

# Variance asymptotics and scaling limits for Gaussian polytopes

Pierre Calka · J. E. Yukich

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**Abstract** Let  $K_n$  be the convex hull of i.i.d. random variables distributed according to the standard normal distribution on  $\mathbb{R}^d$ . We establish variance asymptotics as  $n \rightarrow \infty$  for the re-scaled intrinsic volumes and  $k$ -face functionals of  $K_n$ ,  $k \in \{0, 1, \dots, d-1\}$ , resolving an open problem (Weil and Wieacker, Handbook of Convex Geometry, vol. B, pp. 1391–1438. North-Holland/Elsevier, Amsterdam, 1993). Variance asymptotics are given in terms of functionals of germ-grain models having parabolic grains with apices at a Poisson point process on  $\mathbb{R}^{d-1} \times \mathbb{R}$  with intensity  $e^h dh dv$ . The scaling limit of the boundary of  $K_n$  as  $n \rightarrow \infty$  converges to a festoon of parabolic surfaces, coinciding with that featuring in the geometric construction of the zero viscosity solution to Burgers' equation with random input.

**Keywords** Random polytopes · Parabolic germ-grain models · Convex hulls of Gaussian samples · Poisson point processes · Burgers' equation

**Mathematics Subject Classification** Primary 60F05 · 52A20; Secondary 60D05 · 52A23

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P. Calka

Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, Avenue de l'Université, BP 12, Technopôle du Madrillet, 76801 Saint-Etienne-du-Rouvray, France  
e-mail: pierre.calka@univ-rouen.fr

J. E. Yukich (✉)

Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA  
e-mail: jey0@lehigh.edu

### 1 Main results

For all  $\lambda \in [1, \infty)$ , let  $\mathcal{P}_\lambda$  denote a Poisson point process of intensity  $\lambda\phi(x)dx$ , where

$$\phi(x) := (2\pi)^{-d/2} \exp\left(-\frac{|x|^2}{2}\right)$$

is the standard normal density on  $\mathbb{R}^d$ ,  $d \geq 2$ . Let  $\mathcal{X}_n := \{X_1, \dots, X_n\}$ , where  $X_i$  are i.i.d. with density  $\phi(\cdot)$ . Let  $K_\lambda$  and  $K_n$  be the Gaussian polytopes defined by the convex hull of  $\mathcal{P}_\lambda$  and  $\mathcal{X}_n$ , respectively. The number of  $k$ -faces of  $K_\lambda$  and  $K_n$  are denoted by  $f_k(K_\lambda)$  and  $f_k(K_n)$ , respectively.

In  $d = 2$ , Rényi and Sulanke [20, 21] determined  $\mathbb{E} f_1(K_n)$  and later Raynaud [18] determined  $\mathbb{E} f_{d-1}(K_n)$  for all dimensions. Subsequently, work of Affentranger and Schneider [2] and Baryshnikov and Vitale [6] yielded the general formula

$$\mathbb{E} f_k(K_n) = \frac{2^d}{\sqrt{d}} \binom{d}{k+1} \beta_{k,d-1} (\pi \log n)^{(d-1)/2} (1 + o(1)), \tag{1.1}$$

with  $k \in \{0, \dots, d - 1\}$  and where  $\beta_{k,d-1}$  is the internal angle of a regular  $(d - 1)$ -simplex at one of its  $k$ -dimensional faces. Concerning the volume functional, Affentranger [1] showed that its expectation asymptotics satisfy

$$\mathbb{E} \text{Vol}(K_n) = \kappa_d (2 \log n)^{d/2} (1 + o(1)), \tag{1.2}$$

where  $\kappa_d := \pi^{d/2} / \Gamma(1 + d/2)$  denotes the volume of the  $d$ -dimensional unit ball.

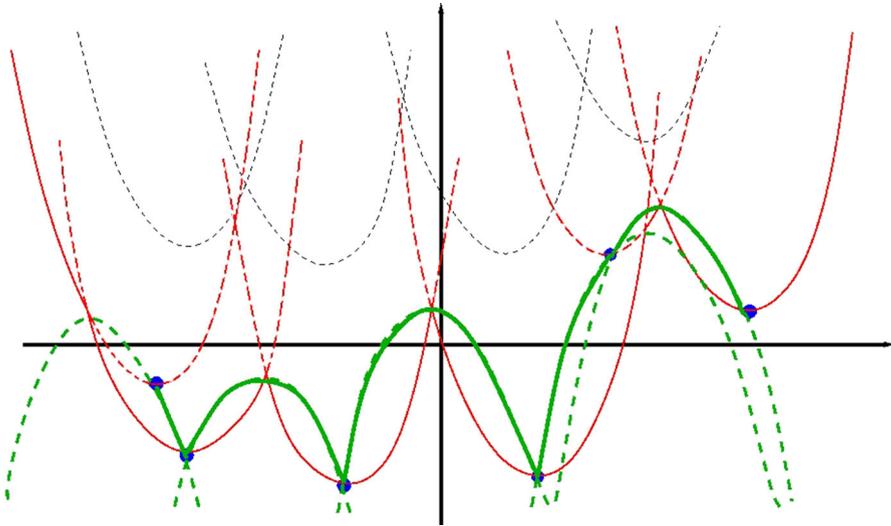
In a remarkable paper, Bárány and Vu [4] use dependency graph methods to establish rates of normal convergence for  $f_k(K_n)$  and  $\text{Vol}(K_n)$ ,  $k \in \{0, \dots, d - 1\}$ . A key part of their work involves obtaining sharp lower bounds for  $\text{Var} f_k(K_n)$  and  $\text{VarVol}(K_n)$ . Their results stop short of determining precise variance asymptotics for  $f_k(K_n)$  and  $\text{Vol}(K_n)$  as  $n \rightarrow \infty$ , an open problem going back to the 1993 survey of Weil and Wieacker (p. 1431 of [25]). We resolve this problem in Theorems 1.3 and 1.4, expressing variance asymptotics in terms of scaling limit functionals of parabolic germ-grain models.

Let  $\mathcal{P}$  be the Poisson point process on  $\mathbb{R}^{d-1} \times \mathbb{R}$  with intensity

$$d\mathcal{P}((v, h)) := e^h dh dv, \quad \text{with } (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}. \tag{1.3}$$

Let  $\Pi^\downarrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}, h \leq -|v|^2/2\}$  and for  $w := (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$  we put  $\Pi^\downarrow(w) := w \oplus \Pi^\downarrow$ , where  $\oplus$  denotes Minkowski addition. The maximal union of parabolic grains  $\Pi^\downarrow(w)$ ,  $w \in \mathbb{R}^{d-1} \times \mathbb{R}$ , whose interior contains no point of  $\mathcal{P}$  is

$$\Phi(\mathcal{P}) := \bigcup_{\substack{w \in \mathbb{R}^{d-1} \times \mathbb{R} \\ \mathcal{P} \cap \text{int}(\Pi^\downarrow(w)) = \emptyset}} \Pi^\downarrow(w).$$



**Fig. 1** The point process  $\text{Ext}(\mathcal{P})$  (blue); the boundary of the germ-grain model  $\partial(\Psi(\mathcal{P}))$  (red); the Burgers' festoon  $\partial(\Phi(\mathcal{P}))$  (green). Points which are not extreme are at apices of gray parabolas (color figure online)

Notice that  $\partial\Phi(\mathcal{P})$  is a union of inverted parabolic surfaces. Remove points of  $\mathcal{P}$  not belonging to  $\partial\Phi(\mathcal{P})$  and call the resulting thinned point set  $\text{Ext}(\mathcal{P})$ . See Fig. 1.

We show that the re-scaled configuration of extreme points in  $\mathcal{P}_\lambda$  (and in  $\mathcal{X}_n$ ) converges to  $\text{Ext}(\mathcal{P})$  and that the scaling limit  $\partial K_\lambda$  as  $\lambda \rightarrow \infty$  (and of  $\partial K_n$  as  $n \rightarrow \infty$ ) coincides with  $\partial\Phi(\mathcal{P})$ . Curiously, this boundary features in the geometric construction of the zero-viscosity solution of Burgers' equation [8]. We consequently obtain a closed form expression for expectation and variance asymptotics for the number of shocks in the solution of the inviscid Burgers' equation, adding to [5].

Fix  $u_0 := (0, 0, \dots, 1) \in \mathbb{R}^d$  and let  $T_{u_0}$  denote the tangent space to the unit sphere  $\mathbb{S}^{d-1}$  at  $u_0$ . The exponential map  $\exp := \exp_{d-1} : T_{u_0} \rightarrow \mathbb{S}^{d-1}$  maps a vector  $v$  of  $T_{u_0}$  to the point  $u \in \mathbb{S}^{d-1}$  such that  $u$  lies at the end of the geodesic of length  $|v|$  starting at  $u_0$  and having initial direction  $v$ . Here and elsewhere  $|\cdot|$  denotes Euclidean norm.

For all  $\lambda \in [1, \infty)$  put

$$R_\lambda := \sqrt{2 \log \lambda - \log(2 \cdot (2\pi)^d \cdot \log \lambda)}. \tag{1.4}$$

Choose  $\lambda_0$  so that for  $\lambda \in [\lambda_0, \infty)$  we have  $R_\lambda \in [1, \infty)$ . Let  $\mathbb{B}_{d-1}(\pi)$  be the closed Euclidean ball of radius  $\pi$  and centered at the origin in the tangent space of  $\mathbb{S}^{d-1}$  at the point  $u_0$ . It is also the closure of the injectivity region of  $\exp_{d-1}$ , i.e.  $\exp(\mathbb{B}_{d-1}(\pi)) = \mathbb{S}^{d-1}$ . For  $\lambda \in [\lambda_0, \infty)$ , define the scaling transform  $T^{(\lambda)} : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1} \times \mathbb{R}$  by

$$T^{(\lambda)}(x) := \left( R_\lambda \exp_{d-1}^{-1} \frac{x}{|x|}, R_\lambda^2 \left( 1 - \frac{|x|}{R_\lambda} \right) \right), \quad x \in \mathbb{R}^d. \tag{1.5}$$

Here  $\exp^{-1}(\cdot)$  is the inverse exponential map, which is well defined on  $\mathbb{S}^{d-1} \setminus \{-u_0\}$  and which takes values in the injectivity region  $\mathbb{B}_{d-1}(\pi)$ . For formal completeness, on the ‘missing’ point  $-u_0$  we let  $\exp^{-1}$  admit an arbitrary value, say  $(0, 0, \dots, \pi)$ , and likewise we put  $T^{(\lambda)}(\mathbf{0}) := (\mathbf{0}, R_\lambda^2)$ , where  $\mathbf{0}$  denotes either the origin of  $\mathbb{R}^{d-1}$  or  $\mathbb{R}^d$ , according to the context.

Postponing the heuristics behind  $T^{(\lambda)}$  until Sect. 3, we state our main results.

**Theorem 1.1** *Under the transformations  $T^{(\lambda)}$  and  $T^{(n)}$ , the extreme points of the convex hull of the respective Gaussian samples  $\mathcal{P}_\lambda$  and  $\mathcal{X}_n$  converge in distribution to the thinned process  $\text{Ext}(\mathcal{P})$  as  $\lambda \rightarrow \infty$  (respectively, as  $n \rightarrow \infty$ ).*

Let  $B_d(v, r)$  be the closed  $d$ -dimensional Euclidean ball centered at  $v \in \mathbb{R}^d$  and with radius  $r \in (0, \infty)$ .  $\mathcal{C}(B_d(v, r))$  is the space of continuous functions on  $B_d(v, r)$  equipped with the supremum norm.

**Theorem 1.2** *Fix  $L \in (0, \infty)$ . As  $\lambda \rightarrow \infty$ , the re-scaled boundary  $T^{(\lambda)}(\partial K_\lambda)$  converges in probability to  $\partial(\Phi(\mathcal{P}))$  in the space  $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$ .*

In a companion paper we shall show that  $\partial(\Phi(\mathcal{P}))$  is also the scaling limit of the boundary of the convex hull of i.i.d. points in polytopes. In  $d = 2$ , the reflection of  $\partial(\Phi(\mathcal{P}))$  about the  $x$ -axis describes a festoon of parabolic arcs featuring in the geometric construction of the zero viscosity solution ( $\mu = 0$ ) to Burgers’ equation

$$\frac{\partial v}{\partial t} + (v, \nabla)v = \mu \Delta v, \quad v = v(t, x), \quad t > 0, \quad (x, t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+, \quad (1.6)$$

subject to Gaussian initial conditions [17]; see Remark (i) below. Given its prominence in the asymptotics of Burgers’ equation and its role in scaling limits of boundaries of random polytopes, we shall henceforth refer to  $\partial(\Phi(\mathcal{P}))$  as *the Burgers’ festoon*.

The transformation  $T^{(\lambda)}$  induces scaling limit  $k$ -face and volume functionals governing the large  $\lambda$  behavior of convex hull functionals, as seen in the next results. These scaling limit functionals are used in the description of the variance asymptotics for the  $k$ -face and volume functionals,  $k \in \{0, 1, \dots, d - 1\}$ .

**Theorem 1.3** *For all  $k \in \{0, 1, \dots, d - 1\}$ , there exists a constant  $F_{k,d} \in (0, \infty)$ , defined in terms of averages of covariances of a scaling limit  $k$ -face functional on  $\mathcal{P}$ , such that*

$$\lim_{\lambda \rightarrow \infty} (2 \log \lambda)^{-(d-1)/2} \text{Var } f_k(K_\lambda) = F_{k,d} \quad (1.7)$$

and

$$\lim_{n \rightarrow \infty} (2 \log n)^{-(d-1)/2} \text{Var } f_k(K_n) = F_{k,d}. \quad (1.8)$$

**Theorem 1.4** *There exists a constant  $V_d \in (0, \infty)$ , defined in terms of averages of covariances of a scaling limit volume functional on  $\mathcal{P}$ , such that*

$$\lim_{\lambda \rightarrow \infty} (2 \log \lambda)^{-(d-3)/2} \text{Var Vol}(K_\lambda) = V_d \quad (1.9)$$

and

$$\lim_{n \rightarrow \infty} (2 \log n)^{-(d-3)/2} \text{Var Vol}(K_n) = V_d. \tag{1.10}$$

We also have

$$\kappa_d^{-1} (2 \log \lambda)^{-d/2} \mathbb{E} \text{Vol}(K_\lambda) = 1 - \frac{d \log(\log \lambda)}{4 \log \lambda} + O\left(\frac{1}{\log \lambda}\right). \tag{1.11}$$

The thinned point set  $\text{Ext}(\mathcal{P})$  features in the description of asymptotic solutions to Burgers’ equation (cf. Remark (i) below) and we next consider its limit theory with respect to the sequence of cylindrical windows  $Q_\lambda := [-\frac{1}{2}\lambda^{1/(d-1)}, \frac{1}{2}\lambda^{1/(d-1)}]^{d-1} \times \mathbb{R}$  as  $\lambda \rightarrow \infty$ . The next result, a by-product of our general methods, yields variance and expectation asymptotics for the number of points in  $\text{Ext}(\mathcal{P})$  over growing windows, adding to [5].

**Corollary 1.1** *There exist constants  $E_d$  and  $N_d \in (0, \infty)$ , defined respectively in terms of averages of means and covariances of a thinning functional on  $\mathcal{P}$ , such that*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} \mathbb{E} [\text{card}(\text{Ext}(\mathcal{P} \cap Q_\lambda))] = E_d \tag{1.12}$$

and

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var}[\text{card}(\text{Ext}(\mathcal{P} \cap Q_\lambda))] = N_d. \tag{1.13}$$

In particular,  $N_d = F_{0,d}$ .

For  $k \in \{1, \dots, d - 1\}$ , we denote by  $V_k(K_\lambda)$  the  $k$ th intrinsic volume of  $K_\lambda$ . In [16], Hug and Reitzner establish expectation asymptotics for  $V_k(K_\lambda)$  as well as an upper-bound for its variance. The analog of Theorem 1.4 holds for  $V_k(K_\lambda)$ , as shown by the next result, proved in Sect. 5.

**Theorem 1.5** *There exists a constant  $v_k \in [0, \infty)$ , defined in terms of averages of covariances of a scaling limit (intrinsic) volume functional on  $\mathcal{P}$ , such that*

$$\lim_{\lambda \rightarrow \infty} (2 \log \lambda)^{-k+(d+3)/2} \text{Var} V_k(K_\lambda) = v_k. \tag{1.14}$$

Moreover, we have

$$\frac{\kappa_{d-k}}{\binom{d}{k} \kappa_d} (2 \log \lambda)^{-k/2} \mathbb{E} V_k(K_\lambda) = 1 - \frac{k \log(\log \lambda)}{4 \log \lambda} + O\left(\frac{1}{\log \lambda}\right).$$

The limit (1.14) improves upon Theorem 1.2 in [16] which shows that  $(\log \lambda)^{-(k-3)/2} \text{Var} V_k(K_\lambda)$  is bounded. In [19], Reitzner remarks ‘it seems that these upper bounds are not best possible’. We are unable to show that the limits  $v_k$ ,  $1 \leq k \leq (d - 1)$ ,

are non-vanishing, that is to say we are unable to show optimality of our bounds. In particular,  $\text{Var}V_k(K_\lambda)$  goes to infinity for  $k > (d + 3)/2$  as soon as  $v_k \neq 0$ .

There are several ways in which this paper differs from [9], which considers functionals of convex hulls on i.i.d. uniform points in  $B_d(\mathbf{0}, 1)$ . First, as the extreme points of a Gaussian sample are concentrated in the vicinity of the critical sphere  $\partial B_d(\mathbf{0}, R_\lambda)$ , we need to calibrate the scaling transform  $T^{(\lambda)}$  accordingly. Second, the Gaussian sample  $\mathcal{P}_\lambda$ , when transformed by  $T^{(\lambda)}$ , converges to a non-homogenous limit point process  $\mathcal{P}$ , which is carried by the whole of  $\mathbb{R}^{d-1} \times \mathbb{R}$ . This contrasts with [9], where the limit point process is simpler in that it is homogeneous and confined to the upper half-space. The non-uniformity of  $\mathcal{P}$ , together with its larger domain, induce spatial dependencies between the re-scaled functionals which are themselves non-uniform, at least with respect to height coordinates. The description of these dependencies is made explicit and may be modified to describe the simpler dependencies of [9]. Non-uniformity of spatial dependencies leads to moment bounds for re-scaled  $k$ -face and volume functionals which are also non-uniform. Third, the scaling limit of the boundary of the Gaussian sample converges to a festoon of parabolic surfaces, coinciding with that given by the geometric solution to Burgers' equation with random input. This correspondence, described more precisely below, merits further investigation as it suggests that some aspects of the convex hull geometry are captured by a stochastic partial differential equation.

*Remarks (i) Burgers' equation.* Let  $\text{Ext}(\mathcal{P})'$  be the reflection of  $\text{Ext}(\mathcal{P})$  about the hyperplane  $\mathbb{R}^{d-1}$ . The point process  $\text{Ext}(\mathcal{P})'$  features in the solution to Burgers' equation (1.6) for  $\mu \in (0, \infty)$  as well as for  $\mu = 0$  (inviscid limit).

When  $\mu = 0$ ,  $d = 2$ , and when the initial conditions are specified by a stationary Gaussian process  $\eta$  having covariance  $\mathbb{E} \eta(\mathbf{0})\eta(x) = o(1/\log x)$ ,  $x \rightarrow \infty$ , the re-scaled local maximum of the solutions converge in distribution to  $\text{Ext}(\mathcal{P})'$  [17]. The abscissas of points in  $\text{Ext}(\mathcal{P})'$  correspond to zeros of the limit velocity process  $v(L^2t, L^2x)$ , as  $L \rightarrow \infty$  (here the initial condition is re-scaled in terms of  $L$ , not  $L^2$ ). See Figure 1 in [17] as well as Figure 13 in the seminal work of Burgers [8]. The shocks in the limit velocity process coincide with the local minima of the festoon  $\partial(\Phi(\tilde{\mathcal{P}}))$ , which are themselves the scaling limit of the projections of the origin onto the hyperplanes containing the hyperfaces of  $K_\lambda$ . By (1.5), when  $d = 2$ , the typical angular difference between consecutive extreme points of  $K_\lambda$ , after scaling by  $R_\lambda$ , converges in probability to the typical distance between abscissas of points in  $\text{Ext}(\mathcal{P})'$ . Thus the re-scaled angular increments between consecutive extreme points in  $K_\lambda$  behave like the spacings between zeros of the zero-viscosity solution to (1.6).

In the case  $\mu \in (0, \infty)$ , the point set  $\text{Ext}(\mathcal{P})'$  is shown to be the scaling limit as  $t \rightarrow \infty$  of centered and re-scaled local maxima of the solutions to Burgers' equation (1.6) when the initial conditions are specified by degenerate shot noise with Poissonian spatial locations; see Theorem 9 and Remark 3 of [3]. Correlation functions for  $\text{Ext}(\mathcal{P})'$  are given in section 5 of [3].

