, there exists a point (v_2, h_2) in $\{(v, h) : |v - v_1| \leq L, h \in (-\infty, -t)\}$ which is on the boundary of an upward paraboloid with apex in $\mathcal{P}^{(\lambda)}$. The apex of this upward paraboloid is contained in the union of all down paraboloids with apex on $B_{d-1}(v_1, L) \times \{h_2\}$. The $d\mathcal{P}^{(\lambda)}$ measure of this union is bounded by $c \exp(h_2/c)$. Consequently, the probability that the union is not devoid of points from $\mathcal{P}^{(\lambda)}$ is less than $1 - \exp(-ce^{h_2/c}) \leq c \exp(h_2/c)$, i.e., the probability that the union contains points from $\mathcal{P}^{(\lambda)}$ is less than $c \exp(h_2/c)$. It remains to discretize and integrate over $h_2 \in (-\infty, -t)$. This goes as follows.

Discretizing $C(v_1, L) \times [-t-1, -t]$ into unit volume subcubes, we get that the probability there exists $(v_2, h_2) \in \mathbb{R}^{d-1} \times (-t-1, -t]$ on the boundary of an up paraboloid, is bounded by $cL^{d-1}\exp(-t/c)$. The probability there exists $(v_2, h_2) \in \mathbb{R}^{d-1} \times (-\infty, -t]$ on the boundary of an up paraboloid is thus bounded by

$$cL^{d-1} \int_{-\infty}^{-t} e^{h_2/c} e^{h_2} dh_2$$

This establishes (4.7). The same argument applies to the dual process $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ and so (4.7) holds for $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$, as claimed. $\square$

4.2. Moment bounds for $\xi^{(\lambda)}, \lambda \in [\lambda_1, \infty]$. For a random variable W and all $p \in (0, \infty)$, we let $||W||_p := (\mathbb{E}\,|W|^p)^{1/p}$.

Lemma 4.3 *For all $p \in (0, \infty)$, $k \in \{0, 1, ..., d-1\}$ there is $c_1 := c_1(p, k, d)$ such that for all $(v, h) \in W_\lambda$, $\lambda \in [\lambda_1, \infty]$, we have*

$$\mathbb{E}\,[|\xi_k^{(\lambda)}((v, h)), \mathcal{P}^{(\lambda)})|^p] \leq c_1(-h \vee 4)^{c_1} \exp(-\frac{e^{h \vee 0}}{c_1}). \tag{4.8}$$

For all $p \in (0, \infty)$, there is $c_2 := c_2(p, d)$ such that for all $(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$ we have

$$\mathbb{E}\,[|\xi_V^{(\infty)}((v, h), \mathcal{P})|^p] \leq c_2(-h \vee 4)^{c_2} \exp(-\frac{e^{h \vee 0}}{c_2}) \tag{4.9}$$

whereas for all $(v, h) \in W_\lambda$, $\lambda \in [\lambda_1, \infty)$, we have

$$\mathbb{E}\,[|R_\lambda \xi_V^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})|^p] \leq c_2(-h \vee 4)^{c_2} \exp(-\frac{e^{h \vee 0}}{c_2}). \tag{4.10}$$

Proof. We start by showing for all $\lambda \in [\lambda_1, \infty]$

$$\sup_{v \in \mathbb{R}^{d-1}} \mathbb{E}\,[|\xi_k^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})|^p] \leq c_1(-h \vee 4)^{c_1}. \tag{4.11}$$

Put $w_0 = (v, h)$. Let $N := N(w_0) := \mathrm{card}\{\mathrm{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(v, R)\}$ with $R := R^{\xi_k^{(\lambda)}}$ the radius of spatial localization for $\xi_k^{(\lambda)}$ at w_0. Clearly

$$\xi_k^{(\lambda)}(w_0, \mathcal{P}) \leq \binom{N}{k-1}.$$

To show (4.11), it suffices to show there is a constant $c_1 := c_1(p, k, d)$ such that

$$\mathbb{E}\,N(w_0)^{p(k-1)} \leq c_1(-h \vee 4)^{c_1}.$$

For fixed $r \in (0, \infty)$ we may write

$$\text{card}\{\text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(v, r)\} = \sum_{x \in \mathcal{P}^{(\lambda)} \cap C(v,r)} e(x, \mathcal{P}^{(\lambda)})$$

where $e(x, \mathcal{P}^{(\lambda)}) = 1$ if $x \in \text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)})$ and otherwise $e(x, \mathcal{P}^{(\lambda)}) = 0$. By Lemma 4.2(b), for all $\lambda \in [\lambda_1, \infty]$ we have $\mathbb{E}\, e((\mathbf{0}, h), \mathcal{P}^{(\lambda)}) \leq c \exp(-e^h/c)$ and it follows that for all $r \in (0, \infty)$ and $p \in (0, \infty)$ that

$$\mathbb{E}\,(\text{card}\{\text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(v, r)\})^p \leq c(p) r^{d-1}. \tag{4.12}$$

Thus for all $\lambda \in [\lambda_1, \infty]$ we have

$$\begin{aligned}
&\mathbb{E}\, N(w_0)^{p(k-1)} \\
&= \sum_{i=0}^{\infty} \mathbb{E}\,[(\text{card}\{\text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(v, R)\})^{p(k-1)} \mathbf{1}(i \leq R < i+1)] \\
&\leq c_1(-h \vee 4)^{c_1} + \sum_{i=(-h\vee 4)}^{\infty} (\mathbb{E}\,[(\text{card}\{\text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(v, i+1)\})^{2p(k-1)})^{1/2} (P[R > i])^{1/2} \\
&\leq c_1(-h \vee 4)^{c_1} + \sum_{i=(-h\vee 4)}^{\infty} c(p,k)(i+1)^{(d-1)p(k-1)} (P[R > i])^{1/2},
\end{aligned}$$

where the last two inequalities follow by the Cauchy-Schwarz inequality and (4.12), respectively. Now (4.11) follows since R has exponentially decaying tails.

To deduce (4.8), we argue as follows. First consider the case $h \in [0, \infty)$. By the Cauchy-Schwarz inequality

$$\begin{aligned}
&\mathbb{E}\,[|\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})|^p] \\
&\leq (\mathbb{E}\,|\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})|^{2p})^{1/2} P[|\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})| > 0]^{1/2} \\
&\leq (c_1(2p, k, d))^{1/2} (-h \vee 4)^{c_1(p,k,d)} P[|\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})| \neq 0]^{1/2}
\end{aligned}$$

by (4.11). The event $\{|\xi^{(\lambda)}((\mathbf{0}, h), v)| \neq 0\}$ is a subset of the event that $(\mathbf{0}, h)$ is extreme in $\mathcal{P}^{(\lambda)}$ and we may now apply (4.6) for $t = h$, which is possible since we have assumed h is positive. This gives (4.8) for $h \in [0, \infty)$. When $h \in (-\infty, 0)$ we bound $P[|\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})| > 0]^{1/2}$ by $c \exp(-e^0/c)$, c large, which shows (4.8) for $h \in (-\infty, 0)$. This concludes the proof of (4.8).