(ii) *Theorems 1.1 and 1.2-related work.* In 1961, Geffroy [14] states that the Hausdorff distance between  $K_n$  and  $B_d(\mathbf{0}, \sqrt{2 \log n})$  converges almost surely to zero. From [4] we also know that the extreme points of the polytope  $K_\lambda$  concentrate around the sphere  $R_\lambda \mathbb{S}^{d-1}$  with high probability. Theorems 1.1–1.2 add to these results, showing conver-

gence of the measure induced by the re-scaled extreme points as well as convergence of the re-scaled boundary.

(iii) *Theorem 1.3-related work.* As mentioned, Bárány and Vu [4] show that  $(\text{Var } f_k(K_n))^{-1/2}(f_k(K_n) - \mathbb{E} f_k(K_n))$  converges to a normal random variable as  $n \rightarrow \infty$ . They also show (Theorem 6.3 of [4]) that  $\text{Var } f_k(K_n) = \Omega((\log n)^{(d-1)/2})$ . These bounds are sharp, as Hug and Reitzner [16] had previously showed that  $\text{Var } f_k(K_n) = O((\log n)^{(d-1)/2})$ . Aside from these variance bounds and work of Hueter [15], asserting that  $\text{Var } f_0(K_n) = c(\log n)^{(d-1)/2} + o(1)$ , the second order issues raised by Weil and Wieacker [25] have largely remained unsettled in the case of Gaussian input. In particular the question of showing

$$\text{Var } f_k(K_n) = c(\log n)^{(d-1)/2}(1 + o(1))$$

for  $k \in \{1, \dots, d - 1\}$  has remained open. On page 298 of [16], Hug and Reitzner, commenting on the likelihood of progress, remarked that ‘Most probably it is difficult to establish such a precise limit relation...’. Theorem 1.3 addresses these issues.

(iv) *Theorem 1.4-related work.* Hug and Reitzner [16] show  $\text{VarVol}(K_n) = O((\log n)^{(d-3)/2})$  and later Bárány and Vu [4] show that  $\text{VarVol}(K_n) = \Theta((\log n)^{(d-3)/2})$ . The asymptotics (1.9) and (1.10) turn these bounds into precise limits. The equivalence (1.11) improves upon (1.2) in the setting of Poisson input.

(v) *Corollary 1.1-related work.* Baryshnikov [5] establishes the asymptotic normality of  $\text{card}(\text{Ext}(\mathcal{P}) \cap Q_\lambda)$  as  $\lambda \rightarrow \infty$ , obtaining expectation and variance asymptotics in Theorem 1.9.2 of [5]. Notice that  $\text{Ext}(\mathcal{P}) \cap Q_\lambda$  restricts extreme points in  $\mathcal{P}$  to  $Q_\lambda$ , whereas  $\text{Ext}(\mathcal{P} \cap Q_\lambda)$  are the extreme points in  $\mathcal{P} \cap Q_\lambda$ , which in general is not the same set, by boundary effects. Baryshnikov left open the question of obtaining explicit limits, remarking that ‘the question of constants is quite tricky’; see p. 180 of *ibid*.

In general, if a point process  $\mathcal{P}_\infty$  is a scaling limit to the solution of (1.6), then  $\text{card}(\mathcal{P}_\infty \cap Q_\lambda)$  coincides with the number of Voronoi cells generated by the abscissas of points in  $\mathcal{P}_\infty \cap Q_\lambda$ ; under conditions on the viscosity and initial input, such cells model the matterless voids in the Universe [3, 5, 17].

(vi) *Goodman-Pollack model.* In view of the Goodman-Pollack model for Gaussian polytopes, it is well-known [2, 6, 16, 19] that asymptotics for functionals of  $K_n$  admit counterparts for functionals of the orthogonal projection of randomly rotated regular simplices in  $\mathbb{R}^{n-1}$ . The proof of (1.1), as given in [2], is actually formulated as a limit result for the Goodman-Pollack model. Theorems 1.3 and 1.4 may be likewise cast in terms of variances of projections of high-dimensional random simplices. For more on the Goodman-Pollack model and its applications to coding theory, see [16, 19].

This paper is organized as follows. Section 2 introduces scaling limit functionals of germ-grain models having parabolic grains. These scaling limit functionals appear in a general theorem which extends and refines Theorems 1.3 and 1.4. In particular the limit constants in Theorems 1.3 and 1.4 are seen to be the averages of scaling limit functionals on parabolic germ-grain models carried by the infinite non-homogenous input  $\mathcal{P}$ . Section 3 shows for each  $\lambda \in [1, \infty)$  that the scaling transform  $T^{(\lambda)}$  maps the Euclidean convex hull geometry into ‘nearly’ parabolic convex geometry, which in the limit  $\lambda \rightarrow \infty$  becomes parabolic convex geometry. We show that the image of  $\mathcal{P}_\lambda$  under  $T^{(\lambda)}$  converges in distribution to  $\mathcal{P}$  and that  $T^{(\lambda)}$  defines re-scaled  $k$ -face

and volume functionals. Section 4 establishes that the re-scaled  $k$ -face and volume functionals localize in space, which is crucial to showing the convergence of their means and covariances to the respective means and covariances of their scaling limits. Finally Sect. 5 provides the proofs of the main results.

## 2 Parabolic germ-grain models and a general result

In this section we define scaling limit functionals of germ-grain models and we use their second order correlations to precisely define the limit constants  $F_{k,d}$  and  $V_d$  in (1.7) and (1.9), respectively. We use the scaling limit functionals to establish variance asymptotics for the empirical measures induced by the  $k$ -face and volume functionals, thereby extending Theorems 1.3 and 1.4. Denote points in  $\mathbb{R}^{d-1} \times \mathbb{R}$  by  $w := (v, h)$ .

### 2.1 Parabolic germ-grain models

Let

$$\Pi^\uparrow := \left\{ (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}^+, h \geq \frac{|v|^2}{2} \right\}.$$

Let  $\Pi^\uparrow(w) := w \oplus \Pi^\uparrow$ . The point set  $\mathcal{P}$  generates a germ-grain model of paraboloids

$$\Psi(\mathcal{P}) := \bigcup_{w \in \mathcal{P}} \Pi^\uparrow(w).$$

A point  $w_0 \in \mathcal{P}$  is *extreme* with respect to  $\Psi(\mathcal{P})$  if the grain  $\Pi^\uparrow(w_0)$  is not a subset of the union of the grains  $\Pi^\uparrow(w)$ ,  $w \in \mathcal{P} \setminus \{w_0\}$ . See Fig 1. It may be verified that the extreme points from this construction coincide with  $\text{Ext}(\mathcal{P})$ , see e.g. section 3 of [9].

### 2.2 Empirical $k$ -face and volume measures

Given a finite point set  $\mathcal{X} \subset \mathbb{R}^d$ , let  $\text{co}(\mathcal{X})$  be its convex hull

**Definition 2.1** Given  $k \in \{0, 1, \dots, d - 1\}$  and  $x$  a vertex of  $\text{co}(\mathcal{X})$ , define the  $k$ -face functional  $\xi_k(x, \mathcal{X})$  to be the product of  $(k + 1)^{-1}$  and the number of  $k$ -faces of  $\text{co}(\mathcal{X})$  which contain  $x$ . Otherwise we put  $\xi_k(x, \mathcal{X}) = 0$ . Thus the total number of  $k$ -faces in  $\text{co}(\mathcal{X})$  is  $\sum_{x \in \mathcal{X}} \xi_k(x, \mathcal{X})$ . Letting  $\delta_x$  be the unit point mass at  $x$ , the empirical  $k$ -face measure for  $\mathcal{P}_\lambda$  is

$$\mu_\lambda^{\xi_k} := \sum_{x \in \mathcal{P}_\lambda} \xi_k(x, \mathcal{P}_\lambda) \delta_x. \tag{2.1}$$

Let  $\mathcal{F}(x, \mathcal{P}_\lambda)$  be the collection of  $(d - 1)$ -dimensional faces in  $K_\lambda$  which contain  $x$  and let  $\text{cone}(x, \mathcal{P}_\lambda) := \{ry, r > 0, y \in \mathcal{F}(x, \mathcal{P}_\lambda)\}$  be the cone generated by  $\mathcal{F}(x, \mathcal{P}_\lambda)$ .

**Definition 2.2** Given  $x$  a vertex of  $\text{co}(\mathcal{P}_\lambda)$ , define the defect volume functional

$$\xi_V(x, \mathcal{P}_\lambda) := d^{-1} R_\lambda [\text{Vol}(\text{cone}(x, \mathcal{P}_\lambda) \cap B_d(\mathbf{0}, R_\lambda)) - \text{Vol}(\text{cone}(x, \mathcal{P}_\lambda) \cap K_\lambda)].$$

When  $x$  is not a vertex of  $\text{co}(\mathcal{P}_\lambda)$ , we put  $\xi_V(x, \mathcal{P}_\lambda) = 0$ . The empirical defect volume measure is

$$\mu_\lambda^{\xi_V} := \sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda) \delta_x. \tag{2.2}$$

Thus the total defect volume of  $K_\lambda$  with respect to the ball  $B_d(\mathbf{0}, R_\lambda)$  is given by  $R_\lambda^{-1} \sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda)$ .

### 2.3 Scaling limit $k$ -face and volume functionals

A set of  $(k + 1)$  extreme points  $\{w_1, \dots, w_{k+1}\} \subset \text{Ext}(\mathcal{P})$ , generates a  $k$ -dimensional parabolic face of the Burgers' festoon  $\partial(\Phi(\mathcal{P}))$  if there exists a translate  $\tilde{\Pi}^\downarrow$  of  $\Pi^\downarrow$  such that  $\{w_1, \dots, w_{k+1}\} = \tilde{\Pi}^\downarrow \cap \text{Ext}(\mathcal{P})$ . When  $k = d - 1$  the parabolic face is a hyperface.

**Definition 2.3** Define the scaling limit  $k$ -face functional  $\xi_k^{(\infty)}(w, \mathcal{P})$ ,  $k \in \{0, 1, \dots, d - 1\}$ , to be the product of  $(k + 1)^{-1}$  and the number of  $k$ -dimensional parabolic faces of the Burgers' festoon  $\partial(\Phi(\mathcal{P}))$  which contain  $w$ , if  $w \in \text{Ext}(\mathcal{P})$  and zero otherwise.

**Definition 2.4** Define the scaling limit defect volume functional  $\xi_V^{(\infty)}(w, \mathcal{P})$ ,  $w \in \text{Ext}(\mathcal{P})$ , by

$$\xi_V^{(\infty)}(w, \mathcal{P}) := d^{-1} \int_{\text{Cyl}(w)} \partial(\Phi(\mathcal{P}))(v) dv,$$

where  $\text{Cyl}(w)$  denotes the projection onto  $\mathbb{R}^{d-1}$  of the hyperfaces of  $\partial(\Phi(\mathcal{P}))$  containing  $w$ . Otherwise, when  $w \notin \text{Ext}(\mathcal{P})$  we put  $\xi_V^{(\infty)}(w, \mathcal{P}) = 0$ .

One of the main features of our approach is that  $\xi_k^{(\infty)}$ ,  $k \in \{0, 1, \dots, d - 1\}$ , are scaling limits of re-scaled  $k$ -face functionals, as defined in Sect. 3.3. A similar statement holds for  $\xi_V^{(\infty)}$ . Lemma 4.6 makes these assertions precise. Let  $\Xi$  denote the collection of functionals  $\xi_k$ ,  $k \in \{0, 1, \dots, d - 1\}$ , together with  $\xi_V$ . Let  $\Xi^{(\infty)}$  denote the collection of scaling limits  $\xi_k^{(\infty)}$ ,  $k \in \{0, 1, \dots, d - 1\}$ , together with  $\xi_V^{(\infty)}$ .

### 2.4 Limit theory for empirical $k$ -face and volume measures

Define the following second order correlation functions for  $\xi^{(\infty)} \in \Xi^{(\infty)}$ .

**Definition 2.5** For all  $w_1, w_2 \in \mathbb{R}^d$  and  $\xi^{(\infty)} \in \Xi^{(\infty)}$  put

$$\begin{aligned}
 c^{\xi^{(\infty)}}(w_1, w_2) &:= c^{\xi^{(\infty)}}(w_1, w_2, \mathcal{P}) \\
 &:= \mathbb{E} \xi^{\xi^{(\infty)}}(w_1, \mathcal{P} \cup \{w_2\}) \xi^{\xi^{(\infty)}}(w_2, \mathcal{P} \cup \{w_1\}) - \mathbb{E} \xi^{\xi^{(\infty)}}(w_1, \mathcal{P}) \mathbb{E} \xi^{\xi^{(\infty)}}(w_2, \mathcal{P})
 \end{aligned}
 \tag{2.3}$$

and

$$\begin{aligned}
 \sigma^2(\xi^{(\infty)}) &:= \int_{-\infty}^{\infty} \mathbb{E} \xi^{\xi^{(\infty)}}((\mathbf{0}, h_0), \mathcal{P})^2 e^{h_0} dh_0 \\
 &\quad + \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^{\xi^{(\infty)}}((\mathbf{0}, h_0), (v_1, h_1)) e^{h_0+h_1} dh_1 dv_1 dh_0.
 \end{aligned}
 \tag{2.4}$$

Theorem 1.3 is a special case of a general result expressing the asymptotic behavior of the empirical  $k$ -face and volume measures in terms of scaling limit functionals  $\xi_k^{(\infty)}$  of parabolic germ-grain models. Let  $\mathcal{C}(\mathbb{S}^{d-1})$  be the class of bounded functions on  $\mathbb{R}^d$  whose set of continuity points includes  $\mathbb{S}^{d-1}$ . Given  $g \in \mathcal{C}(\mathbb{S}^{d-1})$ , let  $g_r(x) := g(x/r)$  and let  $\langle g, \mu_\lambda^\xi \rangle$  denote the integral of  $g$  with respect to  $\mu_\lambda^\xi$ . Let  $\sigma_{d-1}$  be the  $(d - 1)$ -dimensional surface measure on  $\mathbb{S}^{d-1}$ . The following is proved in Sect. 5.

**Theorem 2.1** For all  $\xi \in \Xi$  and  $g \in \mathcal{C}(\mathbb{S}^{d-1})$  we have

$$\begin{aligned}
 &\lim_{\lambda \rightarrow \infty} (2 \log \lambda)^{-(d-1)/2} \mathbb{E} [\langle g_{R_\lambda}, \mu_\lambda^\xi \rangle] \\
 &= \int_{-\infty}^{\infty} \mathbb{E} \xi^{\xi^{(\infty)}}((\mathbf{0}, h_0), \mathcal{P}) e^{h_0} dh_0 \int_{\mathbb{S}^{d-1}} g(u) d\sigma_{d-1}(u)
 \end{aligned}
 \tag{2.5}$$

and

$$\lim_{\lambda \rightarrow \infty} (2 \log \lambda)^{-(d-1)/2} \text{Var}[\langle g_{R_\lambda}, \mu_\lambda^\xi \rangle] = \sigma^2(\xi^{(\infty)}) \int_{\mathbb{S}^{d-1}} g(u)^2 d\sigma_{d-1}(u) \in (0, \infty).
 \tag{2.6}$$

*Remarks* (i) *Deducing Theorems 1.3 and 1.4 from Theorem 2.1.* Setting  $\xi$  to be  $\xi_k$ , the convergence (1.7) is implied by (2.6) with  $F_{k,d} = \sigma^2(\xi_k^{(\infty)}) \cdot d\kappa_d$ , with  $d\kappa_d = d\pi^{d/2} / \Gamma(1 + d/2)$  being the surface area of the unit sphere. Indeed, applying (2.6) to  $g \equiv 1$ , we have

$$\langle 1, \mu_\lambda^{\xi_k} \rangle = \sum_{x \in \mathcal{P}_\lambda} \xi_k(x, \mathcal{P}_\lambda) = f_k(K_\lambda).$$

Likewise, putting  $g \equiv 1$  in (2.6), setting  $\xi$  to be  $\xi_V$ , and recalling that  $\xi_V$  incorporates an extra factor of  $R_\lambda$ , we get the convergence (1.9), with  $V_d := \sigma^2(\xi_V^{(\infty)}) \cdot d\kappa_d$ .

To obtain (1.11), put  $g \equiv 1$  in (2.5) and set  $\xi \equiv \xi_V$  to get  $(2 \log \lambda)^{-d/2} \mathbb{E} [\text{Vol}(B_d(\mathbf{0}, R_\lambda)) - \text{Vol}(K_\lambda)] = O(R_\lambda^{-1} (\log \lambda)^{-1/2}) = O((\log \lambda)^{-1})$ . We have

$$R_\lambda = \sqrt{2 \log \lambda} - \frac{\sqrt{2} \log(\log \lambda)}{4\sqrt{\log \lambda}} + O\left(\frac{1}{\sqrt{\log \lambda}}\right)$$

which gives  $(2 \log \lambda)^{-d/2} R_\lambda^d = 1 - d \log \log \lambda / (4 \log \lambda) + O((\log \lambda)^{-1})$  and thus (1.11) holds. When  $\xi$  is set to  $\xi_V$ , we are unable to show that the right side of (2.5) is non-zero, that is we are unable to show  $(2 \log \lambda)^{-d/2} \mathbb{E} [\text{Vol}(B_d(\mathbf{0}, R_\lambda)) - \text{Vol}(K_\lambda)] = \Omega((\log \lambda)^{-1})$ .

The de-Poissonized limit (1.10) follows from the coupling of binomial and Poisson points used in Bárány and Vu [4], in particular Lemma 8.1 of [4]. The limit (1.8) similarly follows from (1.7) and the same coupling, as described in Section 13.2 of [4].

(ii) *Central limit theorems.* Combining (2.6) with the results of [4] shows the following central limit theorem, as  $\lambda \rightarrow \infty$ :

$$(2 \log \lambda)^{-(d-1)/2} (\langle g_{R_\lambda}, \mu_{\lambda}^{\xi_k} \rangle - \mathbb{E} \langle g_{R_\lambda}, \mu_{\lambda}^{\xi_k} \rangle) \xrightarrow{\mathcal{D}} N(0, \sigma^2), \tag{2.7}$$

where  $N(0, \sigma^2)$  denotes a mean zero normal random variable with variance  $\sigma^2 := \sigma^2(\xi_k^{(\infty)}) \int_{\mathbb{S}^{d-1}} g(u)^2 du$ . Alternatively, using the localization of functionals  $\xi \in \Xi$ , as described in Sect. 4, together with standard stabilization methods as in [9], we obtain another proof of (2.7).

### 2.5 Further extensions

(i) *Brownian limits.* Following the scaling methods of this paper and by appealing to the methods of section 8 of [9] we may deduce that the process given as the integrated version of the defect volume converges to a Brownian sheet process. This goes as follows. For  $\mathcal{X} \subset \mathbb{R}^d$  and  $u \in \mathbb{S}^{d-1}$  we put

$$r(u, \mathcal{X}) := R_\lambda - \sup\{\rho > 0 : \rho u \in \text{co}(\mathcal{X})\}$$

and for all  $\lambda \in [\lambda_0, \infty)$  let  $r_\lambda(u) := r(u, \mathcal{P}_\lambda)$ . Recall that  $\mathbb{B}_{d-1}(\pi)$  is the closure of the injectivity region of  $\exp_{d-1}$ . Define for  $v \in \mathbb{B}_{d-1}(\pi)$  the *defect volume process*

$$V_\lambda(v) := \int_{\exp(\mathbf{0}, v)} r_\lambda(u) d\sigma_{d-1}(u).$$

Here  $[\mathbf{0}, v]$  for  $v \in \mathbb{R}^{d-1}$  is the rectangular solid in  $\mathbb{R}^{d-1}$  with vertices  $\mathbf{0}$  and  $v$ , that is to say  $[\mathbf{0}, v] := \prod_{i=1}^{d-1} [\min(0, v^{(i)}), \max(0, v^{(i)})]$ , with  $v^{(i)}$  standing for the  $i$ th coordinate of  $v$ . When  $[T^{(\lambda)}]^{-1}[\mathbf{0}, v] = \mathbb{B}_{d-1}(\pi)$ , we have that  $V_\lambda(v)$  is the total defect volume of  $K_\lambda$  with respect to  $B_d(\mathbf{0}, R_\lambda)$ . We re-scale  $V_\lambda(v)$  by its standard deviation, which in view of (1.9), gives

$$\hat{V}_\lambda(v) := (2 \log \lambda)^{-(d-3)/4} (V_\lambda(v) - \mathbb{E} V_\lambda(v)), \quad v \in \mathbb{R}^{d-1}.$$

For any  $\sigma^2 > 0$  let  $B^{\sigma^2}(\cdot)$  be the Brownian sheet of variance coefficient  $\sigma^2$  on the injectivity region  $\mathbb{B}_{d-1}(\pi)$ . Extend the domain of  $B^{\sigma^2}$  to all of  $\mathbb{R}^{d-1}$  by defining  $B^{\sigma^2}$  as the mean zero continuous path Gaussian process indexed by  $\mathbb{R}^{d-1}$  with

$$\text{Cov}(B^{\sigma^2}(v), B^{\sigma^2}(w)) = \sigma^2 \cdot \sigma_{d-1}(\exp([\mathbf{0}, v] \cap [\mathbf{0}, w])).$$

**Theorem 2.2** *As  $\lambda \rightarrow \infty$ , the random functions  $\hat{V}_\lambda : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  converge in law to the Brownian sheet  $B^{\sigma_V^2}$  in  $\mathcal{C}(\mathbb{R}^{d-1})$ , where  $\sigma_V^2 := \sigma^2(\xi_V^{(\infty)})$ .*

We shall not prove this result, as it follows closely the proof of Theorem 8.1 of [9].  
 (ii) *Binomial input.* By coupling binomial and Poisson points as in [4], we deduce the binomial analog of Theorem 2.1 for measures  $\sum_{i=1}^n \xi(X_i, \mathcal{X}_n) \delta_{X_i}$ ,  $\xi \in \Xi$ , where we recall that  $X_i$  are i.i.d. with density  $\phi$  and  $\mathcal{X}_n := \{X_j\}_{j=1}^n$ .  
 (iii) *Random polytopes on general Poisson input.* We expect that our main results extend to random polytopes generated by Poisson points having an isotropic intensity density. As shown by Carnal [11] and others, there are qualitative differences in the behavior of  $\mathbb{E} f_k(K_n)$  according to whether the input  $\mathcal{X}_n$  has an exponential tail or an algebraic tail modulated by a slowly varying function. The choice for the critical radius  $R_\lambda$  and the scaling transform  $T^{(\lambda)}$  would thus need to reflect such behavior. For example, if  $X_i, i \geq 1$ , are i.i.d. on  $\mathbb{R}^d$  with an isotropic intensity density decaying exponentially with the distance to the origin and if  $R_\lambda = \log \lambda - \log \log \lambda$ , then  $T^{(\lambda)}(\mathcal{P}_\lambda) \xrightarrow{D} \mathcal{H}_1$ , where  $\mathcal{H}_1$  is a rate one homogenous Poisson point process on  $\mathbb{R}^d$ .

### 3 Scaling transformations

For all  $\lambda \in [\lambda_0, \infty)$ , the scaling transform  $T^{(\lambda)}$  defined at (1.5) maps  $\mathbb{R}^d$  onto the rectangular solid  $W_\lambda \subset \mathbb{R}^{d-1} \times \mathbb{R}$  given by

$$W_\lambda := (R_\lambda \cdot \mathbb{B}_{d-1}(\pi)) \times (-\infty, R_\lambda^2].$$

Let  $(v, h)$  be the coordinates in  $W_\lambda$ , that is

$$v = R_\lambda \exp_{d-1}^{-1} \frac{x}{|x|}, \quad h = R_\lambda^2 \left( 1 - \frac{|x|}{R_\lambda} \right). \tag{3.1}$$

Note that  $\mathbb{S}^{d-1}$  is geodesically complete in that  $\exp_{d-1}$  is well defined on the whole tangent space  $\mathbb{R}^{d-1} \simeq T_{u_0}$ , although it is injective only on  $\{v \in T_{u_0}, |v| < \pi\}$ .

The reader may wonder about the genesis of  $T^{(\lambda)}$  and the parabolic scaling by  $R_\lambda$ . Roughly speaking, the effect of  $T^{(\lambda)}$  is to first re-scale the Gaussian sample by the characteristic scale factor  $R_\lambda^{-1}$  so that  $\partial K_\lambda$  is close to  $\mathbb{S}^{d-1}$ . By considering the distribution of  $\max_{i \leq n} |X_i|$  we see that  $(1 - |x|/R_\lambda)$  is small when  $x \in \partial K_\lambda$ ; cf. [14]. Re-scale again according to the twin desiderata: (i) unit volume image subsets near the hyperplane  $\mathbb{R}^{d-1}$  should host  $\Theta(1)$  re-scaled points, and (ii) radial components of points should scale as the square of angular components  $\exp_{d-1}^{-1} x/|x|$ . Desideratum (ii) preserves the parabolic behavior of the defect support function for  $R_\lambda^{-1} K_\lambda$ , namely the function  $1 - h_{R_\lambda^{-1} K_\lambda}(u)$ ,  $u \in \mathbb{S}^{d-1}$ , where  $h_K$  is the support function of  $K \subset \mathbb{R}^d$ . Extreme value theory [22] for  $|X_i|, i \geq 1$ , suggests (i) is achieved via radial scaling by  $R_\lambda^2$ , whence by (ii) we obtain angular scaling of  $R_\lambda$ , and (1.5) follows. These heuristics

are justified below, particularly through Lemma 3.2. In this and in the following section, our aim is to show:

- (i)  $T^{(\lambda)}$  defines a 1 – 1 correspondence between boundaries of convex hulls of point sets  $\mathcal{X} \subset \mathbb{R}^d$  and a subset of piecewise smooth functions on  $W_\lambda$ ,
- (ii)  $T^{(\lambda)}(\mathcal{P}_\lambda)$  converges in distribution to  $\mathcal{P}$  defined at (1.3), and
- (iii)  $T^{(\lambda)}$  defines re-scaled  $k$ -face and volume functionals on input carried by  $W_\lambda$ ; when the input is  $T^{(\lambda)}(\mathcal{P}_\lambda)$  then as  $\lambda \rightarrow \infty$  the means and covariances converge to the respective means and covariances of the corresponding functionals in  $\Xi^{(\infty)}$ .

### 3.1 The re-scaled boundary of the convex hull under $T^{(\lambda)}$

Abusing notation, we let  $\langle \cdot, \cdot \rangle$  denote inner product on  $\mathbb{R}^d$ . For  $x_0 \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ , consider the ball

$$B_d\left(\frac{x_0}{2}, \frac{|x_0|}{2}\right) = \left\{x \in \mathbb{R}^d \setminus \{\mathbf{0}\} : |x| \leq \left\langle x_0, \frac{x}{|x|} \right\rangle\right\} \cup \{\mathbf{0}\}.$$

Consideration of the support function of  $\text{co}(\mathcal{X})$  shows that  $x_0 \in \mathcal{X}$  is a vertex of  $\text{co}(\mathcal{X})$  iff  $B_d(x_0/2, |x_0|/2)$  is not a subset of  $\bigcup_{x \neq x_0} B_d(x/2, |x|/2)$ . With  $d_{\mathbb{S}^{d-1}}$  standing for the geodesic distance in  $\mathbb{S}^{d-1}$ , let  $\theta := d_{\mathbb{S}^{d-1}}(x/|x|, x_0/|x_0|)$ . We rewrite  $B_d(x_0/2, |x_0|/2)$  as

$$B_d\left(\frac{x_0}{2}, \frac{|x_0|}{2}\right) = \left\{x \in \mathbb{R}^d : |x| \leq |x_0| \cos \theta\right\}.$$

Recalling the change of variable at (3.1), let  $T^{(\lambda)}(x_0) := (v_0, h_0)$ , so that  $h_0 = R_\lambda^2(1 - |x_0|/R_\lambda)$ . We may rewrite  $B_d(x_0/2, |x_0|/2)$  as

$$B_d\left(\frac{x_0}{2}, \frac{|x_0|}{2}\right) := \left\{x \in \mathbb{R}^d : R_\lambda^2\left(1 - \frac{|x|}{R_\lambda \cos \theta}\right) \geq R_\lambda^2\left(1 - \frac{|x_0|}{R_\lambda}\right)\right\}.$$

Thus for all  $\lambda \in [\lambda_0, \infty)$ ,  $T^{(\lambda)}$  transforms  $B_d(x_0/2, |x_0|/2)$  into the upward opening grain

$$[\Pi^\uparrow(v_0, h_0)]^{(\lambda)} := \{(v, h) \in W_\lambda, h \geq R_\lambda^2(1 - \cos[e_\lambda(v, v_0)]) + h_0 \cos[e_\lambda(v, v_0)]\}, \tag{3.2}$$

with

$$e_\lambda(v, v_0) := d_{\mathbb{S}^{d-1}}(\exp_{d-1}(R_\lambda^{-1}v), \exp_{d-1}(R_\lambda^{-1}v_0)). \tag{3.3}$$

Every finite  $\mathcal{X} \subset W_\lambda, \lambda \in [\lambda_0, \infty)$ , generates the germ-grain model

$$\Psi^{(\lambda)}(\mathcal{X}) := \bigcup_{w \in \mathcal{X}} [\Pi^\uparrow(w)]^{(\lambda)}. \tag{3.4}$$

This germ-grain model has a twofold relevance: (i)  $[\Pi^\uparrow(T^{(\lambda)}(x))]^{(\lambda)}$  is the image by  $T^{(\lambda)}$  of  $B_d(x/2, |x|/2)$  and (ii)  $x \in \mathcal{X}$  is a vertex of  $\text{co}(\mathcal{X})$  if and only if  $[\Pi^\uparrow(T^{(\lambda)}(x))]^{(\lambda)}$  is not covered by the union  $\Psi^{(\lambda)}(T^{(\lambda)}(\mathcal{X} \setminus x))$ ,  $\lambda \in [\lambda_0, \infty)$ . Similar germ-grain models have been considered in section 4 of [24], sections 2 and 4 of [9] and section 2 of [10]. We say that  $T^{(\lambda)}(x)$  is extreme in  $T^{(\lambda)}(\mathcal{X})$  if  $x \in \mathcal{X}$  is a vertex of  $\text{co}(\mathcal{X})$ . Given  $T^{(\lambda)}(\mathcal{X}) = \mathcal{X}'$ , write  $\text{Ext}^{(\lambda)}(\mathcal{X}')$  for the set of extreme points in  $\mathcal{X}'$ .

For  $x_0 \in \mathbb{R}^d$  consider the half-space

$$H(x_0) := \left\{ x \in \mathbb{R}^d : \left\langle x, \frac{x_0}{|x_0|} \right\rangle \geq |x_0| \right\}.$$

Taking again  $\theta = d_{\mathbb{S}^{d-1}}\left(\frac{x}{|x|}, \frac{x_0}{|x_0|}\right)$ , we rewrite  $H(x_0)$  as

$$H(x_0) := \left\{ x \in \mathbb{R}^d : R_\lambda^2 \left( 1 - \frac{|x_0|}{R_\lambda \cos \theta} \right) \geq R_\lambda^2 \left( 1 - \frac{|x|}{R_\lambda} \right) \right\},$$

Taking  $T^{(\lambda)}(x_0) = (v_0, h_0)$  and using (3.1), we see that  $T^{(\lambda)}$  transforms  $H(x_0)$  into the downward grain

$$T^{(\lambda)}(H(x_0)) := [\Pi^\downarrow(v_0, h_0)]^{(\lambda)} := \left\{ (v, h) \in W_\lambda, h \leq R_\lambda^2 - \frac{R_\lambda^2 - h_0}{\cos[e_\lambda(v, v_0)]} \right\}. \tag{3.5}$$

Noting that  $\mathbb{R}^d \setminus \text{co}(\mathcal{X})$  is the union of half-spaces not containing points in  $\mathcal{X}$ , it follows that  $T^{(\lambda)}$  transforms  $\mathbb{R}^d \setminus \text{co}(\mathcal{X})$  into the subset of  $W_\lambda$  given by

$$\Phi^{(\lambda)}(T^{(\lambda)}(\mathcal{X})) := \bigcup_{\substack{w \in W_\lambda \\ [\Pi^\downarrow(w)]^{(\lambda)} \cap T^{(\lambda)}(\mathcal{X}) = \emptyset}} [\Pi^\downarrow(w)]^{(\lambda)}.$$

Thus  $T^{(\lambda)}$  sends the boundary of  $\text{co}(\mathcal{X})$  to the continuous function on  $W_\lambda$  whose graph coincides with the boundary of  $\Phi^{(\lambda)}(T^{(\lambda)}(\mathcal{X}))$ . There is thus a 1 – 1 correspondence between convex hull boundaries and a subset of the continuous functions on  $\mathbb{R}^{d-1} \times \mathbb{R}$ . This contrasts with Eddy [12], who mapped support functions of convex hulls into a subset of the continuous functions on  $\mathbb{R}^{d-1} \times \mathbb{R}$ .

The germ-grain models  $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  and  $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  link the geometry of  $K_\lambda$  with that of the limit paraboloid germ-grain models  $\Psi(\mathcal{P})$  and  $\Phi(\mathcal{P})$ . Theorem 1.2 and the upcoming Proposition 5.1 show that the boundaries  $\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  and  $\partial\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  respectively converge in probability to  $\partial(\Psi(\mathcal{P}))$  and to  $\partial(\Phi(\mathcal{P}))$  as  $\lambda \rightarrow \infty$ .

The next lemma is suggestive of this convergence and shows for fixed  $w \in W_\lambda$  that  $[\Pi^\uparrow(w)]^{(\lambda)}$  and  $[\Pi^\downarrow(w)]^{(\lambda)}$  locally approximate the paraboloids  $[\Pi^\uparrow(w)]^{(\infty)} := \Pi^\uparrow(w)$  and  $[\Pi^\downarrow(w)]^{(\infty)} := \Pi^\downarrow(w)$ , respectively. We may henceforth refer to  $[\Pi^\uparrow(w)]^{(\lambda)}$  and  $[\Pi^\downarrow(w)]^{(\lambda)}$  as quasi-paraboloids or sometimes ‘paraboloids’ for short. Recalling that  $B_{d-1}(v, r)$  is the  $(d - 1)$  dimensional ball centered at  $v \in \mathbb{R}^{d-1}$  with radius  $r$ , define the cylinder  $C(v, r) \subset \mathbb{R}^{d-1} \times \mathbb{R}$  by

$$C(v, r) := C_{d-1}(v, r) := B_{d-1}(v, r) \times \mathbb{R}. \tag{3.6}$$

Here and in the sequel, by  $c$  and  $c_1, c_2, \dots$  we mean generic positive constants which may change from line to line.

**Lemma 3.1** For all  $w_1 := (v_1, h_1) \in W_\lambda$ ,  $L \in (0, \infty)$ , and all  $\lambda \in [\lambda_0, \infty)$ , we have

$$\begin{aligned} & \|\partial([\Pi^\uparrow(w_1)]^{(\lambda)} \cap C(v_1, L)) - \partial([\Pi^\uparrow(w_1)]^{(\infty)} \cap C(v_1, L))\|_\infty \\ & \leq cL^3 R_\lambda^{-1} + ch_1 L^2 R_\lambda^{-2} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & \|\partial([\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(v_1, L)) - \partial([\Pi^\downarrow(w_1)]^{(\infty)} \cap C(v_1, L))\|_\infty \\ & \leq cL^3 R_\lambda^{-1} + ch_1 L^2 R_\lambda^{-2}. \end{aligned} \tag{3.8}$$

*Proof* We first prove (3.7). By (3.2) we have

$$\begin{aligned} \partial([\Pi^\uparrow(w_1)]^{(\lambda)}) & := \{(v, h) \in W_\lambda, \\ & h = R_\lambda^2(1 - \cos[e_\lambda(v, v_1)]) + h_1 \cos[e_\lambda(v, v_1)]\}. \end{aligned} \tag{3.9}$$

For  $v \in B_{d-1}(v_1, L)$ , notice that

$$e_\lambda(v, v_1) = |R_\lambda^{-1}v - R_\lambda^{-1}v_1| + O(|R_\lambda^{-1}v - R_\lambda^{-1}v_1|^2) \tag{3.10}$$

and thus

$$1 - \cos(e_\lambda(v, v_1)) = \frac{|R_\lambda^{-1}v - R_\lambda^{-1}v_1|^2}{2} + O(L^3 R_\lambda^{-3}).$$

It follows that

$$R_\lambda^2(1 - \cos(e_\lambda(v, v_1))) = \frac{|v - v_1|^2}{2} + O(L^3 R_\lambda^{-1})$$

and

$$|h_1(1 - \cos(e_\lambda(v, v_1)))| = O(h_1 L^2 R_\lambda^{-2}).$$

Thus the boundary of  $[\Pi^\uparrow(w_1)]^{(\lambda)} \cap C(v_1, L)$  is within  $cL^3 R_\lambda^{-1} + ch_1 L^2 R_\lambda^{-2}$  of the graph of

$$v \mapsto h_1 + \frac{|v - v_1|^2}{2},$$

which establishes (3.7). The proof of (3.8) is similar, and goes as follows. By (3.5) we have

$$\partial([\Pi^\downarrow(w_1)]^{(\lambda)}) := \left\{ (v, h) \in W_\lambda, h = R_\lambda^2 - \frac{R_\lambda^2 - h_1}{\cos[e_\lambda(v, v_1)]} \right\}. \tag{3.11}$$

Using (3.10), Taylor expanding  $\cos \theta$  up to second order, and writing  $1/(1 - r) = 1 + r + r^2 + \dots$  gives

$$\begin{aligned} \partial([\Pi^\downarrow(w_1)]^{(\lambda)}) := & \left\{ (v, h) \in W_\lambda, h = h_1 - \frac{|v - v_1|^2}{2} + \right. \\ & \left. O(R_\lambda^{-1}|v - v_1|^3) + O(h_1 R_\lambda^{-2}|v - v_1|^2) \right\}, \end{aligned} \tag{3.12}$$

and (3.8) follows. □

### 3.2 The weak limit of $T^{(\lambda)}(\mathcal{P}_\lambda)$

Put

$$\mathcal{P}^{(\lambda)} := T^{(\lambda)}(\mathcal{P}_\lambda).$$

Unlike the set-up of [9], the weak limit of  $T^{(\lambda)}(\mathcal{P}_\lambda)$  converges to a point process which is non-homogenous and which is carried by all of  $\mathbb{R}^{d-1} \times \mathbb{R}$ . Let  $\text{Vol}_d$  denote  $d$ -dimensional volume measure on  $\mathbb{R}^d$  and let  $\text{Vol}_d^{(\lambda)}$  be the image of  $R_\lambda \text{Vol}_d$  under  $T^{(\lambda)}$ . Recall the definition of  $\mathcal{P}$  at (1.3).

**Lemma 3.2** *As  $\lambda \rightarrow \infty$ , we have (a)  $\mathcal{P}^{(\lambda)} \xrightarrow{\mathcal{D}} \mathcal{P}$  and (b)  $\text{Vol}_d^{(\lambda)} \xrightarrow{\mathcal{D}} \text{Vol}_d$ . The convergence is in the sense of total variation convergence on compact sets.*

*Remarks* (i) It is likewise the case that the image of the binomial point process  $\sum_{x \in \mathcal{X}_n} \delta_x$  under  $T^{(n)}$  converges in distribution to  $\mathcal{P}$  as  $n \rightarrow \infty$ .

(ii)  $T^{(\lambda)}$  carries  $\mathcal{P}_\lambda$  into a point process on  $\mathbb{R}^{d-1} \times \mathbb{R}$  which in the large  $\lambda$  limit is stationary in the spatial coordinate. This contrasts with the transformation of Eddy [12] (and generalized in Eddy and Gale [13]) which carries  $\sum_{x \in \mathcal{X}_n} \delta_x$  into a point process  $(T_k, Z_k)$  on  $\mathbb{R} \times \mathbb{R}^{d-1}$  where  $T_k, k \geq 1$ , are points of a Poisson point process on  $\mathbb{R}$  with intensity  $e^{-h} dh$  and  $Z_k, k \geq 1$ , are i.i.d. standard Gaussian on  $\mathbb{R}^{d-1}$ .