We now prove (4.9). Put $w_0 := (v, h)$. Notice that $\xi_V^{(\infty)}(w_0, \mathcal{P})$ is bounded by the Lebesgue measure of $B(v, R) \times [-D(R), D(R)]$, where for all $L \in (0, \infty)$ we put $D(L) := d((\partial\Phi(\mathcal{P}) \cap C(v, L)), B_{d-1}(v, L))$. We have

$$\mathbb{E}\,|\xi_V^{(\infty)}(w_0, \mathcal{P})|^p \leq c\mathbb{E}\,(R^{d-1} D(R))^p \leq c||R^{p(d-1)}||_2 ||D(R)^p||_2,$$

by the Cauchy-Schwarz inequality. By the tail behavior for R we have $\mathbb{E}\, R^r = r \int_0^\infty P[R > t]t^{r-1}dt \le c(r)(-h \vee 4)^r$ for all $r \in [1, \infty)$. Also, for all $r \in [1, \infty)$ we have

$$\mathbb{E}\, D(R)^r = \sum_{i=0}^{\infty} \mathbb{E}\, (D(R))^r \mathbf{1}(i \le R < i+1)$$

$$\le \sum_{i=0}^{\infty} ||D(i+1))^r||_2 P[R \ge i]^{1/2}.$$

By Lemma 4.2 we have $||D(i+1))^r||_2 \le c(r)(i+1)^{d-1}$. We also have that $P[R \ge i]$ decays exponentially fast, showing that

$$\mathbb{E}\, D(R)^r \le c(r)(-h \vee 4)^{d-1}.$$

It follows that

$$\mathbb{E}\, |\xi_V^{(\infty)}(w_0, \mathcal{P})|^p \le ||R^{p(d-1)}||_2 ||D(R)^p||_2 \le c(p,d)(-h \vee 4)^{p(d-1)}(-h \vee 4)^{(d-1)/2},$$

which gives

$$\mathbb{E}\, [|\xi_V^{(\infty)}(w_0, \mathcal{P})|^p] \le c_2(-h \vee 4)^{c_2}. \tag{4.13}$$

The bound (4.9) follows from (4.13) in the same way that (4.11) implies (4.8). The proof of (4.10) follows similarly. □

4.3. Scaling limits. The next two lemmas justify the assertion that the functionals in $\Xi^{(\infty)}$ are indeed scaling limits of their counterparts in $\Xi^{(\lambda)}$.

Lemma 4.4 *For all $w \in W_\lambda$, $r \in (0, \infty)$, and ξ a generic k-face functional we have*

$$\lim_{\lambda \to \infty} \mathbb{E}\, \xi_{[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \mathbb{E}\, \xi_{[r]}^{(\infty)}(w, \mathcal{P}). \tag{4.14}$$

If ξ is the volume functional ξ_V then

$$\lim_{\lambda \to \infty} \mathbb{E}\, R_\lambda \xi_{[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \mathbb{E}\, \xi_{[r]}^{(\infty)}(w, \mathcal{P}). \tag{4.15}$$

Proof. We prove (4.14) for $w_0 := (\mathbf{0}, h)$, as the proof for other choices of w is no different. Put $S(r, H) := B_{d-1}(\mathbf{0}, r) \times [-H, H]$, with H a fixed deterministic height. By Lemma 4.2 and the Cauchy-Schwarz inequality, it is enough to show

$$\lim_{\lambda \to \infty} \mathbb{E}\, \xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)} \cap S(r, H))] = \mathbb{E}\, \xi_{[r]}^{(\infty)}(w_0, \mathcal{P} \cap S(r, H)).$$

It is understood that the left-hand side is determined by the geometry of the quasi-paraboloids $\{[\Pi^\uparrow(w)]^{(\lambda)}\}, w \in \mathcal{P}^{(\lambda)} \cap S(r, H))$ and similarly for the right-hand side.

Equip the collection $\mathcal{X}(r, H)$ of locally finite point sets in $S(r, H)$ with the discrete topology. Thus if $\mathcal{X}_i, i \geq 1$, is a sequence in $\mathcal{X}(r, H)$ and if

$$\lim_{i \to \infty} \mathcal{X}_i = \mathcal{X}, \quad \text{then} \quad \mathcal{X}_i = \mathcal{X} \text{ for } \ i \geq i_0. \tag{4.16}$$

Recall that $[\Pi^{\downarrow}(w')]^{(\infty)}$ coincides with $\Pi^{\downarrow}(w')$. For all $\lambda \in [\lambda_0, \infty], w \in W_\lambda$, and $\mathcal{X} \in \mathcal{X}(r, H)$ we define $g_{k,\lambda} : W_\lambda \times \mathcal{X}(r, H) \mapsto \mathbb{R}$ by taking $g_{k,\lambda}(w, \mathcal{X})$ to be the product of $(k+1)^{-1}$ and the number of quasi parabolic k-dimensional faces of $\bigcup_{w' \in \mathcal{X}} [\Pi^{\downarrow}(w')]^{(\lambda)}$ which contain w, if w is a vertex in $\mathcal{X}$, otherwise $g_{k,\lambda}(w, \mathcal{X}) = 0$. Thus $g_{k,\lambda}(w, \mathcal{X}) := \xi_{[r]}^{(\lambda)}(w, \mathcal{X} \cap S(r, H)))$.

Let $\mathcal{X}$ be in regular position, that is to say the intersection of k quasi-paraboloids contains at most $(d-k+1)$ points of $\mathcal{X}$ for all $1 \leq k \leq d$. Thus $\mathcal{P}$ is in regular position with probability one. To apply the continuous mapping theorem (Theorem 5.5 in [9]), by (4.16), it is enough to show that $g_{k,\lambda}(w_0, \mathcal{X})$ coincides with $g_{k,\infty}(w_0, \mathcal{X})$ for λ large enough. Let $\varepsilon > 0$ be the minimal distance between any down paraboloid containing d points of $\mathcal{X}$ and the rest of the point set. Perturbations of the paraboloids within an ε parallel set does not change the number of k-dimensional faces. In particular, for λ large enough, the set $\partial \left(\cup_{w' \in \mathcal{X}} [\Pi^{\downarrow}(w')]^{(\lambda)} \right)$ is included in that parallel set so that the number of k-dimensional faces does not change. Thus $g_{k,\lambda}(w_0, \mathcal{X})$ coincides with $g_{k,\infty}(w_0, \mathcal{X})$ for large λ.

Since $\mathcal{P}^{(\lambda)} \xrightarrow{\mathcal{D}} \mathcal{P}$, we may apply the continuous mapping theorem to get

$$\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) \xrightarrow{\mathcal{D}} \xi_{[r]}^{(\infty)}(w_0, \mathcal{P})$$

as $\lambda \to \infty$. The convergence in distribution extends to convergence of expectations by the uniform integrability of $\xi_{[r]}^{(\lambda)}$, which follows from moment bounds for $\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})$ analogous to those for $\xi_k^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})$ as given in Lemma 4.3. This proves (4.14) when ξ is a generic k-face functional.

Next we show for $\xi := \xi_V$, $r \in (0, \infty)$ that

$$\lim_{\lambda \to \infty} \mathbb{E}\,[R_\lambda \xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)} \cap S(r, H))] = \mathbb{E}\,[\xi_{[r]}^{(\infty)}(w_0, \mathcal{P} \cap S(r, H))].$$

This will yield (4.15). Recall that $\mathrm{Vol}_d^{(\lambda)}$ is the image of Vol_d under $T^{(\lambda)}$. For $\lambda \in [\lambda_0, \infty)]$, we define this time $\tilde{g}_{k,\lambda} : (R^{d-1} \times \mathbb{R}) \times S(r, H) \mapsto \mathbb{R}$ by

$$\tilde{g}_{k,\lambda}(w, \mathcal{X}) = R_\lambda \xi_{[r]}^{(\lambda)}(w, \mathcal{X} \cap S(r, H))$$

$$= R_\lambda \mathrm{Vol}_d^{(\lambda)}(\{(v,h) \in S(r, H) : 0 \leq h \leq \partial \Phi^{(\lambda)}(\mathcal{X})(v), v \in \mathrm{Cyl}^{(\lambda)}(\mathrm{w}), \Phi^{(\lambda)}(\mathcal{X})(\mathrm{v}) \geq 0\})$$
$$- R_\lambda \mathrm{Vol}_d^{(\lambda)}(\{(v,h) \in S(r, H) : \Phi^{(\lambda)}(\mathcal{X})(v) \leq h \leq 0, v \in \mathrm{Cyl}^{(\lambda)}(\mathrm{w}), \Phi^{(\lambda)}(\mathcal{X})(\mathrm{v}) < 0\}).$$

When $\lambda = \infty$ we put

$$\tilde{g}_{k,\infty}(w, \mathcal{X}) = \xi^{(\infty)}_{[r]}(w, \mathcal{X} \cap S(r, H))$$

$$= \mathrm{Vol}_d(\{(v, h) \in S(r, H) : 0 \leq h \leq \partial\Phi(\mathcal{X})(v), v \in \mathrm{Cyl(w)}, \Phi(\mathcal{X})(\mathrm{v}) \geq 0\})$$

$$-\mathrm{Vol}_d(\{(v, h) \in S(r, H) : \Phi(\mathcal{X})(v) \leq h \leq 0, v \in \mathrm{Cyl(w)}, \Phi(\mathcal{X})(\mathrm{v}) < 0\}).$$

Recalling (4.16), it is enough to show for a fixed point set $\mathcal{X}$ in regular position that

$$\lim_{\lambda\to\infty} |\tilde{g}_{k,\lambda}(w, \mathcal{X}) - \tilde{g}_{k,\infty}(w, \mathcal{X})| = 0.$$

We show that the first term comprising $\tilde{g}_{k,\lambda}(w, \mathcal{X})$ converges to the first term comprising $\tilde{g}_{k,\infty}(w, \mathcal{X})$. In other words, setting

$$F(\mathcal{X}, \lambda) := \{(v, h) \in S(r, H) : 0 \leq h \leq \partial\Phi^{(\lambda)}(\mathcal{X})(v), v \in \mathrm{Cyl}^{(\lambda)}(\mathrm{w}), \Phi^{(\lambda)}(\mathcal{X})(\mathrm{v}) \geq 0\}$$

and

$$F(\mathcal{X}) := \{(v, h) \in S(r, H) : 0 \leq h \leq \partial\Phi(\mathcal{X})(v), v \in \mathrm{Cyl(w)}, \Phi(\mathcal{X})(\mathrm{v}) \geq 0\}$$

we show

$$\lim_{\lambda\to\infty} |R_\lambda \mathrm{Vol}^{(\lambda)}_d(F(\mathcal{X}, \lambda)) - \mathrm{Vol}_d(F(\mathcal{X}))| = 0.$$

The proof that the second term comprising $\tilde{g}_{k,\lambda}(w, \mathcal{X})$ converges to the second term comprising $\tilde{g}_{k,\infty}(w, \mathcal{X})$ is identical. We have

$$|R_\lambda \mathrm{Vol}^{(\lambda)}_d(F(\mathcal{X}, \lambda)) - \mathrm{Vol}_d(F(\mathcal{X}))| \leq |R_\lambda \mathrm{Vol}^{(\lambda)}_d(F(\mathcal{X}, \lambda)) - R_\lambda \mathrm{Vol}^{(\lambda)}_d(F(\mathcal{X}))| \tag{4.17}$$

$$+|R_\lambda \mathrm{Vol}^{(\lambda)}_d(F(\mathcal{X})) - \mathrm{Vol}_d(F(\mathcal{X}))|.$$

Since $\partial\Phi^{(\lambda)}(\mathcal{X})$ converges uniformly to $\partial\Phi(\mathcal{X})$ on compacts and since $d^H(\mathrm{Cyl}^{(\lambda)}(w), \mathrm{Cyl}(w))$ decreases to zero as $\lambda \to \infty$ (indeed $\partial(\mathrm{Cyl}^{(\lambda)}(w)) \to \partial\mathrm{Cyl}(w)$ uniformly), we get for $\lambda \geq \lambda_0$ that $F(\mathcal{X}, \lambda)\Delta F(\mathcal{X})$ is a subset of a set $A(\mathcal{X}) \subset \mathbb{R}^d$ of arbitrarily small volume. So $|R_\lambda \mathrm{Vol}^{(\lambda)}_d(F(\mathcal{X}, \lambda)) - R_\lambda \mathrm{Vol}^{(\lambda)}_d(F(\mathcal{X}))| \leq R_\lambda \mathrm{Vol}^{(\lambda)}_d(A(\mathcal{X}))$. By Lemma 3.2, we have $R_\lambda \mathrm{Vol}^{(\lambda)}_{\mathrm{d}} \xrightarrow{\mathcal{D}} \mathrm{Vol}_{\mathrm{d}}$ and thus the first term in (4.17) goes to zero as $\lambda \to \infty$. Appealing again to $R_\lambda \mathrm{Vol}^{(\lambda)}_{\mathrm{d}} \xrightarrow{\mathcal{D}} \mathrm{Vol}_{\mathrm{d}}$, the second term in (4.17) likewise tends to zero, showing (4.15) as desired. □

Lemma 4.5 *For all $h \in \mathbb{R}$ and ξ a generic k-face functional we have*

$$\lim_{\lambda\to\infty} \mathbb{E}\,[\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})] \to \mathbb{E}\,[\xi^{(\infty)}((\mathbf{0}, h), \mathcal{P})].$$

If ξ is the volume functional ξ_V then the left-hand sides require an additional prefactor of R_λ.

Proof. We prove the first limit as follows. Let $w_0 := (\mathbf{0}, h)$. By Lemma 4.4, given $\epsilon > 0$, we have for all $\lambda \geq \lambda_0(\epsilon)$

$$|\mathbb{E}\,\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \mathbb{E}\,\xi_{[r]}^{(\infty)}(w_0, \mathcal{P}))| < \epsilon. \tag{4.18}$$

We now show that replacing $\xi_{[r]}^{(\lambda)}$ and $\xi_{[r]}^{(\infty)}$ by $\xi^{(\lambda)}$ and $\xi^{(\infty)}$, respectively, introduces negligible error in (4.18). Write

$$\begin{aligned}
&|\mathbb{E}\,\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \mathbb{E}\,\xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}))| \\
&= |\mathbb{E}\,(\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}))\mathbf{1}(R^{\xi^{(\lambda)}}[w_0] < r)| \\
&\quad + \ |\mathbb{E}\,(\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}))\mathbf{1}(R^{\xi^{(\lambda)}}[w_0] > r)|.
\end{aligned}$$

The first term vanishes by definition of $R^{\xi^{(\lambda)}}[w_0]$. By the Cauchy-Schwarz inequality, the second term is bounded by

$$||\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})||_2 P[R^{\xi^{(\lambda)}}[w_0] > r]^{1/2} \leq cP[R^{\xi^{(\lambda)}}[w_0] > r]^{1/2} \leq \epsilon \tag{4.19}$$

if $r \geq (-h \vee 4)$ is large enough. It follows that for $r \geq r_0(\epsilon)$ and $\lambda \geq \lambda_0(\epsilon)$

$$|\mathbb{E}\,\xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \mathbb{E}\,\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})| < \epsilon. \tag{4.20}$$

Similarly for $r \geq r_1(\epsilon)$ we have

$$|\mathbb{E}\,\xi^{(\infty)}(w_0, \mathcal{P}) - \mathbb{E}\,\xi_{[r]}^{(\infty)}(w_0, \mathcal{P})| < \epsilon. \tag{4.21}$$

Combining (4.18)-(4.21) and using the triangle inequality we get for $r \geq (r_0(\epsilon) \vee r_1(\epsilon)$ and $\lambda \geq \lambda_0(\epsilon)$

$$|\mathbb{E}\,\xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \mathbb{E}\,\xi^{(\infty)}(w_0, \mathcal{P})| < 3\epsilon.$$

Since ϵ is arbitrary we have shown Lemma 4.5. □

4.4. Two point correlation function for $\xi^{(\lambda)}$. For all $h \in \mathbb{R}$, $(v', h') \in W_\lambda$, and ξ a generic k-face functional, we extend the definition (2.3) by putting for all $\lambda \in [\lambda_0, \infty]$

$$c^{(\lambda)}((\mathbf{0}, h), (v', h')) := c^{\xi_k^{(\lambda)}}((\mathbf{0}, h), (v', h')) :=$$

$$\mathbb{E}\,[\xi_k^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)} \cup (v', h')) \times \xi_k^{(\lambda)}((v', h'), \mathcal{P}^{(\lambda)} \cup (\mathbf{0}, h))] -$$
$$\mathbb{E}\,\xi_k^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})\mathbb{E}\,\xi_k^{(\lambda)}((v', h'), \mathcal{P}^{(\lambda)}).$$

If ξ is the volume functional ξ_V we likewise put for $\lambda \in [\lambda_0, \infty)$

$$c^{\xi_V^{(\lambda)}}((\mathbf{0}, h), (v', h')) :=$$

$$\mathbb{E}\,[R_\lambda \xi_V^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h')) \times R_\lambda \xi_V^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))] -$$
$$\mathbb{E}\,R_\lambda(\xi_V^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)})\mathbb{E}\,R_\lambda \xi_V^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)}).$$

The next lemma shows convergence of the re-scaled two-point correlation functions on re-scaled input $\mathcal{P}^{(\lambda)}$ to their counterpart correlation functions on the limit input $\mathcal{P}$.