*Proof* Representing  $x \in \mathbb{R}^d$  by  $x = ur, u \in \mathbb{S}^{d-1}, r \in [0, \infty)$ , we find the image by  $T^{(\lambda)}$  of the Poisson measure on  $\mathbb{R}^d$  with intensity

$$\lambda \phi(x) dx = \lambda \phi(ur) r^{d-1} dr d\sigma_{d-1}(u). \tag{3.13}$$

Make the change of variables

$$v := R_\lambda \exp_{d-1}^{-1}(u) = R_\lambda v_u, \quad h := R_\lambda^2 \left(1 - \frac{r}{R_\lambda}\right),$$

The exponential map  $\exp_{d-1} : T_{u_0} \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$  has the following expression:

$$\exp_{d-1}(v) = \cos(|v|)(0, \dots, 0, 1) + \sin(|v|) \begin{pmatrix} v \\ |v| \end{pmatrix}, \quad v \in \mathbb{R}^{d-1} \setminus \{0\}. \quad (3.14)$$

Therefore, since  $v_u := \exp_{d-1}^{-1}(u)$  we have

$$d\sigma_{d-1}(u) = \sin^{d-2}(|v_u|)d(|v_u|)d\sigma_{d-2} \left( \frac{v_u}{|v_u|} \right) = \frac{\sin^{d-2}(|v_u|)dv_u}{|v_u|^{d-2}}.$$

Since  $v_u = R_\lambda^{-1}v$ , this gives

$$d\sigma_{d-1}(u) = \frac{\sin^{d-2}(R_\lambda^{-1}|v|)}{|R_\lambda^{-1}v|^{d-2}}(R_\lambda^{-1})^{d-1}dv. \quad (3.15)$$

We also have

$$r^{d-1}dr = \left[ R_\lambda \left(1 - \frac{h}{R_\lambda^2}\right) \right]^{d-1} R_\lambda^{-1}dh \quad (3.16)$$

as well as

$$\lambda\phi(x) = \lambda\phi \left( uR_\lambda \left(1 - \frac{h}{R_\lambda^2}\right) \right) = \sqrt{2 \log \lambda} \exp \left( h - \frac{h^2}{2R_\lambda^2} \right). \quad (3.17)$$

Combining (3.13) and (3.15)–(3.17), we get that  $\mathcal{P}^{(\lambda)}$  has intensity density

$$\begin{aligned} \frac{d\mathcal{P}^{(\lambda)}}{dvdh}((v, h)) &= \frac{\sqrt{2 \log \lambda}}{R_\lambda} \frac{\sin^{d-2}(R_\lambda^{-1}|v|)}{|R_\lambda^{-1}v|^{d-2}} \left(1 - \frac{h}{R_\lambda^2}\right)^{d-1} \\ &\quad \exp \left( h - \frac{h^2}{2R_\lambda^2} \right), \quad (v, h) \in W_\lambda. \end{aligned} \quad (3.18)$$

Given a fixed compact subset  $D$  of  $W_\lambda$ , this intensity converges to the intensity of  $\mathcal{P}$  in  $L^1(D)$ , completing the proof of part (a).

Replacing the intensity  $\lambda\phi dx$  with  $dx$  in the above computations gives

$$\frac{d\text{Vol}_d^{(\lambda)}}{dvdh}((v, h)) = \frac{\sin^{d-2}(R_\lambda^{-1}|v|)}{|R_\lambda^{-1}v|^{d-2}} \left(1 - \frac{h}{R_\lambda^2}\right)^{d-1}, \quad (v, h) \in W_\lambda. \quad (3.19)$$

This intensity density converges pointwise to 1 as  $\lambda \rightarrow \infty$ , showing part (b). □

### 3.3 Re-scaled $k$ -face and volume functionals

Fix  $\lambda \in [\lambda_0, \infty)$ . Let  $\xi_k, k \in \{0, 1, \dots, d - 1\}$ , be a generic  $k$ -face functional, as in Definition 2.1. The inverse transformation  $[T^{(\lambda)}]^{-1}$  defines generic re-scaled functionals  $\xi^{(\lambda)}$  defined for  $\xi \in \Xi, w \in W_\lambda$  and  $\mathcal{X} \subset W_\lambda$  by

$$\xi^{(\lambda)}(w, \mathcal{X}) := \xi^{(\lambda)}(w, \mathcal{X}) := \xi([T^{(\lambda)}]^{-1}(w), [T^{(\lambda)}]^{-1}(\mathcal{X})). \tag{3.20}$$

For all  $\lambda \in [\lambda_0, \infty)$ , it follows that  $\xi(x, \mathcal{P}_\lambda) := \xi^{(\lambda)}(T^{(\lambda)}(x), \mathcal{P}^{(\lambda)})$ . Note for all  $\lambda \in [\lambda_0, \infty), k \in \{0, 1, \dots, d - 1\}, w_1 \in W_\lambda$ , and  $\mathcal{X} \subset W_\lambda$ , that  $\xi_k^{(\lambda)}(w_1, \mathcal{X})$  is the product of  $(k + 1)^{-1}$  and the number of quasi-parabolic  $k$ -dimensional faces of  $\partial(\bigcup_{w \in \mathcal{X}} [\Pi^\perp(w)]^{(\lambda)})$  which contain  $w_1, w_1 \in \text{Ext}^{(\lambda)}(\mathcal{X})$ , otherwise  $\xi_k^{(\lambda)}(w_1, \mathcal{X}) = 0$ .

Similarly, define for  $w \in \text{Ext}^{(\lambda)}(\mathcal{X})$

$$\xi_V^{(\lambda)}(w, \mathcal{X}) = \frac{1}{d} \int_{v \in \text{Cyl}^{(\lambda)}(w)} \int_0^{\Phi^{(\lambda)}(\mathcal{X})(v)} d\text{Vol}_d^{(\lambda)}((v, h)), \tag{3.21}$$

where  $\text{Cyl}^{(\lambda)}(w) := \text{Cyl}^{(\lambda)}(w, \mathcal{X})$  denotes the projection onto  $\mathbb{R}^{d-1}$  of the quasi-parabolic faces of  $\Phi^{(\lambda)}(\mathcal{X})$  containing  $w$ . When  $w \notin \text{Ext}^{(\lambda)}(\mathcal{X})$  we define  $\xi_V^{(\lambda)}(w, \mathcal{X}) = 0$ .

Given  $\lambda \in [\lambda_0, \infty)$ , let  $\Xi^{(\lambda)}$  denote the collection of re-scaled functionals  $\xi_k^{(\lambda)}, k \in \{0, 1, \dots, d - 1\}$ , together with  $\xi_V^{(\lambda)}$ . Our main goal in the next section is to show that, given a generic  $\xi^{(\lambda)} \in \Xi^{(\lambda)}$ , the means and covariances of  $\xi^{(\lambda)}(\cdot, \mathcal{P}^{(\lambda)})$  converge as  $\lambda \rightarrow \infty$  to the respective means and covariances of  $\xi^{(\infty)}(\cdot, \mathcal{P})$ , with  $\xi^{(\infty)} \in \Xi^{(\infty)}$ .

## 4 Properties of re-scaled $k$ -face and volume functionals

To establish convergence of re-scaled functionals  $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \in [\lambda_0, \infty)$ , to their respective counterparts  $\xi^{(\infty)} \in \Xi^{(\infty)}$ , we first need to show that  $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \in [\lambda_0, \infty]$  satisfy a localization in the spatial and time coordinates  $v$  and  $h$ , respectively. These localization results are the analogs of Lemmas 7.2 and 7.3 of [9].

In the following the point process  $\mathcal{P}^{(\lambda)}, \lambda = \infty$ , is taken to be  $\mathcal{P}$  whereas  $W_\lambda, \lambda = \infty$ , is taken to be  $\mathbb{R}^d$ . Many of our proofs for the case  $\lambda \in (0, \infty)$  may be modified to yield explicit proofs of some unproved assertions in [9].

### 4.1 Localization of $\xi^{(\lambda)}$

Recall the definition at (3.6) of the cylinder  $C(v, r) := C_{d-1}(v, r) := B_{d-1}(v, r) \times \mathbb{R}, v \in \mathbb{R}^{d-1}, r > 0$ . Given a generic functional  $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \in [\lambda_0, \infty]$ , and  $w := (v, h) \in W_\lambda$ , we shall write

$$\xi_{[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)} \cap C_{d-1}(v, r)). \tag{4.1}$$

Given  $\xi^{(\lambda)}$ ,  $\lambda \in [\lambda_0, \infty]$ , recall from [9,24] that a random variable  $R := R^{\xi^{(\lambda)}}[w] := R^{\xi^{(\lambda)}}[w, \mathcal{P}^{(\lambda)}]$  is a *spatial localization radius* for  $\xi^{(\lambda)}$  at  $w$  with respect to  $\mathcal{P}^{(\lambda)}$  iff a.s.

$$\xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \xi_{[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \quad \text{for all } r \geq R. \tag{4.2}$$

There are in general more than one  $R$  satisfying (4.2) and we shall henceforth assume  $R$  is the infimum of all reals satisfying (4.2).

We may similarly define a localization radius in the non-rescaled picture. Indeed, given a generic functional  $\xi$  and  $x \in \mathbb{R}^d \setminus \{0\}$ , we shall write

$$\xi_{[r]}(x, \mathcal{P}_\lambda) := \xi(x, \mathcal{P}_\lambda \cap S(x, r)).$$

where  $S(x, r) := \{y \in \mathbb{R}^d \setminus \{0\} : d_{\mathbb{S}^{d-1}}(x/|x|, y/|y|) \leq r\}$ .  $R^\xi[x, \mathcal{P}_\lambda]$  is then the infimum of all  $R \in (0, \infty)$  which satisfy  $\xi(x, \mathcal{P}_\lambda) = \xi_{[r]}(x, \mathcal{P}_\lambda)$  for every  $r \in [R, \infty)$ . In particular, by rotation-invariance of  $\mathcal{P}_\lambda$  and the fact that  $|v - \mathbf{0}| = d_{\mathbb{S}^{d-1}}(\exp_{d-1}(\mathbf{0}), \exp_{d-1}(v))$  for all  $v \in \mathbb{B}_{d-1}(\pi)$ , we have the distributional equality:

$$R^\xi[x, \mathcal{P}_\lambda] \stackrel{D}{=} R^\xi[|x|u_0, \mathcal{P}_\lambda] = R_\lambda^{-1} R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)], \tag{4.3}$$

where  $h_0 = R_\lambda^2 (1 - |x|/R_\lambda)$ . In view of (4.3), it is enough to investigate the distribution tail of  $R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)]$  for any  $h_0 \in \mathbb{R}$ . In the next lemmas, we prove that the functionals  $\xi^{(\lambda)} \in \Xi^{(\lambda)}$ ,  $\lambda \in [\lambda_0, \infty]$ , admit spatial localization radii with tails decaying super-exponentially fast at  $(\mathbf{0}, h_0)$ ,  $h_0 \in \mathbb{R}$ . We first establish a localization radius for  $\xi_0$ . We remark this shows that  $\text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)})$ ,  $\lambda \in [\lambda_0, \infty]$ , is a strongly mixing random point set.

**Lemma 4.1** *There is a constant  $c > 0$  such that the localization radius  $R^{\xi_0^{(\lambda)}}[(\mathbf{0}, h_0)]$  satisfies for all  $\lambda \in [\lambda_0, \infty]$ ,  $h_0 \in (-\infty, R_\lambda^2]$ , and  $t \geq (-h_0 \vee 0)$*

$$P[R^{\xi_0^{(\lambda)}}[(\mathbf{0}, h_0)] > t] \leq c \exp\left(-\frac{t^2}{c}\right). \tag{4.4}$$

*Proof* Abbreviate  $\xi_0$  by  $\xi$ . It suffices to show that (4.4) holds for  $t \geq -h_0 \vee c$ ,  $c$  a positive constant, a simplification used repeatedly in what follows. For  $t \geq (-h_0 \vee 0)$  and  $\lambda \in [\lambda_0, \infty]$ , we have

$$\{R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)] > t\} \subset E_1 \cup E_2, \tag{4.5}$$

where

$$E_1 := \left\{ R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)] > t, (\mathbf{0}, h_0) \notin \text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \right\}$$

and

$$E_2 := \left\{ R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)] > t, (\mathbf{0}, h_0) \in \text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \right\}.$$

Rewrite  $E_1$  as

$$E_1 = \{(\mathbf{0}, h_0) \notin \text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}), (\mathbf{0}, h_0) \in \text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, t))\}.$$

If  $E_1$  occurs then there is a

$$w_1 := (v_1, h_1) \in \partial \left( [\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \right) \cap C(\mathbf{0}, t)$$

belonging to some  $[\Pi^\uparrow(y)]^{(\lambda)}$ ,  $y \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, t)^c$ , but  $w_1 \notin \bigcup_{w \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, t)} [\Pi^\uparrow(w)]^{(\lambda)}$ . In other words,  $w_1$  is covered by paraboloids with apices in  $\mathcal{P}^{(\lambda)}$ , but not by paraboloids with apices in  $\mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, t)$ . This means that the down paraboloid  $[\Pi^\downarrow(w_1)]^{(\lambda)}$  does not contain points in  $C(\mathbf{0}, t) \cap \mathcal{P}^{(\lambda)}$ , but it must contain a point in  $C(\mathbf{0}, t)^c \cap \mathcal{P}^{(\lambda)}$ . In other words, we have  $E_1 \subset F_1 \cup F_2$ , where

$$F_1 := \{\exists w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap C(\mathbf{0}, t) : h_1 \in (-\infty, t), \\ [\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, t) \cap \mathcal{P}^{(\lambda)} = \emptyset, [\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, t)^c \cap \mathcal{P}^{(\lambda)} \neq \emptyset\}$$

and

$$F_2 := \{\exists w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap C(\mathbf{0}, t) : h_1 \in [t, \infty), \\ [\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, t) \cap \mathcal{P}^{(\lambda)} = \emptyset\}.$$

If  $E_2$  happens then there is  $w_1 := (v_1, h_1) \in C(\mathbf{0}, t)^c \cap [\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)}$  which is not covered by paraboloids with apices in  $\mathcal{P}^{(\lambda)}$  and  $(\mathbf{0}, h_0)$  belongs to  $\partial[\Pi^\downarrow(w_1)]^{(\lambda)}$ . Notice that  $w_1 \in C(\mathbf{0}, \pi R_\lambda/2)$  since the ball  $[T^{(\lambda)}]^{-1}([\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)})$  is included in  $\mathbb{R}^{d-1} \times [0, \infty)$ . There is a constant  $c > 0$  such that  $1 - \cos(\theta) \geq c\theta^2$  for  $\theta \in [0, \pi]$  so that in view of (3.2) and  $e_\lambda(v_1, \mathbf{0}) = R_\lambda^{-1}|v_1|$ , we have

$$h_1 \geq h_0 \cos(R_\lambda^{-1}|v_1|) + R_\lambda^2(1 - \cos(R_\lambda^{-1}|v_1|)) \geq (h_0 \wedge 0) + c|v_1|^2 \geq (h_0 \wedge 0) + ct^2.$$

Now  $h_0 \wedge 0 \geq -t$  always holds so we obtain  $h_1 \geq -t + ct^2 \geq t$  for large enough  $t$ . Thus we have  $E_2 \subset \tilde{E}_2$  where

$$\tilde{E}_2 := \{\exists w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} : h_1 \in [t, \infty), [\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset\}. \tag{4.6}$$

By (4.5) and the inclusions  $E_1 \subset F_1 \cup F_2$  and  $E_2 \subset \tilde{E}_2$ , it is enough to show that each term  $P[F_1]$ ,  $P[F_2]$  and  $P[\tilde{E}_2]$  is bounded by  $c \exp(-t^2/c)$ .

*Upper-bound for  $P[F_1]$ .* We start with the case  $\lambda = \infty$ . Consider a fixed  $w_1 \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]$  with  $h_1 = h_0 + \frac{1}{2}|v_1|^2 \leq t$ . The probability that  $[\Pi^\downarrow(w_1)] \cap C(\mathbf{0}, t)^c \cap \mathcal{P} \neq \emptyset$  is bounded by the  $d\mathcal{P}$  measure of  $[\Pi^\downarrow(w_1)] \cap C(\mathbf{0}, t)^c$ . The maximal height of  $[\Pi^\downarrow(w_1)] \cap C(\mathbf{0}, t)^c$  is  $h_1 - \frac{1}{2}(t - \sqrt{2(h_1 - h_0)})^2$ . Consequently, the  $d\mathcal{P}$  measure of  $[\Pi^\downarrow(w_1)] \cap C(\mathbf{0}, t)^c$  is bounded by the  $d\mathcal{P}$  measure of  $[\Pi^\downarrow(w_1)] \cap \{(v, h) : h \leq$

$h_1 - \frac{1}{2}(t - \sqrt{2(h_1 - h_0)})^2$ . Recall that  $c$  is a constant which changes from line to line. Up to a multiplicative constant, the  $d\mathcal{P}$  measure of  $\Pi^\downarrow(w_1) \cap C(\mathbf{0}, t)^c$  is bounded by

$$\int_{-\infty}^{h_1 - \frac{1}{2}(t - \sqrt{2(h_1 - h_0)})^2} e^h (2(h_1 - h))^{\frac{d-1}{2}} dh = e^{h_1} \int_{\frac{1}{2}(t - \sqrt{2(h_1 - h_0)})^2}^{+\infty} e^{-u} (2u)^{\frac{d-1}{2}} du$$

$$\leq c \exp\left(h_1 - \frac{1}{c} \left(\frac{t^2}{2} + h_1 - h_0 - t\sqrt{2(h_1 - h_0)}\right)\right),$$

where we put  $u := h_1 - h$ .

Consequently, discretizing  $\partial\Pi^\uparrow((\mathbf{0}, h_0)) \cap (\mathbb{R}^{d-1} \times (h_0, t])$  and using  $h_0 \leq h_1 \leq t$ , we get

$$P[F_1] \leq ce^{\frac{-t^2}{2c}} \int_{h_0}^t (2(h_1 - h_0))^{\frac{d-2}{2}} \exp\left(\left(1 - \frac{1}{c}\right)h_1 + \frac{h_0}{c} + \frac{t\sqrt{h_1 - h_0}}{c}\right) dh_1$$

$$\leq ce^{\frac{-t^2}{2c}} \int_0^{t-h_0} (2h_1)^{\frac{d-2}{2}} \exp\left(\left(1 - \frac{1}{c}\right)h_1 + h_0 + \frac{t\sqrt{h_1}}{c}\right) dh_1$$

$$\leq ce^{\frac{-(t^2 - t^{3/2} - t)}{c}}$$

$$\leq ce^{\frac{-t^2}{c}},$$

concluding the case  $\lambda = \infty$ .

When  $\lambda \in [\lambda_0, \infty)$ , recall from (3.2) that

$$[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} := \{(v, h) \in W_\lambda, h \geq R_\lambda^2(1 - \cos[e_\lambda(v, \mathbf{0})]) + h_0 \cos[e_\lambda(v, \mathbf{0})]\}. \tag{4.7}$$

We claim that  $[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap (\mathbb{R}^{d-1} \times (-\infty, t])$  has a spatial diameter (in the  $v$  coordinates) bounded by  $c_1\sqrt{t}$ . We see this as follows. Let  $(v, h) \in [\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap (\mathbb{R}^{d-1} \times (-\infty, t])$ . When  $h \leq t$  and  $|h_0| \leq t$ , the display (4.7) yields  $R_\lambda^2(1 - \cos[e_\lambda(v, \mathbf{0})]) \leq 2t$ . Thus  $1 - \cos[e_\lambda(v, \mathbf{0})] \leq 2tR_\lambda^{-2}$ . It follows that

$$ce_\lambda(v, \mathbf{0})^2 \leq 1 - \cos[e_\lambda(v, \mathbf{0})] \leq 2tR_\lambda^{-2}. \tag{4.8}$$

Using the equality  $e_\lambda(v, \mathbf{0}) = R_\lambda^{-1}|v|$ , we deduce  $|v| \leq c_1\sqrt{t}$ , as desired.