Lemma 4.6 *For all $h \in \mathbb{R}$, $(v',h') \in W_\lambda$ and $\xi \in \Xi$ we have*

$$\lim_{\lambda\to\infty} c^{\xi^{(\lambda)}}((\mathbf{0},h),(v',h')) = c^{\xi^{(\infty)}}((\mathbf{0},h),(v',h')).$$

Proof. We prove the convergence only for ξ_k as the method would be the same for ξ_V. We deduce from Lemma 4.5 that

$$\lim_{\lambda\to\infty} \mathbb{E}\,\xi_k^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)})\mathbb{E}\,\xi_k^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)})] = \mathbb{E}\,\xi_k^{(\infty)}((\mathbf{0},h),\mathcal{P}^{(\lambda)})\mathbb{E}\,\xi_k^{(\infty)}((v',h'),\mathcal{P}^{(\lambda)}).$$

Now by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\mathbb{E}\,[\xi_k^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h')) \times \xi_k^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))]& \\ - \mathbb{E}\,[\xi_k^{(\infty)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h')) \times \xi_k^{(\infty)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))]| &\le T_1 + T_2\end{aligned}$$

where

$$T_1 = \mathbb{E}\,[|\xi_k^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h')) - \xi_k^{(\infty)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))|^2]^{1/2}\mathbb{E}\,[|\xi_k^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))|^2]^{1/2},$$
$$T_2 = \mathbb{E}\,[|\xi_k^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h)) - \xi_k^{(\infty)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))|^2]^{1/2}\mathbb{E}\,[|\xi_k^{(\infty)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))|^2]^{1/2}.$$

It is enough to show that the term T_1 goes to zero as the proof would be similar for T_2. We have showed in the proof of Lemma 4.4 that $\xi_{[r]}^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))$ converges in distribution to $\xi_{[r]}^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\infty)} \cup (v',h'))$ for every $r > 0$. Lemma 4.3 implies that this family is uniformly integrable so the convergence occurs in L^2 also. Using the same method as in the proof of Lemma 4.5, we obtain that

$$\lim_{\lambda\to\infty} \mathbb{E}\,[|\xi_k^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h')) - \xi_k^{(\infty)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))|^2] = 0. \tag{4.22}$$

By Lemma 4.3, the variables $\xi_k^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))$ are uniformly bounded in L^2 so we deduce from (4.22) that the term T_1 tends to zero. $\square$

The next lemma shows that the re-scaled and limit two point correlation function decays exponentially fast with the distance between spatial coordinates of the input and super-exponentially fast with respect to positive height coordinates.

Lemma 4.7 *There is a constant $c_3 := c_3(k,d) \in (0,\infty)$ such that for all $h \in \mathbb{R}$, $(v',h') \in W_\lambda$, and all $\xi \in \Xi$, we have for $|v'| \geq 2\max(-h \vee 4, -h' \vee 4)$ and $\lambda \in [\lambda_0, \infty]$*

$$|c^{\xi^{(\lambda)}}((\mathbf{0},h),(v',h'))| \leq c_3(-h \vee 4)^{c_3}(-h' \vee 4)^{c_3} \exp\left(\frac{-1}{c_3}(|v'|^2 + e^{h\vee 0} + e^{h'\vee 0})\right). \quad (4.23)$$

Proof. We first show these bounds for the k-face functional ξ_k. Put $r := |v'|/2$. We have

$$|c^{\xi^{(\lambda)}}((\mathbf{0},h),(v',h'))| \leq |c^{\xi^{(\lambda)}_{[r]}}((\mathbf{0},h),(v',h'))| + T_1 + T_2 \quad (4.24)$$

where

$$\begin{aligned} T_1 = |\mathbb{E}\,[\xi^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))\xi^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))] \\ - \mathbb{E}\,[\xi^{(\lambda)}_{[r]}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))\xi^{(\lambda)}_{[r]}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))]|, \\ T_2 = |\mathbb{E}\,[\xi^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))]\mathbb{E}\,[\xi^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))] \\ - \mathbb{E}\,[\xi^{(\lambda)}_{[r]}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))]\mathbb{E}\,[\xi^{(\lambda)}_{[r]}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))]|. \end{aligned}$$

This choice of r ensures that $C(\mathbf{0},r)$ and $C(v',r)$ do not intersect and therefore by independence we have

$$c^{\xi^{(\lambda)}_{[r]}}((\mathbf{0},h),(v',h')) = 0. \quad (4.25)$$

Since $r \geq \max(-h \vee 4, -h' \vee 4)$, we get from the Cauchy-Schwarz inequality, (4.3) and (4.8) with $p = 2$ that there is a constant $c := c(k) \in (0,\infty)$ with

$$\begin{aligned} &|\mathbb{E}\,[\xi^{(\lambda)}_{[r]}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup \{(v',h')\})] - \mathbb{E}\,[\xi^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup \{(v',h')\})]| \\ &\quad \leq c\|\xi^{(\lambda)}_{[r]}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup \{(v',h')\}) - \xi^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup \{(v',h')\})\|_2 P[R^{\xi^{(\lambda)}}[(\mathbf{0},h)] > r]^{1/2} \\ &\quad \leq c(-h \vee 4)^c \exp(\frac{-1}{c}(|v'|^2 + e^{h\vee 0})), \end{aligned} \quad (4.26)$$

where the last inequality uses $r = |v'|/2$. The same inequality holds when the roles of $(\mathbf{0},h)$ and $(v'h')$ are exchanged. Consequently, using (4.26) and (4.8) for $p = 1$, we deduce that

$$T_2 \leq c(-h \vee 4)^c(-h' \vee 4)^c \exp(\frac{-1}{c}(|v'|^2 + e^{h\vee 0} + e^{h'\vee 0})). \quad (4.27)$$

In the same way, we have

$$\begin{aligned} T_1 \leq \|\xi^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h')) - \xi^{(\lambda)}_{[r]}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))\|_2 \|\xi^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))]\|_2 \\ + \|\xi^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h)) - \xi^{(\lambda)}_{[r]}((v',h'),\mathcal{P}^{(\lambda)} \cup (\mathbf{0},h))\|_2 \|\xi^{(\lambda)}_{[r]}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))\|_2. \end{aligned}$$

Using (4.8) and the same method as for (4.26) with L^2 norms instead of expectation, we get that

$$T_1 \le c(-h \vee 4)^c(-h' \vee 4)^c \exp(\frac{-1}{c}(|v'|^2 + e^{h\vee 0} + e^{h'\vee 0})). \tag{4.28}$$

Combining (4.24), (4.27) and (4.28), we get the required result. To show these bounds hold for the volume functional ξ_V, we may follow the above arguments verbatim, replacing ξ_k by $R_\lambda \xi_V$. □

5 Proofs of main results

5.1. Proof of Theorems 1.1 and 1.2. The next result contains Theorem 1.2 and it yields Theorem 1.1, since it implies that the extreme points of $\mathcal{P}^{(\lambda)}$ converge in law to $\mathrm{Ext}(\mathcal{P})$ as $\lambda \to \infty$.

Proposition 5.1 *Fix $L \in (0, \infty)$. The boundary of $\Psi(\mathcal{P}^{(\lambda)})$ converges in probability as $\lambda \to \infty$ to the boundary of $\Psi(\mathcal{P})$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, L)))$ equipped with the supremum norm. Similarly, the boundary of $\Phi(\mathcal{P}^{(\lambda)})$ converges in probability as $\lambda \to \infty$ to the Burgers festoon $\partial(\Phi(\mathcal{P}))$.*

Proof. We only prove the convergence of $\Psi(\mathcal{P}^{(\lambda)})$, as the convergence of $\Phi(\mathcal{P}^{(\lambda)})$ is handled similarly. We show for fixed $L \in (0, \infty)$ that the boundary of $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ converges in law to $\partial(\Psi(\mathcal{P}))$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, L)))$. With L fixed, for all $H \in [0, \infty)$ and $\lambda \in [0, \infty)$, let $E(L, H, \lambda)$ be the event that the heights of $\partial(\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}))$ and $\partial(\Psi(\mathcal{P}))$ belong to $[-H, H]$ over the spatial region $B_{d-1}(\mathbf{0}, L)$. By Lemma 4.2, we have that $P[E(L, H, \lambda)]^c$ decays exponentially fast in H, uniformly in λ, and so it is enough to show, conditional on $E(L, H, \lambda)$, that $\partial(\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}))$ is close to $\partial(\Psi(\mathcal{P}))$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$, λ large.

Recalling the definition of $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ at (3.4), we need to show, conditional on $E(L, H, \lambda)$, that the boundary of

$$\bigcup_{w \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, L)} ([\Pi^{\uparrow}(w)]^{(\lambda)} \cap C(\mathbf{0}, L))$$

is close to the boundary of

$$\bigcup_{w \in \mathcal{P} \cap C(\mathbf{0}, L)} (\Pi^{\uparrow}(w) \cap C(\mathbf{0}, L)). \tag{5.1}$$

By Lemma 3.1, given $w_1 := (v_1, h_1) \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, L)$, it follows that on $E(L, H, \lambda)$ the boundary of $[\Pi^{\uparrow}(w_1)]^{(\lambda)} \cap C(\mathbf{0}, L)$ coincides with the graph of

$$v \mapsto h_1 + \frac{|v - v_1|^2}{2} + O(R_\lambda^{-1}),$$

that is coincides to within $O(R_\lambda^{-1})$ of the boundary of $\Pi^\uparrow(w_1)$. The boundary of $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(\mathbf{0}, L)$ is a.s. the finite union of graphs of the above form and is thus a.s. within $O(R_\lambda^{-1})$ of the boundary of

$$\bigcup_{w \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, L)} (\Pi^\uparrow(w) \cap C(\mathbf{0}, L)).$$

It therefore suffices to show that the boundary of $\bigcup_{w \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, L)} (\Pi^\uparrow(w) \cap C(\mathbf{0}, L))$ is close to the boundary of the set given at (5.1). However, we may couple $\mathcal{P}^{(\lambda)}$ and $\mathcal{P}$ on $B_{d-1}(\mathbf{0}, L) \times [-H, H]$ so that they coincide except on a set with probability less than ϵ, showing the desired closeness with probability at least $1 - \epsilon$. $\square$

5.2. Proof of expectation asymptotics (2.5)**.** For $g \in \mathcal{C}(\mathbb{S}^{d-1})$, we have

$$\mathbb{E}\left[\langle g_{R_\lambda}, \mu_\lambda^\xi \rangle\right] = \int_{\mathbb{R}^d} g(\frac{x}{R_\lambda}) \mathbb{E}\left[\xi(x, \mathcal{P}_\lambda)\right] \lambda \phi(x) dx. \tag{5.2}$$

Given $x \in \mathbb{R}^d$, let $\theta := \theta_x$ be the rotation sending $x/|x|$ to $u_0 := (0, 0, ..., 1) \in \mathbb{S}^{d-1}$. Let x^θ denote the point x rotated by the angle θ, and similarly for $\mathcal{P}_\lambda^\theta$. Thus $\mathbb{E}\, \xi(x, \mathcal{P}_\lambda) = \mathbb{E}\, \xi(x^\theta, \mathcal{P}_\lambda^\theta)$. Concerning the expectation inside the integral, we may without loss of generality assume $x/|x| = u_0$ and $T^{(\lambda)}(x) = ((\mathbf{0}, h))$. Thus $\mathbb{E}\, \xi(x, \mathcal{P}_\lambda) = \mathbb{E}\left[\xi^{(\lambda)}\left((\mathbf{0}, h), \mathcal{P}^{(\lambda)}\right)\right]$. Recalling $x = u \cdot R_\lambda(1 - h/R_\lambda^2)$, and writing $dx = |R_\lambda(1 - h/R_\lambda^2)|^{d-1} R_\lambda^{-1} dh d\sigma_{d-1}(u)$, and recalling (3.16), we see that $R_\lambda^{-(d-1)} \mathbb{E}\left[\langle g_{R_\lambda}, \mu_\lambda^\xi \rangle\right]$ transforms to

$$= \int_{u \in \mathbb{S}^{d-1}} \int_{h \in (-\infty, R_\lambda^2]} g(u(1 - \frac{h}{R_\lambda^2})) \mathbb{E}\left[\xi^{(\lambda)}\left((\mathbf{0}, h), \mathcal{P}^{(\lambda)}\right)\right] \tilde{\phi}_\lambda(u, h) |R_\lambda(1 - \frac{h}{R_\lambda^2})|^{d-1} dh d\sigma_{d-1}(u),$$

where $\tilde{\phi} : \mathbb{S}^{d-1} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\tilde{\phi}_\lambda(u, h) := \frac{\lambda}{R_\lambda} \phi(u \cdot R_\lambda(1 - \frac{h}{R_\lambda^2})). \tag{5.3}$$

By (3.17) we have for all $u \in \mathbb{S}^{d-1}$ that

$$\tilde{\phi}_\lambda(u, h) = \frac{\sqrt{2 \log \lambda}}{R_\lambda} \exp\left(h - \frac{h^2}{2R_\lambda^2}\right).$$

Thus there is $c \in (0, \infty)$ such that for all $h \in \mathbb{R}$ we have

$$\sup_{u \in \mathbb{S}^{d-1}} \sup_{\lambda \geq 3} \tilde{\phi}_\lambda(u, h) \leq c e^h \tag{5.4}$$

and for all $u \in \mathbb{S}^{d-1}$

$$\lim_{\lambda \to \infty} \tilde{\phi}_\lambda(u, h) = e^h. \tag{5.5}$$

By the continuity of g, Lemma 4.5 and the limit (5.5), we have for $h \in (-\infty, R_\lambda^2)$ that the integrand converges to $g(u)\mathbb{E}\left[\xi^{(\infty)}((\mathbf{0},h),\mathcal{P})\right]e^h$ as $\lambda \to \infty$. Moreover, by (5.4) and the moment bounds of Lemma 4.3, the integrand is dominated by the product of a polynomial in h and an exponentially decaying function of h. The dominated convergence theorem gives the claimed result (2.5).

5.3. Proof of variance asymptotics (2.6)**.** For $g \in \mathcal{C}(\mathbb{S}^{d-1})$, we have

$$\begin{aligned}
&\mathrm{Var}[\langle g_{R_\lambda}, \mu_\lambda^\xi\rangle] \\
&= \int_{\mathbb{R}^d} g(\frac{x}{R_\lambda})^2 \mathbb{E}\left[\xi(x,\mathcal{P}_\lambda)^2\right]\lambda\phi(x)dx \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(\frac{x}{R_\lambda})g(\frac{y}{R_\lambda})[\mathbb{E}\,\xi(x,\mathcal{P}_\lambda\cup y)\xi(y,\mathcal{P}_\lambda\cup x) - \mathbb{E}\,\xi(x,\mathcal{P}_\lambda)\mathbb{E}\,\xi(y,\mathcal{P}_\lambda)]\lambda^2\phi(y)\phi(x)dydx \\
&:= I_1(\lambda) + I_2(\lambda).
\end{aligned} \tag{5.6}$$

We examine $\lim_{\lambda\to\infty} R_\lambda^{-(d-1)} I_1(\lambda)$ and $\lim_{\lambda\to\infty} R_\lambda^{-(d-1)} I_2(\lambda)$ separately. As in the proof of expectation asymptotics (2.5), we have

$$\lim_{\lambda\to\infty} R_\lambda^{-(d-1)} I_1(\lambda) = \int_{-\infty}^{\infty} \mathbb{E}\,\xi_k^{(\infty)}((\mathbf{0},h),\mathcal{P})^2 e^h dh \int_{\mathbb{S}^{d-1}} g(u)^2 du. \tag{5.7}$$

Next consider $\lim_{\lambda\to\infty} R_\lambda^{-(d-1)} I_2(\lambda)$. For $x \in \mathbb{R}^d$ we write

$$x = uR_\lambda(1 - h/R_\lambda^2),\ (u,h) \in \mathbb{S}^{d-1}\times\mathbb{R}. \tag{5.8}$$

We now re-scale the integrand in $I_2(\lambda)$ as follows. Given $u := u_x \in \mathbb{S}^{d-1}$ in the definition of x, define $T^{(\lambda)}$ as in (1.3), but with u_0 there replaced by u. Write $T_u^{(\lambda)}$ to denote the dependency on u. Denoting by $(\mathbf{0},h)$ and $(v'h')$ the images under $T_u^{(\lambda)}(x)$ of x and y respectively, we notice that $R_\lambda^{-(d-1)} I_2(\lambda)$ is transformed as follows.