Let

$$w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap C(\mathbf{0}, t).$$

We now estimate the maximal height of  $[\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, t)^c$ . If  $(v, h)$  belongs to the boundary of  $[\Pi^\downarrow(w_1)]^{(\lambda)}$  then we have from (3.11) that

$$h = R_\lambda^2 - \frac{R_\lambda^2 - h_1}{\cos[e_\lambda(v, v_1)]}$$

which gives

$$ce_{\lambda}(v, v_1)^2 \leq 1 - \cos[e_{\lambda}(v, v_1)] = \frac{h_1 - h}{R_{\lambda}^2 - h} \leq \frac{h_1 - h}{R_{\lambda}^2 - t} \leq \frac{h_1 - h}{R_{\lambda}^2 - 2\pi R_{\lambda}} \tag{4.9}$$

where we use  $h \leq t \leq 2\pi R_{\lambda}$ . Indeed, we may without loss of generality assume  $t \in [0, 2\pi R_{\lambda}]$ , since the stabilization radius never exceeds the spatial diameter of  $W_{\lambda}$ . Consequently, we have

$$h \leq h_1 - c_2(R_{\lambda}^2 - 2\pi R_{\lambda})e_{\lambda}(v, v_1)^2. \tag{4.10}$$

The maximal height of  $[(\Pi^{\downarrow}(w_1))^{\lambda}] \cap C(\mathbf{0}, t)^c$  is found by letting  $v$  belong to the boundary of  $C(\mathbf{0}, t)$ . In particular, we have  $e_{\lambda}(v, \mathbf{0}) = R_{\lambda}^{-1}|v| = R_{\lambda}^{-1}t$ . Moreover, we deduce from (4.8) that  $e_{\lambda}(v_1, \mathbf{0}) \leq c_1 R_{\lambda}^{-1}\sqrt{t}$ . Consequently, we have

$$e_{\lambda}(v, v_1) \geq e_{\lambda}(v, \mathbf{0}) - e_{\lambda}(v_1, \mathbf{0}) \geq R_{\lambda}^{-1}(t - c_1\sqrt{t}).$$

Combining the last inequality above with (4.10) shows for any  $(v, h) \in \partial[(\Pi^{\downarrow}(w_1))^{\lambda}] \cap \partial C(\mathbf{0}, t)$  that

$$h \leq h_1 - c_2 \frac{R_{\lambda}^2 - 2\pi R_{\lambda}}{R_{\lambda}^2} (t - c_1\sqrt{t})^2 \leq t - c_3(t - c_1\sqrt{t})^2.$$

Now we follow the proof for the case  $\lambda = \infty$ . We have

$$d\mathcal{P}^{(\lambda)}([\Pi^{\downarrow}(w_1))^{\lambda}] \cap C(\mathbf{0}, t)^c \leq d\mathcal{P}^{(\lambda)}([\Pi^{\downarrow}(w_1))^{\lambda}] \cap \{(v, h) : h \leq t - c_2(t - c_1\sqrt{t})^2\},$$

In view of (3.18) and (4.9),  $d\mathcal{P}^{(\lambda)}([\Pi^{\downarrow}(w_1))^{\lambda}] \cap C(\mathbf{0}, t)^c$  is bounded by

$$c \int_{-\infty}^{t-c_3(t-c_1\sqrt{t})^2} \left(1 - \frac{h}{R_{\lambda}^2}\right)^{d-1} e^h \left[ \int \mathbf{1}(e_{\lambda}(v, v_1) \leq \frac{c\sqrt{h_1-h}}{R_{\lambda}} \frac{\sin^{d-2}(R_{\lambda}^{-1}|v|)}{|R_{\lambda}^{-1}v|^{d-2}}) dv \right] dh.$$

Using the change of variables  $u = \exp_{d-1}^{-1}(R_{\lambda}^{-1}v)$  with  $u_1 = \exp_{d-1}^{-1}(R_{\lambda}^{-1}v_1)$  gives

$$\begin{aligned} & d\mathcal{P}^{(\lambda)}([\Pi^{\downarrow}(w_1))^{\lambda}] \cap C(\mathbf{0}, t)^c \\ & \leq c \int_{-\infty}^{t-c_3(t-c_1\sqrt{t})^2} \left(1 - \frac{h}{R_{\lambda}^2}\right)^{d-1} e^h \left[ \int_{\mathbb{S}^{d-1}} \mathbf{1}(d_{\mathbb{S}^{d-1}}(u, u_1) \leq \frac{c\sqrt{h_1-h}}{R_{\lambda}}) R_{\lambda}^{d-1} d\sigma_{d-1}(u) \right] dh \\ & \leq c \int_{-\infty}^{t-c_3(t-c_1\sqrt{t})^2} \left(1 - \frac{h}{R_{\lambda}^2}\right)^{d-1} e^h (h_1 - h)^{(d-1)/2} dh. \end{aligned} \tag{4.11}$$

For  $t$  large the upper limit of integration is at most  $-c_4t^2$  where  $c_4 = c_3/2$ . There is a positive constant  $c_5$  such that  $(1 - h/R_{\lambda}^2)^{d-1} e^{h/2} \leq e^{c_5h}$  holds for all  $h \in (-\infty, 0]$ .

Also,

$$(h_1 - h)^{(d-1)/2} \leq c(t^{(d-1)/2} + |h|^{(d-1)/2}).$$

Putting these estimates together yields

$$\begin{aligned} & d\mathcal{P}^{(\lambda)}([\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, t)^c) \\ & \leq c \int_{-\infty}^{-c_4 t^2} e^{c_5 h} (t^{(d-1)/2} + |h|^{(d-1)/2}) dh \leq c_6 \exp(-t^2/c_6). \end{aligned}$$

Consequently, discretizing  $\partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap C(\mathbf{0}, t) \cap (\mathbb{R}^{d-1} \times (-\infty, t])$  we get for  $\lambda \in [\lambda_0, \infty)$

$$P[F_1] \leq c_6 e^{-t^2/c_6} \int_{h_0}^t t^{d-2} dh_1 \leq c_7 e^{-t^2/c_7}.$$

*Upper-bound for  $P[F_2]$ .* We again start with the case  $\lambda = \infty$ . Suppose  $h_1 \in [t, \infty)$  with  $t$  large. As noted,  $\Pi^\downarrow \cap C(\mathbf{0}, t)$  does not contain points in  $\mathcal{P}$ . The  $d\mathcal{P}$  measure of  $\Pi^\downarrow(w_1) \cap C(\mathbf{0}, t)$  is bounded below by the  $d\mathcal{P}$  measure of  $\Pi^\downarrow(w_1) \cap C(\mathbf{0}, t) \cap (\mathbb{R}^{d-1} \times [0, \infty))$ , which we generously bound below by  $e^{h_1/2}$ . Thus the probability that  $\Pi^\downarrow \cap C(\mathbf{0}, t)$  does not contain points in  $\mathcal{P} \cap C(\mathbf{0}, t)$  is bounded above by  $\exp(-e^{h_1/2})$ .

Discretizing  $\partial(\Pi^\uparrow((\mathbf{0}, h_0))) \cap (\mathbb{R}^{d-1} \times [t, \infty)) \cap C(\mathbf{0}, t)$  into unit cubes, we see that the probability that there is  $w_1 := (v_1, h_1) \in \partial(\Pi^\uparrow((\mathbf{0}, h_0))) \cap C(\mathbf{0}, t)$  such that  $\Pi^\downarrow(w_1)$  does not contain points in  $\mathcal{P} \cap C(\mathbf{0}, t)$  is bounded by

$$c \int_t^\infty t^{d-2} \exp(-e^{h_1/2}) dh_1 \leq c t^{d-2} \exp\left(\frac{-e^{t/2}}{c}\right).$$

Thus there is a constant  $c$  such that  $P[F_2] \leq c \exp(-t^2/c)$  for  $t \geq (-h_0 \vee c)$ .

When  $\lambda \in [\lambda_0, \infty)$ , we proceed as follows. Let  $w_1$  be the point defined in event  $F_2$ . Let  $S$  be the unit volume cube centered at  $(v_1 - \sqrt{d-1}v_1/(2|v_1|), (h_1 + 1)/2)$ . We claim that for  $t$  large enough,  $S$  is included in  $[\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, t \wedge 3\pi R_\lambda/4)$ . Indeed,  $S$  is clearly included in  $[\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, t) \cap (\mathbb{R}^{d-1} \times [h_1/4, \infty))$  and since  $v_1 \in B_{d-1}(\mathbf{0}, \pi R_\lambda/2)$ ,  $S$  is included in  $C(\mathbf{0}, 3\pi R_\lambda/4)$ . By (3.18) there exists a constant  $c > 0$  such that for all  $(v, h) \in S$

$$\frac{d\mathcal{P}^{(\lambda)}((v, h))}{dv dh} \geq c \left(1 - \frac{h_1}{4R_\lambda^2}\right)^{d-1} \exp\left(\frac{h_1}{4} - \frac{h_1^2}{32R_\lambda^2}\right).$$

Now using  $h_1 \in (-\infty, R_\lambda^2]$ , we obtain  $d\mathcal{P}^{(\lambda)}(S) \geq c \exp(7h_1/32)$ . Consequently, the probability that  $[\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, t) \cap \mathcal{P}^{(\lambda)} = \emptyset$  is bounded above by  $\exp(-ce^{h_1/c})$ . Discretizing  $\partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap (\mathbb{R}^{d-1} \times [t, \infty)) \cap C(\mathbf{0}, t)$ , we see that the probability

that there is  $w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap C(\mathbf{0}, t)$  such that  $[\Pi^\downarrow(w_1)]^{(\lambda)}$  does not contain points in  $\mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, t)$  is bounded by

$$c \int_t^\infty t^{d-2} \exp(-ce^{h_1/c}) dh_1 \leq c \exp(-ce^{t/c}).$$

*Upper-bound for  $P[\tilde{E}_2]$ .* The arguments closely follow those for  $P[F_2]$  and we sketch the proof only for finite  $\lambda$  as the case  $\lambda = \infty$  is similar. As above, consideration of the cube  $S$  shows that  $P[[\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset]$  is bounded above by  $\exp(-ce^{h_1/c})$ . Only the discretization differs from the case of  $P[F_2]$ . Indeed, we need now to discretize  $\partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap (\mathbb{R}^{d-1} \times [t, \infty))$ . We use the fact that  $|v_1| \leq cR_\lambda \sqrt{h_1 - h_0} / \sqrt{R_\lambda^2 - h_0}$  as soon as  $(v_1, h_1) \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)}$  (by (3.2) and the arguments as in (4.8)). We obtain

$$P[\tilde{E}_2] \leq c \int_{t \vee h_0}^{R_\lambda^2} \left[ \frac{R_\lambda}{\sqrt{R_\lambda^2 - h_0}} \sqrt{h_1 - h_0} \right]^{d-2} \exp(-ce^{h_1/c}) dh_1.$$

When  $h_0 \in (-\infty, R_\lambda^2/2]$ , we bound the ratio  $R_\lambda / \sqrt{R_\lambda^2 - h_0}$  by  $\sqrt{2}$  to obtain  $P[\tilde{E}_2] \leq c \exp(-ce^{t/c})$ . When  $h_0 \in (R_\lambda^2/2, R_\lambda^2]$ , we bound  $(h_1 - h_0) / (R_\lambda^2 - h_0)$  by 1 and we bound  $\exp(-ce^{h_1/c})$  by  $\exp(-ce^{h_1/c}/2 - ce^{R_\lambda^2/(2c)}/2)$  and we also obtain  $P[\tilde{E}_2] \leq c \exp(-ce^{t/c})$ , as desired.

Combining the above bounds for  $P[F_1]$ ,  $P[F_2]$ , and  $P[\tilde{E}_2]$  thus yields

$$P[E_1] + P[E_2] \leq P[F_1] + P[F_2] + P[\tilde{E}_2] \leq c \exp\left(-\frac{t^2}{c}\right),$$

showing Lemma 4.1 as desired. □

Whereas Lemma 4.1 localizes  $k$ -face and volume functionals in the spatial domain, we now localize in the height/time domain. We show that the boundaries of the paraboloid germ-grain processes  $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  and  $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ ,  $\lambda \in [\lambda_0, \infty]$ , are not far from  $\mathbb{R}^{d-1}$ . Recall that  $\mathcal{P}^{(\lambda)}$ ,  $\lambda = \infty$ , is taken to be  $\mathcal{P}$  and we also write  $\Psi(\mathcal{P})$  for  $\Psi^{(\infty)}(\mathcal{P}^\infty)$ . If  $w \in \text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)})$  we put  $H(w) := H(w, \mathcal{P}^{(\lambda)})$  to be the maximal height coordinate (with respect to  $\mathbb{R}^{d-1}$ ) of an apex of a down paraboloid which contains a parabolic face in  $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  containing  $w$ , otherwise we put  $H(w) = 0$ .

**Lemma 4.2** (a) *There is a constant  $c$  such that for all  $\lambda \in [\lambda_0, \infty]$ ,  $h_0 \in (-\infty, R_\lambda^2]$ , and  $t \in [h_0 \vee 0, \infty)$  we have*

$$P[H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \geq t] \leq c \exp\left(-\frac{e^t}{c}\right). \tag{4.12}$$

(b) *There is a constant  $c$  such that for all  $L \in (0, \infty)$ ,  $t \in (0, \infty)$ , and  $\lambda \in [\lambda_0, \infty]$  we have*

$$P[|\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(\mathbf{0}, L)|_\infty > t] \leq cL^{2(d-1)}e^{-\frac{t}{c}}. \tag{4.13}$$

The bound (4.13) also holds for the dual process  $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ .

*Proof* Let us first prove (4.12). We do this for  $\lambda \in [\lambda_0, \infty)$  and we claim that a similar proof holds for  $\lambda = \infty$ . Rewrite the event  $\{H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \geq t\}$  as

$$\{H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \geq t\} = \{\exists w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} : h_1 \in [t, \infty), \\ [\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset\}.$$

Let us consider  $w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)}$  and put  $[T^{(\lambda)}]^{-1}(\mathbf{0}, h_0) := \rho u_0$ ,  $\rho \in [0, \infty)$ . Since  $[\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)}$  is the image by  $T^{(\lambda)}$  of the ball  $B_d(\rho u_0/2, \rho/2)$ , it is a subset of the image of the upper-half space, i.e. a subset of  $C(\mathbf{0}, \pi R_\lambda/2)$ . Consequently, the unit-volume cube centered at  $(v_1, h_1 - 1)$  is included in  $[\Pi^\downarrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, 3\pi R_\lambda/4)$ . The proof now follows along the same lines as for the bound for  $P[\tilde{E}_2]$  in the proof of Lemma 4.1. The  $\mathcal{P}^{(\lambda)}$ -measure of that cube exceeds  $c \exp(h_1/c)$ . The probability that  $[\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset$  is bounded above by  $c \exp(-ce^{h_1/c})$ . Discretizing  $(\mathbb{R}^{d-1} \times [t, \infty))$  into unit cubes, we obtain (4.12).

We now prove (4.13). We bound the probability of the two events

$$E_3 := \{\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap \{(v, h) : |v| \leq L, h > t\} \neq \emptyset\}$$

and

$$E_4 := \{\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap \{(v, h) : |v| \leq L, h < -t\} \neq \emptyset\}.$$

When in  $E_3$ , there is a point  $w_1 := (v_1, h_1)$  with  $h_1 \in [t, \infty)$ ,  $|v_1| \leq L$ , and such that  $[\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset$ . Following the proof of (4.12), we construct a unit-volume cube in  $C(\mathbf{0}, L)$  which is a domain where the density of the  $d\mathcal{P}^{(\lambda)}$  measure exceeds  $ce^{h_1/c}$ . Discretization of  $\{(v, h) : |v| \leq L, h \in [t, \infty)\}$  into unit volume sub-cubes gives

$$P[E_3] \leq cL^{d-1} \exp\left(-\frac{e^t}{c}\right).$$

On the event  $E_4$ , there exists a point  $(v_1, h_1)$  with  $|v_1| \leq L$  and  $h_1 \in (-\infty, -t]$  which is on the boundary of an upward paraboloid with apex in  $\mathcal{P}^{(\lambda)}$ . The apex of this upward paraboloid is contained in the union of all down paraboloids with apex on  $B_{d-1}(\mathbf{0}, L) \times \{h_1\}$ . The  $d\mathcal{P}^{(\lambda)}$  measure of this union is bounded by  $cL^{d-1} \exp(h_1/c)$ . Consequently, the probability that the union contains points from  $\mathcal{P}^{(\lambda)}$  is less than  $1 - \exp(-cL^{d-1}e^{h_1/c}) \leq cL^{d-1} \exp(h_1/c)$ . It remains to discretize and integrate over  $h_1 \in (-\infty, -t)$ . This goes as follows.

Discretizing  $C(\mathbf{0}, L) \times (-\infty, -t]$  into unit volume subcubes and using (3.18), we find that the probability there exists  $(v_1, h_1) \in \mathbb{R}^{d-1} \times (-\infty, -t]$  on the boundary of an up paraboloid is thus bounded by

$$cL^{2(d-1)} \int_{-\infty}^{-t} e^{h_1/c} \left(1 - \frac{h}{R_\lambda^2}\right)^{d-1} e^{h_1} dh_1$$

This establishes (4.13). The same argument applies to the dual process  $\Phi^{(\lambda)}$  ( $\mathcal{P}^{(\lambda)}$ ). □

We now extend Lemma 4.1 to all  $\xi \in \Xi$ .

**Lemma 4.3** *There is a constant  $c > 0$  such that for all  $\xi \in \Xi$ ,  $\lambda \in [\lambda_0, \infty]$ , and  $h_0 \in (-\infty, R_\lambda^2]$ , the localization radius  $R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)]$  satisfies for all  $t \in [|h_0|, \infty)$*

$$P[R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)] > t] \leq c \exp\left(-\frac{t^2}{c}\right). \tag{4.14}$$

*Proof* We show (4.14) for  $\lambda \in [\lambda_0, \infty)$ , as the proof is analogous for  $\lambda = \infty$ . When  $H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \leq t$ , then  $\xi^{(\lambda)}((\mathbf{0}, h_0))$  only depends on points of  $\mathcal{P}^{(\lambda)}$  in

$$\mathcal{U} := \bigcup_{w_1 \in [\Pi^\uparrow((\mathbf{0}, h_0))]^{(\lambda)} \cap \mathbb{R}^{d-1} \times (-\infty, t]} [\Pi^\downarrow(w_1)]^{(\lambda)}.$$

Let  $w = (v, h) \in \mathcal{U}$  and  $w_1 = (v_1, h_1)$ ,  $h_1 \leq t$ , be such that  $\partial[\Pi^\downarrow(w_1)]^{(\lambda)}$  contains both  $(\mathbf{0}, h_0)$  and  $w$ . Thanks to (4.8) and (4.9), which are valid for  $t \geq -h_0$ , we have  $e_\lambda(v_1, \mathbf{0}) \leq c\sqrt{t}/R_\lambda$  and  $e_\lambda(v, v_1) \leq c\sqrt{h_1 - h}/R_\lambda \leq c\sqrt{t - h}/R_\lambda$ . Consequently, there exists a constant  $c > 0$  such that

$$R_\lambda e_\lambda(v, \mathbf{0}) \leq c(\sqrt{2t} + \sqrt{t - h}). \tag{4.15}$$

There is a constant  $c > 0$  such that if  $h \in [-ct^2, \infty)$ , then  $|v| = R_\lambda e_\lambda(v, \mathbf{0}) \leq t$ .

Consequently, when  $\mathcal{P}^{(\lambda)} \cap \mathcal{U} \cap (\mathbb{R}^{d-1} \times (-\infty, -ct^2)) = \emptyset$ , then the localization radius of  $\xi^{(\lambda)}$  is less than  $t$ . This means that

$$P[R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)] > t] \leq P[H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \geq t] + P[\mathcal{P}^{(\lambda)} \cap \mathcal{U} \cap (\mathbb{R}^{d-1} \times (-\infty, -ct^2)) \neq \emptyset].$$

Given (4.15), we may use the same method as in (4.11) to obtain

$$\begin{aligned} & d\mathcal{P}^{(\lambda)}(\mathcal{P}^{(\lambda)} \cap \mathcal{U} \cap (\mathbb{R}^{d-1} \times (-\infty, -ct^2))) \\ & \leq c \int_{ct^2}^\infty e^{-ch} (\sqrt{t} + \sqrt{t+h})^{(d-1)} dh \leq ce^{-t^2/c}. \end{aligned}$$

□

4.2 Moment bounds for  $\xi^{(\lambda)}$ ,  $\lambda \in [\lambda_0, \infty]$

For a random variable  $X$  and  $p \in (0, \infty)$ , we let  $\|X\|_p := (\mathbb{E} |X|^p)^{1/p}$ .

**Lemma 4.4** *For all  $p \in [1, \infty)$  and  $\xi \in \Xi$ , there is a constant  $c > 0$  such that for all  $(v, h) \in W_\lambda$ ,  $\lambda \in [\lambda_0, \infty]$ , we have*

$$\mathbb{E} [|\xi^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})|^p] \leq c|h|^c \exp\left(-\frac{e^{h \vee 0}}{c}\right). \tag{4.16}$$

*Proof* We first prove (4.16) for a  $k$ -face functional  $\xi^{(\lambda)} := \xi_k^{(\lambda)}$ ,  $k \in \{0, 1, \dots, d-1\}$ . We start by showing for all  $\lambda \in [\lambda_0, \infty]$  and  $h \in \mathbb{R}$

$$\sup_{v \in \mathbb{R}^{d-1}} \mathbb{E} [|\xi^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})|^p] \leq c|h|^c. \tag{4.17}$$

Since  $\xi(x, \mathcal{P}_\lambda) \stackrel{D}{=} \xi(y, \mathcal{P}_\lambda)$  whenever  $|x| = |y|$ , it follows that for all  $(v, h) \in W_\lambda$ ,

$$\xi^{(\lambda)}(\mathbf{0}, h), \mathcal{P}^{(\lambda)} \stackrel{D}{=} \xi^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)}).$$

Consequently, without loss of generality we may put  $(v, h)$  to be  $(\mathbf{0}, h_0)$ .