(i) The 'covariance' term $[\mathbb{E}\,\xi(x,\mathcal{P}_\lambda\cup y)\xi(y,\mathcal{P}_\lambda\cup x) - \mathbb{E}\,\xi(x,\mathcal{P}_\lambda)\mathbb{E}\,\xi(y,\mathcal{P}_\lambda)]$ transforms to $c^{(\lambda)}((\mathbf{0},h),(v',h'))$. By Lemma 4.6 we have uniformly in $v' \in T_u^{(\lambda)}(\mathbb{S}^{d-1})$ and $h,h' \in \mathbb{R}$ that

$$\lim_{\lambda\to\infty} c^{(\lambda)}((\mathbf{0},h),(v',h')) = c^{\xi^{(\infty)}}((\mathbf{0},h),(v',h')). \tag{5.9}$$

(ii) The product $g(\frac{x}{R_\lambda})g(\frac{y}{R_\lambda})$ becomes

$$f_{1,\lambda}(u,h,v',h') := g(u(1-\frac{h}{R_\lambda^2}))g(R_\lambda^{-1}[T_u^{(\lambda)}]^{-1}((v',h'))).$$

Using (1.3) and (5.8), we notice that $[T_u^{(\lambda)}]^{-1}((v',h')) = \left(1-\frac{h'}{R_\lambda^2}\right)\exp_{d-1} R_\lambda^{-1}v'$ and consequently

$$\lim_{\lambda\to\infty} R_\lambda^{-1}[T_u^{(\lambda)}]^{-1}((v',h')) = u.$$

By continuity of g, we then have uniformly in $v' \in T_u^{(\lambda)}(\mathbb{S}^{d-1})$ and $h, h' \in \mathbb{R}$ that

$$\lim_{\lambda\to\infty} f_{1,\lambda}(u,h,v',h') = g(u)^2. \tag{5.10}$$

(iii) The double integral over $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ transforms into a quadruple integral over $(u,h,v',h') \in \mathbb{S}^{d-1} \times (-\infty, R_\lambda^2] \times T_u^{(\lambda)}(\mathbb{S}^{d-1}) \times (-\infty, R_\lambda^2]$.

(iv) By (3.18), the differential $\lambda\phi(y)dy$ transforms to

$$\frac{\sin^{d-2}(R_\lambda^{-1}|v'|)}{|R_\lambda^{-1}v'|^{d-2}}\frac{\sqrt{2\log\lambda}}{R_\lambda}|(1-\frac{h'}{R_\lambda^2})|^{d-1}e^{h'-\frac{h'^2}{2R_\lambda^2}}dv'dh'$$

whereas $R_\lambda^{-(d-1)}\lambda\phi(x)dx$ transforms to

$$\tilde{\phi}_\lambda(u,h)|(1-\frac{h}{R_\lambda^2})|^{d-1}dh d\sigma_{d-1}(u).$$

Thus the product $R_\lambda^{-(d-1)}\lambda^2\phi(y)\phi(x)dydx$ transforms to

$$f_{2,\lambda}(u,h,v',h')d\sigma_{d-1}(u)dh(v)dv'dh'$$

where

$$f_{2,\lambda}(u,h,v',h') := \frac{\sin^{d-2}(R_\lambda^{-1}|v'|)}{|R_\lambda^{-1}v'|^{d-2}}\frac{\sqrt{2\log\lambda}}{R_\lambda}|(1-\frac{h'}{R_\lambda^2})|^{d-1}e^{h'-\frac{h'^2}{2R_\lambda^2}}\tilde{\phi}_\lambda(u,h)|(1-\frac{h}{R_\lambda^2})|^{d-1}.$$

By Lemma 3.2 and (5.5) we have uniformly in $u \in \mathbb{S}^{d-1}, v' \in T^{(\lambda)}(\mathbb{S}^{d-1})$ and $h, h' \in \mathbb{R}$ that

$$\lim_{\lambda\to\infty} f_{2,\lambda}(u,h,v',h') = e^{h+h'}. \tag{5.11}$$

We re-write $R_\lambda^{-(d-1)}I_2(\lambda)$ as

$$= \int_{u\in\mathbb{S}^{d-1}}\int_{h\in(-\infty,R_\lambda^2]}\int_{T_u^{(\lambda)}(\mathbb{S}^{d-1})}\int_{h'\in(-\infty,R_\lambda^2]} F_\lambda(u,h,v',h')dh'dv'dhd\sigma_{d-1}(u),$$

where

$$F_\lambda(u,h,v',h') := f_{1,\lambda}(u,h,v',h')c^{(\lambda)}((\mathbf{0},h),(v',h'))f_{2,\lambda}(u,h,v',h').$$

Combining the limits (5.10)- (5.11), it follows for all $u \in \mathbb{S}^{d-1}, h, h' \in \mathbb{R}$ and $v' \in T^{(\lambda)}(\mathbb{S}^{d-1})$, that we have the pointwise convergence

$$\lim_{\lambda\to\infty} F_\lambda(u, h, v', h') = g(u)^2 c^{\xi^{(\infty)}}((\mathbf{0}, h), (v', h'))e^{h+h'}.$$

By Lemma 4.7, we get that

$$|F_\lambda(u, h, v', h')| \le c(-h \vee 4)^{c_3}(-h' \vee 4)^{c_3} \exp\left(\frac{-1}{c_3}(|v'|^2 + e^{h\vee 0} + e^{h'\vee 0}) + h + h'\right).$$

Using that there exists $c' > 0$ such that $\frac{1}{c_3}e^{h\vee 0} - h \ge \frac{1}{c'}e^{h\vee 0}$, we obtain that $F_\lambda(u, h, v', h')$ is dominated by an exponentially decaying function of all arguments which is integrable. The dominated convergence theorem gives

$$\lim_{\lambda\to\infty} R_\lambda^{-(d-1)} I_2(\lambda) = \tag{5.12}$$

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^{d-1}} \int_{h\in(-\infty,\infty)} \int_{h'\in(-\infty,\infty)} g(u)^2 c^{\xi^{(\infty)}}((\mathbf{0}, h), (v', h'))e^{h+h'} dh dh' dv' d\sigma_{d-1}(u).$$

Combining (5.7) and (5.12) gives the claimed variance asymptotics (2.6). The positivity of $F_{k,d}$ is established in [8], concluding the proof of (2.6). $\square$

5.4. Proof of Theorem 2.2. It suffices to follow the proof of Theorem 2.1 nearly verbatim. As in (5.2), we have

$$\mathbb{E}\,[\langle g_{R_\lambda}, \mu_\lambda^{\xi_V}\rangle] = \int_{\mathbb{R}^d} g(\frac{x}{R_\lambda})\mathbb{E}\,[\xi_V(x, \mathcal{P}_\lambda)]\,\lambda\phi(x)dx. \tag{5.13}$$

Multiplying through by $R_\lambda^{-(d-2)}$ gives that $R_\lambda^{-(d-2)}\mathbb{E}\,[\langle g_{R_\lambda}, \mu_\lambda^{\xi_V}\rangle]$ becomes

$$= \int_{u\in\mathbb{S}^{d-1}} \int_{h\in(-\infty,R_\lambda^2]} g(u(1-\frac{h}{R_\lambda^2}))\mathbb{E}\left[R_\lambda\xi_V^{(\lambda)}\left((\mathbf{0}, h), \mathcal{P}^{(\lambda)}\right)\right]\tilde{\phi}_\lambda(u, h)|(1-\frac{h}{R_\lambda^2})|^{d-1} dh d\sigma_{d-1}(u).$$

We obtain (2.7) by following the proof of (2.5) for the k-face functional nearly verbatim, appealing to the limit $\lim_{\lambda\to\infty}\mathbb{E}\,[R_\lambda\xi_V^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})] = \mathbb{E}\,[\xi_V^{(\infty)}((\mathbf{0}, h), \mathcal{P})]$ as given in Lemma 4.5.

The proof of the asserted variance asymptotics (2.8) for $\mathrm{Var}[\langle g_{R_\lambda}, \mu_\lambda^{\xi_V}\rangle]$, follows verbatim the proof of (2.6), replacing $\xi_k(\cdot)$ by $R_\lambda\xi_V(\cdot)$. $\square$

5.5. Proof of Corollary 1.1. Define $\xi(x, \mathcal{P})$ to be one if $x \in \mathrm{Ext}(P)$, otherwise put $\xi(x, \mathcal{P}) = 0$. Put

$$\mu_\lambda := \sum_{x\in\mathrm{Ext}(P)\cap\tilde{Q}_\lambda} \xi(x, \mathcal{P}\cap\tilde{Q}_\lambda)\delta_x.$$

Note that $\mathbb{E}\,[\mathrm{card}(\mathrm{Ext}(\mathcal{P}\cap\tilde{Q}_\lambda))] = \mathbb{E}\,[\langle 1,\mu_\lambda\rangle]$.

Then writing points in $\mathbb{R}^{d-1}\times\mathbb{R}$ as (v,h) we have

$$(\mathrm{vol}Q_\lambda)^{-1}\mathbb{E}\,[\langle 1,\mu_\lambda\rangle] = (\mathrm{vol}Q_\lambda)^{-1}\int_{Q_\lambda}\int_{-\infty}^{\infty}\mathbb{E}\,\xi((v,h),\mathcal{P}\cap\tilde{Q}_\lambda)e^h dhdv.$$

Put $\tilde{\mathcal{P}}_\lambda := \mathcal{P}\cap\tilde{Q}_\lambda$. By translation invariance of ξ we have

$$\xi((v,h),\tilde{\mathcal{P}}_\lambda) = \xi((\mathbf{0},h),\tilde{\mathcal{P}}_\lambda-(v,0))$$

that is to say

$$(\mathrm{vol}Q_\lambda)^{-1}\mathbb{E}\,[\langle 1,\mu_\lambda\rangle] = \int_{-\infty}^{\infty}\mathbb{E}\,\xi((\mathbf{0},h),\tilde{\mathcal{P}}_\lambda-(v,0))e^h dhdv.$$

The functional ξ satisfies the localization and moment conditions of those functionals in $\Xi^{(\infty)}$ and consequently we have

$$\lim_{\lambda\to\infty}\mathbb{E}\,[\xi((\mathbf{0},h),\tilde{\mathcal{P}}_\lambda-(v,0))] = \mathbb{E}\,[\xi((\mathbf{0},h),\mathcal{P})].$$

As in Lemma 4.3, we may show that $\mathbb{E}\,[\xi((\mathbf{0},h),\tilde{\mathcal{P}}_\lambda)]e^h$ is dominated by an exponentially decaying function of h, uniformly in λ. Thus by the dominated convergence theorem we get

$$\lim_{\lambda\to\infty}(\mathrm{vol}Q_\lambda)^{-1}\mathbb{E}\,[\langle 1,\mu_\lambda\rangle] = \int_{-\infty}^{\infty}\mathbb{E}\,[\xi((\mathbf{0},h),\mathcal{P}))]e^h dh.$$

To prove variance asymptotics, we argue as follows. For all $h\in\mathbb{R}$, and $(v',h')\in\mathbb{R}^{d-1}\times\mathbb{R}$, we abuse notation and put

$$c^\xi((\mathbf{0},h),(v',h'),\tilde{\mathcal{P}}_\lambda) := \mathbb{E}\,[\xi((\mathbf{0},h),\tilde{\mathcal{P}}_\lambda\cup(v',h'))\times\xi((v',h'),\tilde{\mathcal{P}}_\lambda\cup(\mathbf{0},h))]-$$
$$\mathbb{E}\,[(\xi((\mathbf{0},h),\tilde{\mathcal{P}}_\lambda)]\mathbb{E}\,[\xi((v',h'),\tilde{\mathcal{P}}_\lambda)].$$

Then we have

$$\begin{aligned}
&(\mathrm{vol}Q_\lambda)^{-1}\mathrm{Var}[\langle 1,\mu_\lambda\rangle]\\
&= (\mathrm{vol}Q_\lambda)^{-1}\int_{Q_\lambda}\int_{-\infty}^{\infty}\mathbb{E}\,\xi((v,h),\tilde{\mathcal{P}}_\lambda)e^h dhdv\\
&\quad + (\mathrm{vol}Q_\lambda)^{-1}\int_{Q_\lambda}\int_{-\infty}^{\infty}\int_{Q_\lambda-v}\int_{-\infty}^{\infty}c^\xi((\mathbf{0},h),(v',h'),\tilde{\mathcal{P}}_\lambda)e^{h+h'}dhdh'dv'dv\\
&= \int_{-\infty}^{\infty}\mathbb{E}\,\xi((v,h),\tilde{\mathcal{P}}_\lambda)e^h dhdv\\
&\quad + \int_{-\infty}^{\infty}\int_{Q_\lambda-v}\int_{-\infty}^{\infty}c^\xi((\mathbf{0},h),(v',h'),\tilde{\mathcal{P}}_\lambda)e^{h+h'}dhdh'dv'dv.
\end{aligned}$$

Now for all h, h' and v' we have

$$\lim_{\lambda\to\infty} c^\xi((\mathbf{0},h),(v',h'),\tilde{\mathcal{P}}_\lambda)e^{h+h'} = c^\xi((\mathbf{0},h),(v',h'),\mathcal{P})e^{h+h'},$$

where for all $w_1, w_2 \in \mathbb{R}^d$, we put

$$c^\xi(w_1,w_2,\mathcal{P}) := \mathbb{E}\,\xi(w_1,\mathcal{P}\cup\{w_2\})\xi(w_2,\mathcal{P}\cup\{w_1\}) - \mathbb{E}\,\xi(w_1,\mathcal{P})\mathbb{E}\,\xi(w_2,\mathcal{P}). \tag{5.14}$$

Moreover, as in the proof of Lemma 4.7, we may show that $c^\xi((\mathbf{0},h),(v',h'),\tilde{\mathcal{P}}_\lambda)e^{h+h'}$ is dominated by an exponentially decaying function of h, h', and v', uniformly in $\lambda \in [\lambda_0,\infty)$. The dominated convergence theorem shows that

$$\lim_{\lambda\to\infty}(\mathrm{vol}Q_\lambda)^{-1}\mathrm{Var}[\langle 1,\mu_\lambda\rangle] = \int_{-\infty}^{\infty}\mathbb{E}\,[\xi((\mathbf{0},h),\mathcal{P}))]e^h dh +$$

$$+\int_{-\infty}^{\infty}\int_{\mathbb{R}^{d-1}}\int_{-\infty}^{\infty} c^\xi((\mathbf{0},h),(v',h'),\mathcal{P})e^{h+h'}dhdv'dh'.$$

This concludes the proof of Corollary 1.1. □

5.6 Proof of Theorem 2.4. To prove Theorem 2.4, we follow the same method and notations as on pages 54-55 in [12]: for any k-dimensional linear subspace L and any convex set K, we denote by $K|L$ the orthogonal projection of K onto L. We consider the function $\vartheta_L(x,K) = \mathbf{1}(\{x \notin K|L\})$ and the so-called projection avoidance functional

$$\vartheta_k(x,K) = \int_{G(\mathrm{lin}[x],k)} \vartheta_L(x,K)d\nu_k^{\mathrm{lin}[x]}(L)$$

where $\mathrm{lin}[x]$ is the one-dimensional linear space spanned by x, $G(\mathrm{lin}[x],k)$ is the set of k-dimensional linear subspaces of $\mathbb{R}^d$ containing $\mathrm{lin}[x]$ and $\nu_k^{\mathrm{lin}[x]}$ is the normalized Haar measure on $G(\mathrm{lin}[x],k)$ (see (2.7) in [12]). We obtain the following rewriting of the defect intrinsic volume of K_λ:

$$V_k(B_d(0,R_\lambda)) - V_k(K_\lambda) = \frac{\binom{d-1}{k-1}}{\kappa_{d-k}}\int_{\mathbb{R}^d}[\vartheta_k(x,K_\lambda) - \vartheta_k(x,B_d(0,R_\lambda))]\frac{dx}{|x|^{d-k}}.$$