Let  $N^{(\lambda)} := N^{(\lambda)}((\mathbf{0}, h_0)) := \text{card}\{\text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(\mathbf{0}, R)\}$  with  $R := R^{\xi^{(\lambda)}}[(\mathbf{0}, h_0)]$  the radius of spatial localization for  $\xi^{(\lambda)}$  at  $(\mathbf{0}, h_0)$ . Clearly

$$\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \leq \frac{1}{k+1} \binom{N^{(\lambda)}((\mathbf{0}, h_0))}{k}.$$

To show (4.17), given  $p \in [1, \infty)$ , it suffices to show there is a constant  $c := c(p, k, d)$  such that for all  $\lambda \in [\lambda_0, \infty]$

$$\mathbb{E} N^{(\lambda)}((\mathbf{0}, h_0))^{pk} \leq c|h_0|^c. \tag{4.18}$$

By (3.18), for all  $r \in [0, \pi R_\lambda]$  and  $\ell \in (-\infty, R_\lambda^2]$  we have

$$d\mathcal{P}^{(\lambda)}(C(\mathbf{0}, r) \cap (-\infty, \ell)) \leq cr^{d-1}(-\ell \vee 1)^c e^\ell.$$

Consequently, with  $H := H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})$  as in Lemma 4.2 and  $\text{Po}(\alpha)$  denoting a Poisson random variable with mean  $\alpha$ , we have for  $\lambda \in [\lambda_0, \infty]$

$$\begin{aligned} \mathbb{E} N^{(\lambda)}((\mathbf{0}, h_0))^{pk} &\leq \mathbb{E} [\text{card}(\mathcal{P}^{(\lambda)} \cap [C(\mathbf{0}, R) \cap (-\infty, H)])^{pk}] \\ &= \sum_{i=0}^\infty \sum_{j=h_0}^\infty \mathbb{E} [\text{Po}(d\mathcal{P}^{(\lambda)}(C(\mathbf{0}, R) \cap (-\infty, H)))^{pk} \mathbf{1}(i \leq R < i+1, j \leq H < j+1)] \\ &\leq \sum_{i=0}^\infty \sum_{j=h_0}^\infty \mathbb{E} [\text{Po}(c(i+1)^{d-1}(-j+1) \vee 1)^c e^{(j+1)}]^{pk} \mathbf{1}(R \geq i, H \geq j). \end{aligned}$$

We shall repeatedly use the moment bounds for Poisson random variables, namely  $\mathbb{E} [\text{Po}(\alpha)^r] \leq c(r)\alpha^r, r \in [1, \infty)$ . Using Hölder’s inequality, we get

$$\mathbb{E} N^{(\lambda)}((\mathbf{0}, h_0))^{pk} \leq c \sum_{i=0}^{\infty} \sum_{j=h_0}^{\infty} (i+1)^{d-1} (-j+1)^{\vee 1} e^{cpk} e^{(j+1)pk} P[R \geq i]^{1/3} P[H \geq j]^{1/3}.$$

Splitting the sum on the  $i$  indices into  $i \in [0, |h_0|]$  and  $i \in (|h_0|, \infty)$  yields with the help of Lemmas 4.1 and 4.2(a)

$$\begin{aligned} \mathbb{E} N^{(\lambda)}((\mathbf{0}, h_0))^{pk} &\leq c|h_0|^c \sum_{j=0}^{\infty} e^{(j+1)pk} e^{-e^j/c} \\ &\quad + c \sum_{i=|h_0|}^{\infty} \sum_{j=0}^{\infty} i^c e^{-i^2/c} e^{(j+1)pk} e^{-e^j/c} \leq c|h_0|^c. \end{aligned}$$

This yields the required bound (4.18).

To deduce (4.16), we argue as follows. First consider the case  $h_0 \in [0, \infty)$ . By the Cauchy-Schwarz inequality and (4.17)

$$\begin{aligned} \mathbb{E} [|\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})|^p] &\leq (\mathbb{E} |\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})|^{2p})^{1/2} P[|\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})| > 0]^{1/2} \\ &\leq (c(2p, k, d))^{1/2} |h_0|^{c_1(p,k,d)} P[|\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})| \neq 0]^{1/2}. \end{aligned}$$

The event  $\{|\xi^{(\lambda)}((\mathbf{0}, h_0), v)| \neq 0\}$  is a subset of the event that  $(\mathbf{0}, h_0)$  is extreme in  $\mathcal{P}^{(\lambda)}$  and we may now apply (4.12) for  $t = h_0$ , which is possible since we have assumed  $h_0$  is positive. This gives (4.16) for  $h_0 \in [0, \infty)$ . When  $h_0 \in (-\infty, 0)$  we bound  $P[|\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})| > 0]^{1/2}$  by  $c \exp(-e^0/c)$ ,  $c$  large, which shows (4.16) for  $h_0 \in (-\infty, 0)$ . This concludes the proof of (4.16) when  $\xi$  is a  $k$ -face functional.

We now prove (4.16) for  $\xi_v$ . For all  $L \in (0, \infty)$  and  $\lambda \in [\lambda_0, \infty)$ , we put  $D^{(\lambda)}(L) := \|\partial\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(\mathbf{0}, L)\|_{\infty}$ . Put  $R := R^{\xi_v^{(\lambda)}}[(\mathbf{0}, h_0)]$ . The identity (3.19) shows that  $|\xi_v^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})|$  is bounded by the product of  $c(1 + D^{(\lambda)}(R)/R_{\lambda}^2)^{d-1}$  and the Lebesgue measure of  $B(\mathbf{0}, R) \times [-D^{(\lambda)}(R), D^{(\lambda)}(R)]$ . We have

$$\mathbb{E} |\xi_v^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})|^p \leq c \mathbb{E} (R^{d-1} D^{(\lambda)}(R)^d)^p \leq c \|R^{p(d-1)}\|_2 \|D^{(\lambda)}(R)^{pd}\|_2,$$

by the Cauchy-Schwarz inequality. By the tail behavior for  $R$  we have  $\mathbb{E} R^r = r \int_0^{\infty} P[R > t] t^{r-1} dt \leq c(r)|h_0|^r$  for all  $r \in [1, \infty)$ . Also, for all  $r \in [1, \infty)$  we have

$$\mathbb{E} (D^{(\lambda)}(R))^r = \sum_{i=0}^{\infty} \mathbb{E} (D^{(\lambda)}(R))^r \mathbf{1}(i \leq R < i+1) \leq \sum_{i=0}^{\infty} \|D^{(\lambda)}(i+1)\|_2 P[R \geq i]^{1/2}.$$

By Lemma 4.2 we have  $\|D^{(\lambda)}(i+1)\|_2 \leq c(r)(i+1)^{2(d-1)r}, \lambda \in [\lambda_0, \infty)$ . We also have that  $P[R \geq i], i \geq |h_0|$ , decays exponentially fast, showing that  $\mathbb{E} (D^{(\lambda)}(R))^r \leq c(r)|h_0|^{2(d-1)r}$ . It follows that

$$\mathbb{E} |\xi_V^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})|^p \leq \|R^{p(d-1)}\|_2 \|D^{(\lambda)}(R)^{pd}\|_2 \leq c(p, d) |h_0|^{2pd(d-1)} |h_0|^{(d-1)/2},$$

which gives

$$\mathbb{E} |\xi_V^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})|^p \leq c|h_0|^c. \tag{4.19}$$

The bound (4.16) for  $\xi_V^{(\lambda)}$  follows from (4.19) in the same way that (4.17) implies (4.16) for  $\xi_k^{(\lambda)}$ . □

### 4.3 Scaling limits

The next two lemmas justify the assertion that functionals in  $\Xi^{(\infty)}$  are indeed scaling limits of their counterparts in  $\Xi^{(\lambda)}$ .

**Lemma 4.5** *For all  $h_0 \in \mathbb{R}$ ,  $r \in (0, \infty)$ , and  $\xi \in \Xi$  we have*

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \xi_{[r]}^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) = \mathbb{E} \xi_{[r]}^{(\infty)}((\mathbf{0}, h_0), \mathcal{P}). \tag{4.20}$$

*Proof* Put  $w_0 := (\mathbf{0}, h_0)$  and put  $S(r, l) := B_{d-1}(\mathbf{0}, r) \times [-l, l]$ , with  $l$  a fixed deterministic height. By Lemma 4.2 and the Cauchy-Schwarz inequality, it is enough to show

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)} \cap S(r, l)) = \mathbb{E} \xi_{[r]}^{(\infty)}(w_0, \mathcal{P} \cap S(r, l)).$$

It is understood that the left-hand side is determined by the geometry of the quasi-paraboloids  $\{[\Pi^\uparrow(w)]^{(\lambda)}, w \in \mathcal{P}^{(\lambda)} \cap S(r, l)$ , and similarly for the right-hand side. Equip the collection  $\mathcal{X}(r, l)$  of locally finite point sets in  $S(r, l)$  with the discrete topology. Thus if  $\mathcal{X}_i, i \geq 1$ , is a sequence in  $\mathcal{X}(r, l)$  and if

$$\lim_{i \rightarrow \infty} \mathcal{X}_i = \mathcal{X}, \text{ then } \mathcal{X}_i = \mathcal{X} \text{ for } i \geq i_0. \tag{4.21}$$

Recall that  $[\Pi^\downarrow(w)]^{(\infty)}$  coincides with  $\Pi^\downarrow(w)$ . For all  $\lambda \in [\lambda_0, \infty]$ ,  $w_1 \in W_\lambda$ , and  $\mathcal{X} \in \mathcal{X}(r, l)$  we define  $g_{k,\lambda} : W_\lambda \times \mathcal{X}(r, l) \rightarrow \mathbb{R}$  by taking  $g_{k,\lambda}(w_1, \mathcal{X})$  to be the product of  $(k + 1)^{-1}$  and the number of quasi parabolic  $k$ -dimensional faces of  $\bigcup_{w \in \mathcal{X}} [\Pi^\downarrow(w)]^{(\lambda)}$  which contain  $w_1$ , if  $w_1$  is a vertex in  $\mathcal{X}$ , otherwise  $g_{k,\lambda}(w_1, \mathcal{X}) = 0$ . Thus  $g_{k,\lambda}(w_1, \mathcal{X}) := \xi_{[r]}^{(\lambda)}(w_1, \mathcal{X} \cap S(r, l))$ .

Let  $\mathcal{X}$  be in regular position, that is to say the intersection of  $k$  quasi-paraboloids contains at most  $(d - k + 1)$  points of  $\mathcal{X}$  for all  $1 \leq k \leq d$ . Thus  $\mathcal{P}$  is in regular position with probability one. To apply the continuous mapping theorem (Theorem 5.5 in [7]), by (4.21), it is enough to show that  $g_{k,\lambda}(w_0, \mathcal{X})$  coincides with  $g_{k,\infty}(w_0, \mathcal{X})$  for  $\lambda$  large enough. Let  $\varepsilon > 0$  be the minimal distance between any down paraboloid containing  $d$  points of  $\mathcal{X}$  and the rest of the point set. Perturbations of the paraboloids within an  $\varepsilon$  parallel set do not change the number of  $k$ -dimensional faces. In particular, for  $\lambda$  large enough, the set  $\partial(\bigcup_{w \in \mathcal{X}} [\Pi^\downarrow(w)]^{(\lambda)})$  is included in that parallel set so that

the number of  $k$ -dimensional faces does not change. Thus  $g_{k,\lambda}(w_0, \mathcal{X})$  coincides with  $g_{k,\infty}(w_0, \mathcal{X})$  for large  $\lambda$ .

Since  $\mathcal{P}^{(\lambda)} \xrightarrow{\mathcal{D}} \mathcal{P}$ , we may apply the continuous mapping theorem to get

$$\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) \xrightarrow{\mathcal{D}} \xi_{[r]}^{(\infty)}(w_0, \mathcal{P})$$

as  $\lambda \rightarrow \infty$ . The convergence in distribution extends to convergence of expectations by the uniform integrability of  $\xi_{[r]}^{(\lambda)}$ , which follows from moment bounds for  $\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})$  analogous to those for  $\xi_k^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})$  as given in Lemma 4.4. This proves (4.20) when  $\xi$  is a generic  $k$ -face functional.

Next we show for  $\xi := \xi_V, r \in (0, \infty)$  that

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)} \cap S(r, l)) = \mathbb{E} [\xi_{[r]}^{(\infty)}(w_0, \mathcal{P} \cap S(r, l))].$$

This will yield (4.20). Recall that  $\text{Vol}_d^{(\lambda)}$  is the image of  $R_\lambda \text{Vol}_d$  under  $T^{(\lambda)}$ . Recall from (3.21) the definition of  $\text{Cyl}^{(\lambda)}(w)$ . For  $\lambda \in [\lambda_0, \infty]$ , we define this time  $\tilde{g}_{k,\lambda} : \mathbb{R}^{d-1} \times \mathbb{R} \times S(r, H) \mapsto \mathbb{R}$  by

$$\begin{aligned} \tilde{g}_{k,\lambda}(w, \mathcal{X}) &= \xi_{[r]}^{(\lambda)}(w, \mathcal{X} \cap S(r, l)) \\ &= \text{Vol}_d^{(\lambda)}(\{(v, h) \in S(r, l) : 0 \leq h \leq \partial \Phi^{(\lambda)}(\mathcal{X})(v), v \in \text{Cyl}^{(\lambda)}(w), \Phi^{(\lambda)}(\mathcal{X})(v) \geq 0\}) \\ &\quad - \text{Vol}_d^{(\lambda)}(\{(v, h) \in S(r, l) : \Phi^{(\lambda)}(\mathcal{X})(v) \leq h \leq 0, v \in \text{Cyl}^{(\lambda)}(w), \Phi^{(\lambda)}(\mathcal{X})(v) < 0\}). \end{aligned} \tag{4.22}$$

Recalling (4.21), it is enough to show for a fixed point set  $\mathcal{X}$  in regular position that

$$\lim_{\lambda \rightarrow \infty} |\tilde{g}_{k,\lambda}(w, \mathcal{X}) - \tilde{g}_{k,\infty}(w, \mathcal{X})| = 0.$$

We show that the first term in (4.22) comprising  $\tilde{g}_{k,\lambda}(w, \mathcal{X})$  converges to the first term comprising  $\tilde{g}_{k,\infty}(w, \mathcal{X})$ . In other words, setting for all  $\lambda \in [\lambda_0, \infty)$

$$F^{(\lambda)}(\mathcal{X}) := \{(v, h) \in S(r, l) : 0 \leq h \leq \partial \Phi^{(\lambda)}(\mathcal{X})(v), v \in \text{Cyl}^{(\lambda)}(w), \Phi^{(\lambda)}(\mathcal{X})(v) \geq 0\}$$

and writing  $F(\mathcal{X})$  for  $F^{(\infty)}(\mathcal{X})$ , we show

$$\lim_{\lambda \rightarrow \infty} |\text{Vol}_d^{(\lambda)}(F^{(\lambda)}(\mathcal{X})) - \text{Vol}_d(F(\mathcal{X}))| = 0.$$

The proof that the second term comprising  $\tilde{g}_{k,\lambda}(w, \mathcal{X})$  converges to the second term comprising  $\tilde{g}_{k,\infty}(w, \mathcal{X})$  is identical. We have

$$\begin{aligned} |\text{Vol}_d^{(\lambda)}(F^{(\lambda)}(\mathcal{X})) - \text{Vol}_d(F(\mathcal{X}))| &\leq |\text{Vol}_d^{(\lambda)}(F^{(\lambda)}(\mathcal{X})) - \text{Vol}_d^{(\lambda)}(F(\mathcal{X}))| \\ &\quad + |\text{Vol}_d^{(\lambda)}(F(\mathcal{X})) - \text{Vol}_d(F(\mathcal{X}))|. \end{aligned} \tag{4.23}$$

Since  $\partial\Phi^{(\lambda)}(\mathcal{X})$  converges uniformly to  $\partial\Phi(\mathcal{X})$  on compacts (recall Lemma 3.1; see also the proof of Proposition 5.1 below) and since  $d^H(\text{Cyl}^{(\lambda)}(w), \text{Cyl}(w))$  decreases to zero as  $\lambda \rightarrow \infty$  (indeed  $\partial(\text{Cyl}^{(\lambda)}(w)) \rightarrow \partial\text{Cyl}(w)$  uniformly), we get for  $\lambda \in [\lambda_0, \infty)$  that  $F^{(\lambda)}(\mathcal{X}) \Delta F(\mathcal{X})$  is a subset of a set  $A(\mathcal{X}) \subset \mathbb{R}^d$  of arbitrarily small volume. So  $|\text{Vol}_d^{(\lambda)}(F^{(\lambda)}(\mathcal{X})) - \text{Vol}_d^{(\lambda)}(F(\mathcal{X}))| \leq \text{Vol}_d^{(\lambda)}(A(\mathcal{X}))$ . By Lemma 3.2, we have  $\text{Vol}_d^{(\lambda)} \xrightarrow{\mathcal{D}} \text{Vol}_d$  and thus the first term in (4.23) goes to zero as  $\lambda \rightarrow \infty$ . Appealing again to  $\text{Vol}_d^{(\lambda)} \xrightarrow{\mathcal{D}} \text{Vol}_d$ , the second term in (4.23) likewise tends to zero, showing (4.20) as desired.  $\square$

**Lemma 4.6** *For all  $h_0 \in \mathbb{R}$  and  $\xi \in \Xi$  we have*

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) = \mathbb{E} \xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P}).$$

*Proof* Let  $w_0 := (\mathbf{0}, h_0)$ . By Lemma 4.5, given  $\epsilon > 0$ , we have for all  $\lambda \in [\lambda_0(\epsilon), \infty)$

$$|\mathbb{E} \xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \mathbb{E} \xi_{[r]}^{(\infty)}(w_0, \mathcal{P})| < \epsilon. \tag{4.24}$$

We now show that replacing  $\xi_{[r]}^{(\lambda)}$  and  $\xi_{[r]}^{(\infty)}$  by  $\xi^{(\lambda)}$  and  $\xi^{(\infty)}$ , respectively, introduces negligible error in (4.24). Write

$$\begin{aligned} & |\mathbb{E} \xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \mathbb{E} \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})| \\ &= |\mathbb{E} (\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})) \mathbf{1}(R^{\xi^{(\lambda)}}[w_0] < r)| \\ &\quad + |\mathbb{E} (\xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})) \mathbf{1}(R^{\xi^{(\lambda)}}[w_0] > r)|. \end{aligned}$$

The first term vanishes by definition of  $R^{\xi^{(\lambda)}}[w_0]$ . By the Cauchy-Schwarz inequality and Lemma 4.3, the second term is bounded by

$$\begin{aligned} & \| \xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) \|_2 \\ & P[R^{\xi^{(\lambda)}}[w_0] > r]^{1/2} \leq c P[R^{\xi^{(\lambda)}}[w_0] > r]^{1/2} \leq \epsilon \end{aligned} \tag{4.25}$$

if  $r \in [|h_0|, \infty)$  is large enough. For  $r \in [r_0(\epsilon), h_0), \infty)$  and  $\lambda \in [\lambda_0(\epsilon), \infty)$  it follows that

$$|\mathbb{E} \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \mathbb{E} \xi_{[r]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})| < \epsilon. \tag{4.26}$$

Similarly for  $r \in [r_1(\epsilon), h_0), \infty)$  we have

$$|\mathbb{E} \xi^{(\infty)}(w_0, \mathcal{P}) - \mathbb{E} \xi_{[r]}^{(\infty)}(w_0, \mathcal{P})| < \epsilon. \tag{4.27}$$

Combining (4.24)–(4.27) and using the triangle inequality we get for  $r \geq (r_0(\epsilon) \vee r_1(\epsilon))$  and  $\lambda \in [\lambda_0(\epsilon), \infty)$

$$|\mathbb{E} \xi^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)}) - \mathbb{E} \xi^{(\infty)}(w_0, \mathcal{P})| < 3\epsilon.$$

Lemma 4.6 follows since  $\epsilon$  is arbitrary.  $\square$

#### 4.4 Two point correlation function for $\xi^{(\lambda)}$

For all  $h \in \mathbb{R}$ ,  $(v_1, h_1) \in W_\lambda$ , and  $\xi \in \Xi$  we extend definition (2.3) by putting for all  $\lambda \in [\lambda_0, \infty]$

$$\begin{aligned}
 c^{\xi^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1)) &:= \mathbb{E} [\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\}) \xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(\mathbf{0}, h_0)\})] \\
 &\quad - \mathbb{E} \xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \mathbb{E} \xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)}).
 \end{aligned}$$

The next lemma shows convergence of the re-scaled two-point correlation functions on re-scaled input  $\mathcal{P}^{(\lambda)}$  to their counterpart correlation functions on the limit input  $\mathcal{P}$ .