In particular, we have the decomposition :

$$V_k(B_d(0,R_\lambda)) - V_k(K_\lambda) = \sum_{x\in\mathcal{P}_\lambda}\xi_{V,k}(x,\mathcal{P}_\lambda)$$

where

$$\xi_{V,k}(x,\mathcal{P}_\lambda) = d^{-1}\frac{\binom{d-1}{k-1}}{\kappa_{d-k}}\int_{\mathcal{C}(x,\mathcal{P}_\lambda)}[\vartheta_k(y,K_\lambda) - \vartheta_k(y,B_d(0,R_\lambda))]\frac{dy}{|y|^{d-k}}$$

if x is extreme and $\xi_{V,k}(x, \mathcal{P}_\lambda) = 0$ otherwise.

We notice that the equalities (5.2) and (5.6) hold for $\xi = \xi_{V,k}$. Let us define

$$\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \xi_{V,k}([T^{(\lambda)}]^{-1}(w), \mathcal{P}_\lambda), \quad w \in \mathbb{R}^d.$$

We observe from the proof of Theorem 2.1 that it is enough to show the convergence up to a multiplicative rescaling of each of the quantities $\mathbb{E}\,[\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})]$, $\mathbb{E}\,[\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})^2]$ and $c^{\xi_{V,k}^{(\lambda)}}(w, w')$ where $w, w' \in \mathbb{R}^d$ as well as bounds similar to those in Lemmas 4.3 and 4.7.

Let us explain for instance how to show the convergence of $\mathbb{E}\,[\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})]$. We first notice that the localization radius associated with $\xi_{V,k}^{(\lambda)}$ is the same as for $\xi_V^{(\lambda)}$. We also need to introduce

$$\xi_{V,k}^{(\infty)}(w, \mathcal{P}) = d^{-1} \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\mathrm{Cyl}(w)} [\vartheta_k^\infty(y, \Phi) - \mathbf{1}(\{y \in \mathbb{R}^{d-1} \times \mathbb{R}_-\})] dy$$

where $\vartheta_k^\infty(y)$ is defined in equality (3.19) from [12]. Let us denote by m_k the measure $\frac{1}{|x|^{d-k}} dx$. Using that $T^\lambda(R_\lambda^{d+1-k} dm_k)$ converges to the Lebesgue measure of $\mathbb{R}^d$, we get a statement similar to Lemma 4.4 and we deduce from Lemma 4.1 that

$$\lim_{\lambda \to \infty} \mathbb{E}\,[R_\lambda^{d+1-k} \xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})] = \mathbb{E}\,[\xi_{V,k}^{(\infty)}(w, \mathcal{P})].$$

References

[1] F. Affentranger (1991), The convex hull of random points with spherically symmetric distributions, *Rend. Sem. Mat. Univ. Politec. Torino*, **49**, 359-383.

[2] F. Affentranger and R. Schneider (1992), Random projections of regular simplices *Discrete Comp. Geom.* **7**, 219226.

[3] S. Albeverio, S. Molchanov, D. Surgailis (1994), Stratified structure of the Universe and Burgers' equation - a probabilistic approach, *Probab. Theory Relat. Fields*, **100**, 457-484.

[4] I. Bárány, F. Fodor, and V. Vigh (2010), Intrinsic volumes of inscribed random polytopes in smooth convex bodies, *Advances in Applied Probability*, **42**, 605-619.

[5] I. Bárány and M. Reitzner (2010), The variance of random polytopes, *Advances in Mathematics*, **225**, 1986-2001.

[6] I. Bárány and Van Vu (2007), Central limit theorems for Gaussian polytopes, *Ann. Probab.*, **35**, 1593-1621.

[7] Yu. Baryshnikov (2000), Supporting-points processes and some of their applications, *Probab. Theory Relat. Fields*, **117**, 163-182.

[8] Yu. Baryshnikov and R. Vitale (1994), Regular simplices and Gaussian samples, *Discrete Comp. Geom.*, **11**, 141-147.

[9] P. Billingsley (1968), *Convergence of Probability Measures*, John Wiley, New York.

[10] J. Burgers (1974), The Nonlinear Diffusion Equation, Dordrecht, Amsterdam.

[11] P. Calka and J. E. Yukich (2013), Variance asymptotics for random polytopes in smooth convex bodies, *Prob. Theory and Related Fields*, to appear.

[12] P. Calka, T. Schreiber, and J. E. Yukich (2013), Brownian limits, local limits, and variance asymptotics for convex hulls in the ball, *Ann. Probab.*, **41**, 50-108.

[13] H. Carnal (1970), Die konvexe Hülle von n rotationssymmetrisch verteilten Punkten, *Z. Wahr. verw. Gebiete*, **15**, 168-176.

[14] D. Hug and M. Reitzner (2005), Gaussian polytopes: Variances and limit theorems, *Adv. Appl. Probab.*, **37**, 297-320.

[15] W. F. Eddy (1980), The distribution of the convex hull of a Gaussian sample, *J. Appl. Probab.* **17**, 686-695.

[16] W. F. Eddy and J. D. Gale (1981), The convex hull of a spherically symmetric sample, *Adv. Appl. Prob.* **13**, 751-763.

[17] J. Geffroy (1961), Localisation asymptotique du polyèdre d'appui d'un échantillon Laplacien à k dimensions. *Publ. Inst. Statist. Univ. Paris* **10**, 213-228.

[18] I. Hueter (1994), Limit theorems for the convex hull of random points in higher dimensions *Trans. Amer. Math. Soc.*, **351**, 4337-4363.

[19] S. A. Molchanov, D. Surgailis, W. A. Woyczynski (1995) Hyperbolic asymptotics in Burger's turbulence and extremal processes, *Comm. Math. Phys.*, **168**, 209-226.

[20] M. D. Penrose (2007), Laws of large numbers in stochastic geometry with statistical applications, *Bernoulli*, **13**, 1124-1150.

[21] H. Raynaud (1970), Sur l'enveloppe convexe des nuages de points aléatoires dans $\mathbb{R}^n$, *J. Appl. Probab.*, **7**, 35-48.

[22] M. Reitzner (2010), Random Polytopes, New Perspectives in Stochastic Geometry, Oxford University Press, 45-77.

[23] A. Rényi and R. Sulanke (1963), Über die konvexe Hülle von n zufällig gewählten Punkten. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, **2**, 75-84.

[24] A. Rényi and R. Sulanke (1964), Über die konvexe Hülle von n zufällig gewählten Punkten II. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, **3**, 138-147.

[25] S. Resnick (1987), *Extreme values, regular variation, and point processes*, Springer-Verlag.

[26] T. Schreiber and J. E. Yukich (2008), Variance asymptotics and central limit theorems for generalized growth processes with applications to convex hulls and maximal points, *Ann. Probab.*, **36**, 363-396.

[27] Ya. G. Sinai (1992), Statistics of shocks in solutions of inviscid Burgers equation, *Commun. Math. Phys.*, **148**, 601-621.

[28] W. Weil and J. A. Wieacker (1993), Stochastic geometry, in *Handbook of Convex Geometry* (P. M. Gruber and J. M. Wills, eds.), vol. B, 1391-1438, North-Holland/Elsevier, Amsterdam.

Pierre Calka, Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, Avenue de l'Université, BP.12, Technopôle du Madrillet, F76801 Saint-Etienne-du-Rouvray France; pierre.calka@univ-rouen.fr

J. E. Yukich, Department of Mathematics, Lehigh University, Bethlehem PA 18015; joseph.yukich@lehigh.edu