**Lemma 4.7** *For all  $h_0 \in \mathbb{R}$ ,  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}$ , and  $\xi \in \Xi$  we have*

$$\lim_{\lambda \rightarrow \infty} c^{\xi^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1)) = c^{\xi^{(\infty)}}((\mathbf{0}, h_0), (v_1, h_1)).$$

*Proof* We deduce from Lemma 4.6 that

$$\begin{aligned}
 &\lim_{\lambda \rightarrow \infty} \mathbb{E} \xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \mathbb{E} \xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)}) \\
 &= \mathbb{E} \xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P}) \mathbb{E} \xi^{(\infty)}((v_1, h_1), \mathcal{P}).
 \end{aligned}$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
 &|\mathbb{E} [\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\}) \xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(\mathbf{0}, h_0)\})] \\
 &\quad - \mathbb{E} [\xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P} \cup \{(v_1, h_1)\}) \xi^{(\infty)}((v_1, h_1), \mathcal{P} \cup \{(\mathbf{0}, h_0)\})]| \\
 &\leq T_1(\lambda) + T_2(\lambda),
 \end{aligned}$$

where

$$\begin{aligned}
 T_1(\lambda) &:= \mathbb{E} [|\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\}) - \xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P} \cup \{(v_1, h_1)\})|^2]^{1/2} \\
 &\quad \times \mathbb{E} [|\xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(\mathbf{0}, h_0)\})|^2]^{1/2}
 \end{aligned}$$

and

$$\begin{aligned}
 T_2(\lambda) &:= \mathbb{E} [|\xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(\mathbf{0}, h_0)\}) - \xi^{(\infty)}((v_1, h_1), \mathcal{P} \cup \{(\mathbf{0}, h_0)\})|^2]^{1/2} \\
 &\quad \times \mathbb{E} [|\xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\})|^2]^{1/2}.
 \end{aligned}$$

Throughout we let  $\mathcal{P}^{(\lambda)}$ ,  $\lambda \geq 1$ , and  $\mathcal{P}$  be defined on the same probability space, with  $\mathcal{P}^{(\lambda)}$  independent of  $\mathcal{P}$  for all  $\lambda \geq 1$ . We couple  $\mathcal{P}^{(\lambda)}$  and  $\mathcal{P}$  so that

$$\begin{aligned} & (\xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(\mathbf{0}, h_0)\}), \xi^{(\infty)}((v_1, h_1), \mathcal{P} \cup \{(\mathbf{0}, h_0)\})) \\ & \stackrel{D}{=} (\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup \{(-v_1, h_0)\}), \xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P} \cup \{(-v_1, h_0)\})). \end{aligned} \quad (4.28)$$

We show first  $\lim_{\lambda \rightarrow \infty} T_1(\lambda) = 0$ . We have seen in the proof of Lemma 4.5 that  $\xi_{[r]}^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\})$  converges in distribution to  $\xi_{[r]}^{(\infty)}((\mathbf{0}, h_0), \mathcal{P} \cup \{(v_1, h_1)\})$  for every  $r > 0$ . Lemma 4.4 implies that this family is uniformly integrable so the convergence occurs in  $L^2$ , that is to say

$$\lim_{\lambda \rightarrow \infty} (\mathbb{E} |\xi_{[r]}^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\}) - \xi_{[r]}^{(\infty)}((\mathbf{0}, h_0), \mathcal{P} \cup \{(v_1, h_1)\})|^2)^{1/2} = 0.$$

Using the same method as in the proof of Lemma 4.6, we obtain

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} [|\xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\}) - \xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P} \cup \{(v_1, h_1)\})|^2] = 0. \quad (4.29)$$

By Lemma 4.4, the variables  $\xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(\mathbf{0}, h_0)\})$  are uniformly bounded in  $L^2$  so we deduce from (4.29) that  $\lim_{\lambda \rightarrow \infty} T_1(\lambda) = 0$ . To see that  $\lim_{\lambda \rightarrow \infty} T_2(\lambda) = 0$ , we use (4.28) and we follow the proof that  $\lim_{\lambda \rightarrow \infty} T_1(\lambda) = 0$ .  $\square$

The next lemma shows that the re-scaled and limit two point correlation function decays exponentially fast with the distance between spatial coordinates of the input and super-exponentially fast with respect to positive height coordinates.

**Lemma 4.8** *For all  $\xi \in \Xi$  there is a constant  $c := c(\xi, d) \in (0, \infty)$  such that for all  $(v_1, h_1) \in W_\lambda$  satisfying  $|v_1| \geq 2 \max(|h_0|, |h_1|)$  and  $\lambda \in [\lambda_0, \infty]$  we have*

$$|c^{\xi^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1))| \leq c|h_0|^c|h_1|^{c^3} \exp\left(\frac{-1}{c}(|v_1|^2 + e^{h_0 \vee 0} + e^{h_1 \vee 0})\right). \quad (4.30)$$

*Proof* Let  $x_\lambda := [T^{(\lambda)}]^{-1}((\mathbf{0}, h_0))$  and  $y_\lambda := [T^{(\lambda)}]^{-1}((v_1, h_1))$ . Put

$$\begin{aligned} X_\lambda & := \xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)} \cup (v_1, h_1)) = \xi(x_\lambda, \mathcal{P}_\lambda \cup y_\lambda), \\ Y_\lambda & := \xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup (\mathbf{0}, h_0)) = \xi(y_\lambda, \mathcal{P}_\lambda \cup x_\lambda), \\ \tilde{X}_\lambda & := \xi^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) = \xi(x_\lambda, \mathcal{P}_\lambda), \\ \text{and } \tilde{Y}_\lambda & := \xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)}) = \xi(y_\lambda, \mathcal{P}_\lambda). \end{aligned}$$

We have

$$|c^{\xi^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1))| = |\mathbb{E} X_\lambda Y_\lambda - \mathbb{E} \tilde{X}_\lambda \mathbb{E} \tilde{Y}_\lambda|$$

which gives for all  $r \in (0, \infty)$

$$\begin{aligned} |c^{\xi^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1))| & \leq |\mathbb{E} X_\lambda Y_\lambda \mathbf{1}(R^\xi(x_\lambda, \mathcal{P}_\lambda) \leq r, R^\xi(y_\lambda, \mathcal{P}_\lambda) \leq r) \\ & \quad - \mathbb{E} \tilde{X}_\lambda \mathbf{1}(R^\xi(x_\lambda, \mathcal{P}_\lambda) \leq r) \mathbb{E} \tilde{Y}_\lambda \mathbf{1}(R^\xi(y_\lambda, \mathcal{P}_\lambda) \leq r)| \end{aligned}$$

$$\begin{aligned}
 &+ |\mathbb{E} X_\lambda Y_\lambda [\mathbf{1}(R^\xi(x_\lambda, \mathcal{P}_\lambda) \geq r) + \mathbf{1}(R^\xi(y_\lambda, \mathcal{P}_\lambda) \geq r)]| \\
 &+ |\mathbb{E} \tilde{X}_\lambda \mathbb{E} \tilde{Y}_\lambda \mathbf{1}(R^\xi(x_\lambda, \mathcal{P}_\lambda) \geq r)| \\
 &+ |\mathbb{E} \tilde{X}_\lambda \mathbb{E} \tilde{Y}_\lambda \mathbf{1}(R^\xi(y_\lambda, \mathcal{P}_\lambda) \geq r)|.
 \end{aligned}$$

Put  $r := |v_1|/2R_\lambda$ . This choice of  $r$  ensures that the difference of the first two terms is zero by independence of  $X_\lambda \mathbf{1}(R^\xi(x_\lambda, \mathcal{P}_\lambda) \leq r)$  and  $Y_\lambda \mathbf{1}(R^\xi(y_\lambda, \mathcal{P}_\lambda) \leq r)$ . Recall that  $R^\xi(x_\lambda, \mathcal{P}_\lambda)$  and  $R^\xi(y_\lambda, \mathcal{P}_\lambda)$  have the same distribution. When  $|v_1| \geq 2 \max(|h_0|, |h_1|)$ , Hölder’s inequality implies that the third term is bounded by

$$\begin{aligned}
 &||X_\lambda||_3 ||Y_\lambda||_3 [P[R^\xi(x_\lambda, \mathcal{P}_\lambda) \geq r]^{1/3} + P[R^\xi(y_\lambda, \mathcal{P}_\lambda) \geq r]^{1/3}] \\
 &\leq c|h_0|^c |h_1|^c \exp\left(\frac{-1}{c}(e^{h_0 \vee 0} + e^{h_1 \vee 0})\right) P[R^\xi(x_\lambda, \mathcal{P}_\lambda) \geq r]^{1/3} \tag{4.31} \\
 &\leq c|h_0|^c |h_1|^c \exp\left(\frac{-1}{c}(|v_1|^2 + e^{h_0 \vee 0} + e^{h_1 \vee 0})\right).
 \end{aligned}$$

The fourth and fifth terms are bounded similarly, giving (4.30). □

### 5 Proofs of main results

#### 5.1 Proof of Theorems 1.1 and 1.2

The next result contains Theorem 1.2 and it yields Theorem 1.1, since it implies that the set  $\text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)})$  of extreme points of  $\mathcal{P}^{(\lambda)}$  converges in law to  $\text{Ext}(\mathcal{P})$  as  $\lambda \rightarrow \infty$  (indeed, the set  $\text{Ext}^{(\lambda)}(\mathcal{P}^{(\lambda)})$  is also the set of local minima of the function  $\partial\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ ).

**Proposition 5.1** *Fix  $L \in (0, \infty)$ . The boundary of  $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  converges in probability as  $\lambda \rightarrow \infty$  to the boundary of  $\Psi(\mathcal{P})$  in the space  $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$  equipped with the supremum norm. Similarly, the boundary of  $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  converges in probability as  $\lambda \rightarrow \infty$  to the Burgers’ festoon  $\partial(\Phi(\mathcal{P}))$ .*

*Proof* We only prove the first convergence statement as the second is handled similarly. We show for fixed  $L \in (0, \infty)$  that the boundary of  $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  converges in law to  $\partial(\Psi(\mathcal{P}))$  in the space  $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$ . With  $L$  fixed, for all  $l \in [0, \infty)$  and  $\lambda \in [0, \infty)$ , let  $E(L, l, \lambda)$  be the event that the heights of  $\partial(\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}))$  and  $\partial(\Psi(\mathcal{P}))$  belong to  $[-l, l]$  over the spatial region  $B_{d-1}(\mathbf{0}, L)$ . By Lemma 4.2, we have that  $P[E(L, l, \lambda)^c]$  decays exponentially fast in  $l$ , uniformly in  $\lambda$ , and so it is enough to show, conditional on  $E(L, l, \lambda)$ , that  $\partial(\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}))$  is close to  $\partial(\Psi(\mathcal{P}))$  in the space  $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$ ,  $\lambda$  large.

Recalling the definition of  $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$  at (3.4), we need to show, conditional on  $E(L, l, \lambda)$ , that the boundary of

$$\bigcup_{w \in \mathcal{P}^{(\lambda)} \cap \mathcal{C}(\mathbf{0}, L)} ([\Pi^\uparrow(w)]^{(\lambda)} \cap \mathcal{C}(\mathbf{0}, L))$$

is close to the boundary of

$$\bigcup_{w \in \mathcal{P} \cap C(\mathbf{0}, L)} (\Pi^\uparrow(w) \cap C(\mathbf{0}, L)). \tag{5.1}$$

By Lemma 3.1, given  $w_1 := (v_1, h_1) \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, L)$ , it follows that on  $E(L, l, \lambda)$  the boundary of  $[\Pi^\uparrow(w_1)]^{(\lambda)} \cap C(\mathbf{0}, L)$  is within  $O(R_\lambda^{-1})$  of the boundary of  $\Pi^\uparrow(w_1) \cap C(\mathbf{0}, L)$ . The boundary of  $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(\mathbf{0}, L)$  is a.s. the finite union of graphs of the above form and is thus a.s. within  $O(R_\lambda^{-1})$  of the boundary of

$$\bigcup_{w \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, L)} (\Pi^\uparrow(w) \cap C(\mathbf{0}, L)).$$

It therefore suffices to show that the boundary of  $\bigcup_{w \in \mathcal{P}^{(\lambda)} \cap C(\mathbf{0}, L)} (\Pi^\uparrow(w) \cap C(\mathbf{0}, L))$  is close to the boundary of the set given at (5.1). However, we may couple  $\mathcal{P}^{(\lambda)}$  and  $\mathcal{P}$  on  $B_{d-1}(\mathbf{0}, L) \times [-l, l]$  so that they coincide except on a set with probability less than  $\epsilon$ , showing the desired closeness with probability at least  $1 - \epsilon$ .  $\square$

### 5.2 Proof of expectation asymptotics (2.5)

For  $g \in \mathcal{C}(\mathbb{S}^{d-1})$  and  $\xi \in \Xi$  we have

$$\mathbb{E}[\langle g_{R_\lambda}, \mu_\lambda^\xi \rangle] = \int_{\mathbb{R}^d} g\left(\frac{x}{R_\lambda}\right) \mathbb{E}[\xi(x, \mathcal{P}_\lambda)] \lambda \phi(x) dx. \tag{5.2}$$

Since  $\xi(x, \mathcal{P}_\lambda) \stackrel{D}{=} \xi(y, \mathcal{P}_\lambda)$  as soon as  $|x| = |y|$ , we have

$$\mathbb{E} \xi(x, \mathcal{P}_\lambda) = \mathbb{E} \xi(|x|u_0, \mathcal{P}_\lambda) = \mathbb{E} [\xi^{(\lambda)}\left(\mathbf{0}, h_0, \mathcal{P}^{(\lambda)}\right)]$$

where  $h_0$  is defined by  $|x| = R_\lambda(1 - h_0/R_\lambda^2)$ . Writing  $u = x/|x|$ , we have by (3.16)  $dx = [R_\lambda(1 - h_0/R_\lambda^2)]^{d-1} R_\lambda^{-1} dh_0 d\sigma_{d-1}(u)$ . Consequently, we see that  $R_\lambda^{-(d-1)} \mathbb{E}[\langle g_{R_\lambda}, \mu_\lambda^\xi \rangle]$  transforms to

$$\int_{u \in \mathbb{S}^{d-1}} \int_{h_0 \in (-\infty, R_\lambda^2]} g\left(u\left(1 - \frac{h_0}{R_\lambda^2}\right)\right) \mathbb{E} \left[ \xi^{(\lambda)}\left(\mathbf{0}, h_0, \mathcal{P}^{(\lambda)}\right) \right] \tilde{\phi}_\lambda(u, h_0) \left(1 - \frac{h_0}{R_\lambda^2}\right)^{d-1} dh_0 d\sigma_{d-1}(u),$$

where  $\tilde{\phi} : \mathbb{S}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}$  is given by

$$\tilde{\phi}_\lambda(u, h_0) := \frac{\lambda}{R_\lambda} \phi\left(u \cdot R_\lambda\left(1 - \frac{h_0}{R_\lambda^2}\right)\right). \tag{5.3}$$

By (3.17) we have for all  $u \in \mathbb{S}^{d-1}$  that

$$\tilde{\phi}_\lambda(u, h_0) = \frac{\sqrt{2 \log \lambda}}{R_\lambda} \exp\left(h_0 - \frac{h_0^2}{2R_\lambda^2}\right).$$

Thus there is  $c \in (0, \infty)$  such that for all  $h_0 \in \mathbb{R}$  we have

$$\sup_{u \in \mathbb{S}^{d-1}} \sup_{\lambda \geq 3} \tilde{\phi}_\lambda(u, h_0) \leq ce^{h_0} \tag{5.4}$$

and for all  $u \in \mathbb{S}^{d-1}$

$$\lim_{\lambda \rightarrow \infty} \tilde{\phi}_\lambda(u, h_0) = e^{h_0}. \tag{5.5}$$

By the continuity of  $g$ , Lemma 4.6 and the limit (5.5), we have for  $h_0 \in (-\infty, R_\lambda^2)$  that the integrand inside the double integral converges to  $g(u) \mathbb{E} [\xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P})] e^{h_0}$  as  $\lambda \rightarrow \infty$ . Moreover, by (5.4) and the moment bounds of Lemma 4.4, the integrand is dominated by the product of a polynomial in  $h_0$  and an exponentially decaying function of  $h_0$ . The dominated convergence theorem gives the claimed result (2.5).  $\square$

### 5.3 Proof of variance asymptotics (2.6)

For  $g \in \mathcal{C}(\mathbb{S}^{d-1})$ , using the Mecke-Slivnyak formula (Corollary 3.2.3 in [23]), we have

$$\text{Var}[\langle g_{R_\lambda}, \mu_\lambda^\xi \rangle] := I_1(\lambda) + I_2(\lambda), \tag{5.6}$$

where

$$I_1(\lambda) := \int_{\mathbb{R}^d} g\left(\frac{x}{R_\lambda}\right)^2 \mathbb{E} [\xi(x, \mathcal{P}_\lambda)^2] \lambda \phi(x) dx$$

and

$$I_2(\lambda) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left(\frac{x}{R_\lambda}\right) g\left(\frac{y}{R_\lambda}\right) [\mathbb{E} \xi(x, \mathcal{P}_\lambda \cup y) \xi(y, \mathcal{P}_\lambda \cup x) - \mathbb{E} \xi(x, \mathcal{P}_\lambda) \mathbb{E} \xi(y, \mathcal{P}_\lambda)] \lambda^2 \phi(y) \phi(x) dy dx.$$

We examine  $\lim_{\lambda \rightarrow \infty} R_\lambda^{-(d-1)} I_1(\lambda)$  and  $\lim_{\lambda \rightarrow \infty} R_\lambda^{-(d-1)} I_2(\lambda)$  separately. As in the proof of expectation asymptotics (2.5), we have

$$\lim_{\lambda \rightarrow \infty} R_\lambda^{-(d-1)} I_1(\lambda) = \int_{-\infty}^\infty \mathbb{E} \xi_k^{(\infty)}((\mathbf{0}, h_0), \mathcal{P})^2 e^{h_0} dh_0 \int_{\mathbb{S}^{d-1}} g(u)^2 du. \tag{5.7}$$

Next consider  $\lim_{\lambda \rightarrow \infty} R_\lambda^{-(d-1)} I_2(\lambda)$ . For  $x \in \mathbb{R}^d$  we write

$$x = u R_\lambda \left( 1 - \frac{h_0}{R_\lambda^2} \right), \quad (u, h_0) \in \mathbb{S}^{d-1} \times \mathbb{R}. \tag{5.8}$$

We now re-scale the integrand in  $I_2(\lambda)$  as follows. Given  $u := u_x \in \mathbb{S}^{d-1}$  in the definition of  $x$ , define  $T^{(\lambda)}$  as in (1.5), but with  $u_0$  there replaced by  $u$ . Write  $T_u^{(\lambda)}$  to denote the dependency on  $u$ . Denoting by  $(\mathbf{0}, h_0)$  and  $(v_1, h_1)$  the images under  $T_u^{(\lambda)}(x)$  of  $x$  and  $y$  respectively, we notice that  $R_\lambda^{-(d-1)} I_2(\lambda)$  is transformed as follows.

- (i) The ‘covariance’ term  $[\mathbb{E} \xi(x, \mathcal{P}_\lambda \cup y) \xi(y, \mathcal{P}_\lambda \cup x) - \mathbb{E} \xi(x, \mathcal{P}_\lambda) \mathbb{E} \xi(y, \mathcal{P}_\lambda)]$  transforms to  $c^{\xi^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1))$ . By Lemma 4.7 we have uniformly in  $v_1 \in T_u^{(\lambda)}(\mathbb{S}^{d-1})$  and  $h_0, h_1 \in \mathbb{R}$  that

$$\lim_{\lambda \rightarrow \infty} c^{\xi^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1)) = c^{\xi^{(\infty)}}((\mathbf{0}, h_0), (v_1, h_1)). \tag{5.9}$$

- (ii) The product  $g\left(\frac{x}{R_\lambda}\right) g\left(\frac{y}{R_\lambda}\right)$  becomes

$$f_{1,\lambda}(u, h_0, v_1, h_1) := g\left(u \left(1 - \frac{h_0}{R_\lambda^2}\right)\right) g(R_\lambda^{-1} [T_u^{(\lambda)}]^{-1}((v_1, h_1))).$$

Using (1.5) and (5.8), we notice that  $[T_u^{(\lambda)}]^{-1}((v_1, h_1)) = R_\lambda \left(1 - \frac{h_1}{R_\lambda^2}\right) \exp_{d-1} R_\lambda^{-1} v_1$  and consequently

$$\lim_{\lambda \rightarrow \infty} R_\lambda^{-1} [T_u^{(\lambda)}]^{-1}((v_1, h_1)) = u.$$

By continuity of  $g$ , we then have uniformly in  $v_1 \in T_u^{(\lambda)}(\mathbb{S}^{d-1})$  and  $h_0, h_1 \in \mathbb{R}$  that

$$\lim_{\lambda \rightarrow \infty} f_{1,\lambda}(u, h_0, v_1, h_1) = g(u)^2. \tag{5.10}$$

- (iii) The double integral over  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$  transforms into a quadruple integral over  $(u, h_0, v_1, h_1) \in \mathbb{S}^{d-1} \times (-\infty, R_\lambda^2] \times T_u^{(\lambda)}(\mathbb{S}^{d-1}) \times (-\infty, R_\lambda^2]$ .

- (iv) By (3.18), the differential  $\lambda \phi(y) dy$  transforms to

$$\frac{\sin^{d-2}(R_\lambda^{-1} |v_1|)}{|R_\lambda^{-1} v_1|^{d-2}} \frac{\sqrt{2 \log \lambda}}{R_\lambda} \left(1 - \frac{h_1}{R_\lambda^2}\right)^{d-1} \exp\left(h_1 - \frac{h_1^2}{2R_\lambda^2}\right) dv_1 dh_1$$

whereas  $R_\lambda^{-(d-1)}\lambda\phi(x)dx$  transforms to

$$\tilde{\phi}_\lambda(u, h_0) \left(1 - \frac{h_0}{R_\lambda^2}\right)^{d-1} dh_0 d\sigma_{d-1}(u).$$

Thus the product  $R_\lambda^{-(d-1)}\lambda^2\phi(y)\phi(x)dydx$  transforms to

$$f_{2,\lambda}(u, h_0, v_1, h_1)d\sigma_{d-1}(u)dh_0dv_1dh_1$$

where

$$f_{2,\lambda}(u, h_0, v_1, h_1) := \frac{\sin^{d-2}(R_\lambda^{-1}|v_1|)}{|R_\lambda^{-1}v_1|^{d-2}} \frac{\sqrt{2\log\lambda}}{R_\lambda} \left(1 - \frac{h_1}{R_\lambda^2}\right)^{d-1} \exp\left(h_1 - \frac{h_1^2}{2R_\lambda^2}\right) \tilde{\phi}_\lambda(u, h_0) \left(1 - \frac{h_0}{R_\lambda^2}\right)^{d-1}.$$

As in the proof of Lemma 3.2 and by (5.5) we have uniformly in  $u \in \mathbb{S}^{d-1}$ ,  $v_1 \in T^{(\lambda)}(\mathbb{S}^{d-1})$  and  $h_0, h_1 \in \mathbb{R}$  that

$$\lim_{\lambda \rightarrow \infty} f_{2,\lambda}(u, h_0, v_1, h_1) = e^{h_0+h_1}. \tag{5.11}$$

We re-write  $R_\lambda^{-(d-1)}I_2(\lambda)$  as

$$= \int_{u \in \mathbb{S}^{d-1}} \int_{h_0 \in (-\infty, R_\lambda^2]} \int_{T_u^{(\lambda)}(\mathbb{S}^{d-1})} \int_{h_1 \in (-\infty, R_\lambda^2]} F_\lambda(u, h_0, v_1, h_1) dh_1 dv_1 dh_0 d\sigma_{d-1}(u),$$

where

$$F_\lambda(u, h_0, v_1, h_1) := f_{1,\lambda}(u, h_0, v_1, h_1) c^{\xi(\lambda)}((\mathbf{0}, h_0), (v_1, h_1)) f_{2,\lambda}(u, h_0, v_1, h_1).$$

Combining the limits (5.9)–(5.11), it follows for all  $u \in \mathbb{S}^{d-1}$ ,  $h_0, h_1 \in \mathbb{R}$  and  $v_1 \in T^{(\lambda)}(\mathbb{S}^{d-1})$ , that we have the pointwise convergence

$$\lim_{\lambda \rightarrow \infty} F_\lambda(u, h_0, v_1, h_1) = g(u)^2 c^{\xi(\infty)}((\mathbf{0}, h_0), (v_1, h_1)) e^{h_0+h_1}.$$

By Lemma 4.8, (4.31) and (5.4), we get that

$$|F_\lambda(u, h_0, v_1, h_1)| \leq c|h_0|^c|h_1|^c \exp\left(\frac{-1}{c}(e^{h_0 \vee 0} + e^{h_1 \vee 0}) + h_0 + h_1\right) \cdot \left(\exp\left(\frac{-1}{c}|v_1|^2\right) + \mathbf{1}(|v_1| \leq 2 \max(|h_0|, |h_1|))\right).$$

Using that there exists  $c' > 0$  such that  $\frac{1}{c}e^{h \vee 0} - h \geq \frac{1}{c'}e^{h \vee 0}$  holds for all  $h \in \mathbb{R}$ , we obtain that  $F_\lambda(u, h_0, v_1, h_1)$  is dominated by an exponentially decaying function of all arguments and is therefore integrable. The dominated convergence theorem gives

$$\lim_{\lambda \rightarrow \infty} R_\lambda^{-(d-1)} I_2(\lambda) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^{d-1}} \int_{h_0 \in (-\infty, \infty)} \int_{h_1 \in (-\infty, \infty)} g(u)^2 e^{\xi^{(\infty)}}((\mathbf{0}, h_0), (v_1, h_1)) e^{h_0+h_1} dh_0 dh_1 dv_1 d\sigma_{d-1}(u). \tag{5.12}$$

Combining (5.7) and (5.12) gives the claimed variance asymptotics (2.6). The positivity of  $F_{k,d}$  is established in [4], concluding the proof of (2.6).  $\square$

### 5.4 Proof of Corollary 1.1

Define  $\xi(x, \mathcal{P})$  to be one if  $x \in \text{Ext}(P)$ , otherwise put  $\xi(x, \mathcal{P}) = 0$ . Put

$$\mu_\lambda := \sum_{x \in \text{Ext}(P) \cap Q_\lambda} \xi(x, \mathcal{P} \cap Q_\lambda) \delta_x.$$

Note that  $\mathbb{E}[\text{card}(\text{Ext}(\mathcal{P} \cap Q_\lambda))] = \mathbb{E}[\langle 1, \mu_\lambda \rangle]$ . Put  $\tilde{Q}_\lambda := [-\frac{1}{2}\lambda^{1/(d-1)}, \frac{1}{2}\lambda^{1/(d-1)}]^{d-1}$  so that  $Q_\lambda = \tilde{Q}_\lambda \times \mathbb{R}$ . Writing generic points in  $\mathbb{R}^{d-1} \times \mathbb{R}$  as  $(v, h)$  we have

$$\lambda^{-1} \mathbb{E}[\langle 1, \mu_\lambda \rangle] = \lambda^{-1} \int_{\tilde{Q}_\lambda} \int_{-\infty}^{\infty} \mathbb{E} \xi((v, h), \mathcal{P} \cap Q_\lambda) e^h dh dv.$$

Put  $\tilde{\mathcal{P}}_\lambda$  to be a Poisson point process on  $\mathbb{R}^{d-1} \times \mathbb{R}$  with intensity  $\lambda e^h dh dv$ . For any  $\mathcal{X} \subset \mathbb{R}^{d-1} \times \mathbb{R}$ , let  $\alpha \mathcal{X} := \{(\alpha v, h) : (v, h) \in \mathcal{X}\}$ . Thus  $\mathcal{P} \cap Q_\lambda \stackrel{\mathcal{D}}{=} \lambda^{1/(d-1)} [\tilde{\mathcal{P}}_\lambda \cap Q_1]$  and

$$\mathcal{P} \cap Q_\lambda - \{(v, 0)\} \stackrel{\mathcal{D}}{=} \lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v', 0)\}),$$

where  $v = \lambda^{1/(d-1)} v'$ . We thus obtain by translation invariance of  $\xi$

$$\lambda^{-1} \mathbb{E}[\langle 1, \mu_\lambda \rangle] = \int_{\tilde{Q}_1} \int_{-\infty}^{\infty} \mathbb{E} \xi((\mathbf{0}, h_0), \lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v'_0, 0)\})) e^{h_0} dh_0 dv'_0.$$

Notice that for all  $v'_0 \in \tilde{Q}_1$  we have  $\lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v'_0, 0)\}) \xrightarrow{\mathcal{D}} \mathcal{P}$  as  $\lambda \rightarrow \infty$ . Also, the functional  $\xi$  satisfies the spatial localization and moment conditions of those functionals in  $\Xi^{(\infty)}$  and consequently we have for all  $v'_0 \in \tilde{Q}_1$

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \xi((\mathbf{0}, h_0), \lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v'_0, 0)\})) = \mathbb{E}[\xi((\mathbf{0}, h_0), \mathcal{P})].$$

As in Lemma 4.4, we may show that  $\mathbb{E}[\xi((\mathbf{0}, h_0), \lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v'_0, 0)\}))] e^{h_0}$  is dominated by an exponentially decaying function of  $h_0$ , uniformly in  $\lambda$ . Thus by the dominated convergence theorem we get

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} \mathbb{E} [ \langle 1, \mu_\lambda \rangle ] = \int_{-\infty}^{\infty} \mathbb{E} [ \xi((\mathbf{0}, h_0), \mathcal{P}) ] e^{h_0} dh_0. \tag{5.13}$$

To prove variance asymptotics, we argue as follows. For all  $h_0 \in \mathbb{R}$ , and  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}$ , we abuse notation and put

$$\begin{aligned} c^\xi((\mathbf{0}, h_0), (v_1, h_1), \mathcal{P} \cap Q_\lambda) &:= \mathbb{E} [ \xi((\mathbf{0}, h_0), (\mathcal{P} \cap Q_\lambda) \cup \{(v_1, h_1)\}) \\ &\quad \times \xi((v_1, h_1), (\mathcal{P} \cap Q_\lambda) \cup \{(\mathbf{0}, h_0)\}) ] \\ &- \mathbb{E} [ (\xi((\mathbf{0}, h_0), \mathcal{P} \cap Q_\lambda)) \mathbb{E} [ \xi((v_1, h_1), \mathcal{P} \cap Q_\lambda) ]. \end{aligned}$$

Then we have

$$\begin{aligned} \lambda^{-1} \text{Var} [ \langle 1, \mu_\lambda \rangle ] &= \lambda^{-1} \int_{\tilde{Q}_\lambda} \int_{-\infty}^{\infty} \mathbb{E} [ \xi((v, h), P \cap Q_\lambda) ] e^h dh_0 dv_0 \\ &\quad + \lambda^{-1} \int_{\tilde{Q}_\lambda} \int_{-\infty}^{\infty} \int_{\tilde{Q}_\lambda} \int_{-\infty}^{\infty} c^\xi((v_0, h_0), (v_1, h_1), P \cap Q_\lambda) e^{h_0+h_1} dh_0 dh_1 dv_1 dv_0 \\ &= \int_{\tilde{Q}_1} \int_{-\infty}^{\infty} \mathbb{E} [ \xi((\mathbf{0}, h_0), \lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v'_0, 0)\})) ] e^{h_0} dh_0 dv'_0 \\ &\quad + \int_{\tilde{Q}_1} \int_{-\infty}^{\infty} \int_{\tilde{Q}_\lambda - \lambda^{1/(d-1)} v'_0} \int_{-\infty}^{\infty} c^\xi((\mathbf{0}, h_0), (v_1, h_1), \lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v'_0, 0)\})) \\ &\quad e^{h_0+h_1} dh_0 dh_1 dv_1 dv'_0. \end{aligned} \tag{5.14}$$

The first integral in (5.14) converges to the limit (5.13). We show that the second term in (5.14) converges. For all  $h_0, h_1, v_1$  and  $v'_0$  we have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} c^\xi((\mathbf{0}, h_0), (v_1, h_1), \lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v'_0, 0)\})) e^{h_0+h_1} \\ = c^\xi((\mathbf{0}, h_0), (v_1, h_1), \mathcal{P}) e^{h_0+h_1}, \end{aligned}$$

where  $c^\xi$  is defined as in (2.3). As in the proof of Lemma 4.8, we have that

$$c^\xi((\mathbf{0}, h_0), (v_1, h_1), \lambda^{1/(d-1)} ([\tilde{\mathcal{P}}_\lambda \cap Q_1] - \{(v'_0, 0)\})) e^{h_0+h_1}$$

is dominated by an exponentially decaying function of  $h_0, h_1$ , and  $v_1$ , uniformly in  $\lambda \in [\lambda_0, \infty)$ . Consequently, by the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var} [ \langle 1, \mu_\lambda \rangle ] &= \int_{-\infty}^{\infty} \mathbb{E} [ \xi((\mathbf{0}, h_0), \mathcal{P}) ] e^{h_0} dh_0 \\ &\quad + \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^\xi((\mathbf{0}, h_0), (v_1, h_1), \mathcal{P}) e^{h_0+h_1} dh_0 dv_1 dh_1. \end{aligned}$$

This concludes the proof of Corollary 1.1. □

5.5 Proof of Theorem 1.5

Let us fix  $k \in \{1, \dots, d - 1\}$ . We follow the same method and notation as on pages 54–55 in [9]. The key idea is to use Kubota’s formula (see (6.11) in [23]) which says roughly that the  $k$ th intrinsic volume of  $K_\lambda$  is, up to a multiplicative constant, equal to the mean over the set  $G(d, k)$  of all  $k$ -dimensional linear subspaces of  $\mathbb{R}^d$  of the  $k$ -dimensional Lebesgue measure of the projection of  $K_\lambda$  onto  $L$ , denoted by  $K_\lambda|L$ . In other words, we have

$$V_k(K_\lambda) = \frac{d! \kappa_d}{k! \kappa_k (d - k)! \kappa_{d-k}} \int_{G(d,k)} \text{Vol}_k(K_\lambda|L) d\nu_k(L), \tag{5.15}$$

where  $\nu_k$  is the normalized Haar measure on the  $k$ th Grassmannian  $G(d, k)$  of  $\mathbb{R}^d$ .

For every  $x \in \mathbb{R}^d \setminus \{0\}$ , We consider now the function  $\vartheta_L(x, K_\lambda) = \mathbf{1}(x \notin K_\lambda|L)$  and the so-called projection avoidance functional

$$\vartheta_k(x, K_\lambda) = \int_{G(\text{lin}[x],k)} \vartheta_L(x, K_\lambda) d\nu_k^{\text{lin}[x]}(L)$$

where  $\text{lin}[x]$  is the one-dimensional linear space spanned by  $x$ ,  $G(\text{lin}[x], k)$  is the set of  $k$ -dimensional linear subspaces of  $\mathbb{R}^d$  containing  $\text{lin}[x]$  and  $\nu_k^{\text{lin}[x]}$  is the normalized Haar measure on  $G(\text{lin}[x], k)$  (see (2.7) in [9]). Using (5.15) and Fubini’s theorem, we rewrite the defect intrinsic volume of  $K_\lambda$  as

$$V_k(B_d(\mathbf{0}, R_\lambda)) - V_k(K_\lambda) = \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\mathbb{R}^d} [\vartheta_k(x, K_\lambda) - \vartheta_k(x, B_d(\mathbf{0}, R_\lambda))] \frac{dx}{|x|^{d-k}}.$$

In particular, we have the decomposition

$$V_k(B_d(\mathbf{0}, R_\lambda)) - V_k(K_\lambda) = R_\lambda^{-(d+1-k)} \sum_{x \in \mathcal{P}_\lambda} \xi_{V,k}(x, \mathcal{P}_\lambda)$$

where

$$\xi_{V,k}(x, \mathcal{P}_\lambda) := d^{-1} \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\text{cone}(x, \mathcal{P}_\lambda)} [\vartheta_k(y, K_\lambda) - \vartheta_k(y, B_d(\mathbf{0}, R_\lambda))] \frac{R_\lambda^{d+1-k} dy}{|y|^{d-k}} \tag{5.16}$$

if  $x$  is extreme and  $\xi_{V,k}(x, \mathcal{P}_\lambda) = 0$  otherwise. The corresponding empirical measure is

$$\mu_\lambda^{\xi_{V,k}} := \sum_{x \in \mathcal{P}_\lambda} \xi_{V,k}(x, \mathcal{P}_\lambda) \delta_x.$$

In the same spirit as in the definition of the defect volume functional (see Definition 2.2), the scaling factor  $R_\lambda^{d+1-k}$  is artificially inserted inside the integral in (5.16) in order to use later the convergence (5.17). In view of the equality

$$\langle 1, \mu_{\lambda}^{\xi_{V,k}} \rangle = R_{\lambda}^{d+1-k} [V_k(B_d(\mathbf{0}, R_{\lambda})) - V_k(K_{\lambda})],$$

we observe that it is enough to show that (2.5) and (2.6) are satisfied when  $\xi = \xi_{V,k}$ .

We notice that the equalities (5.2) and (5.6) hold when  $\xi$  is put to be  $\xi_{V,k}$ . Let us define

$$\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \xi_{V,k}([T^{(\lambda)}]^{-1}(w), \mathcal{P}_{\lambda}), \quad w \in \mathbb{R}^d.$$

Observe from the proof of Theorem 2.1 that it is enough to show the convergence up to a multiplicative rescaling of each of the terms  $\mathbb{E}[\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})], \mathbb{E}[\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})^2]$  and  $c^{\xi_{V,k}^{(\lambda)}}(w, w')$  where  $w, w' \in \mathbb{R}^d$ , as well as bounds similar to those in Lemmas 4.4 and 4.8.

Let us show for instance the convergence of  $\mathbb{E} \xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$ ,  $w \in W_{\lambda}$ . The localization radius associated with  $\xi_{V,k}^{(\lambda)}$  is the same as that for  $\xi_{V,k}$ . Moreover, a moment bound similar to (4.16) can be obtained when  $\xi^{(\lambda)}$  is replaced by  $\xi_{V,k}^{(\lambda)}$ . We need to introduce now the scaling limit  $\xi_{V,k}^{(\infty)}(w, \mathcal{P})$  of  $\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$ , i.e. the exact analogue of (5.16) when Euclidean convex geometry is replaced by parabolic convex geometry (see e.g. section 3 in [9]). For every  $w = (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$ , we denote by  $w^{\dagger}$  the set  $\{v\} \times \mathbb{R}$  and by  $\mu_k^{w^{\dagger}}$  the normalized Haar measure on the set  $A(w^{\dagger}, k)$  of all  $k$ -dimensional affine spaces in  $\mathbb{R}^d$  containing  $w^{\dagger}$ . For every affine space  $L$  containing  $w^{\dagger}$ , we define the corresponding orthogonal paraboloid volume  $\Pi^{\perp}[w; L]$  as the set  $\{w' = (v', h') \in \mathbb{R}^{d-1} \times \mathbb{R} : (w - w') \perp L, h' \leq h - \frac{\|v-v'\|^2}{2}\}$ . In other words,  $\Pi^{\perp}[w; L]$  is the set of points of  $(w \oplus L^{\perp})$  which are under the paraboloid surface  $\partial \Pi^{\downarrow}(w)$  with apex at  $w$ . We put  $\vartheta_L^{(\infty)}(w) = 1$  when  $\Pi^{\perp}[w; L] \cap T^{(\lambda)}(K_{\lambda}) = \emptyset$  and 0 otherwise. In particular, when  $L = w^{\dagger}$ , we have  $\vartheta_{w^{\dagger}}^{(\infty)}(w) = \mathbf{1}(\Pi^{\perp}[w; L] \cap \Phi = \emptyset)$ . Now for every  $w \in \mathbb{R}^d$ , we define

$$\vartheta_k^{(\infty)}(w) := \int_{A(w^{\dagger}, k)} \vartheta_L^{(\infty)}(w) d\mu_k^{w^{\dagger}}(L).$$

We finally define the limit score

$$\xi_{V,k}^{(\infty)}(w, \mathcal{P}) := d^{-1} \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\text{Cyl}(w) \times \mathbb{R}} [\vartheta_k^{(\infty)}(w') - \mathbf{1}(\{w' \in \mathbb{R}^{d-1} \times \mathbb{R}_{-}\})] dw'$$

if  $w \in \text{Ext}(\mathcal{P})$  and  $\xi_{V,k}^{(\infty)}(w, \mathcal{P}) = 0$  otherwise. Denote by  $m_k$  the measure  $|x|^{k-d} dx$ . In the same spirit as Lemma 3.2, have as  $\lambda \rightarrow \infty$

$$T^{(\lambda)}(R_{\lambda}^{d+1-k} dm_k) \xrightarrow{\mathcal{D}} \text{Vol}_d. \tag{5.17}$$

As in the proof of Lemma 4.5, we may show via the continuous mapping theorem that for every  $w \in \mathbb{R}^d$ , we have the convergence in distribution  $\xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \xrightarrow{\mathcal{D}}$

$\xi_{V,k}^{(\infty)}(w, \mathcal{P})$ . We deduce from the moment bound the analog of Lemma 4.5 for  $\xi_{V,k}^{(\lambda)}$ . In the spirit of Lemma 4.6, we finally show that  $\lim_{\lambda \rightarrow \infty} \mathbb{E} \xi_{V,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \mathbb{E} \xi_{V,k}^{(\infty)}(w, \mathcal{P})$ .

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