The Annals of Probability
2008, Vol. 36, No. 1, 363–396
DOI: 10.1214/009117907000000259

VARIANCE ASYMPTOTICS AND CENTRAL LIMIT THEOREMS FOR GENERALIZED GROWTH PROCESSES WITH APPLICATIONS TO CONVEX HULLS AND MAXIMAL POINTS

BY T. SCHREIBER[1] AND J. E. YUKICH[2]

Nicholas Copernicus University and Lehigh University

We show that the random point measures induced by vertices in the convex hull of a Poisson sample on the unit ball, when properly scaled and centered, converge to those of a mean zero Gaussian field. We establish limiting variance and covariance asymptotics in terms of the density of the Poisson sample. Similar results hold for the point measures induced by the maximal points in a Poisson sample. The approach involves introducing a generalized spatial birth growth process allowing for cell overlap.

1. Introduction, main results. Given X_i, $i \geq 1$, i.i.d. random variables with values in a d-dimensional convex set S, $d \geq 2$, a classic problem in convex geometry involves determining the distribution of the number of points in the set of extreme points $\mathcal{V}(\{X_i\}_{i=1}^n)$, defined as the vertices in the convex hull of $\{X_i\}_{i=1}^n$. This problem was first considered by Rényi and Sulanke [33], with recent notable progress by Reitzner [28–31] and Vu [37].

A closely related problem involves determining, for a given $K \subset \mathbb{R}^d$, the distribution of the number of points in the set $\mathcal{M}_K(\{X_i\}_{i=1}^n)$ of K-maximal points, where a point X_j belongs to $\mathcal{M}_K(\{X_i\}_{i=1}^n)$ iff $(X_j \oplus K) \cap \{X_i\}_{i=1}^n = X_j$, where here and henceforth, for all $B \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we write $x \oplus B := \{x + y : y \in B\}$. When K is $(\mathbb{R}_+)^d$, then $\mathcal{M}_K(\{X_i\}_{i=1}^n)$ is simply the set of maximal points, that is, those points X_j in $\{X_i\}_{i=1}^n$ having the property that no point X_i, $i \neq j$, exceeds it in all coordinates. The limit theory for the number of maximal points in $\mathcal{M}_K(\{X_i\}_{i=1}^n)$ was first considered by Rényi [32] and Barndorff-Nielsen and Sobel [5]. Chen, Hwang and Tsai [11] surveys the vast literature, which includes books by Ehrgott [17], Pomerol and Barba-Romero [27], and recent papers of [1, 2, 4, 8, 16].

In this paper we establish convergence of the finite-dimensional distributions of the re-scaled point measures induced by the random point sets $\mathcal{V}(\mathcal{P}_{\lambda\rho})$, where

Received May 2006; revised February 2007.

[1] Supported in part by Polish Minister of Scientific Research and Information Technology Grant 1 P03A 018 28 (2005–2007).

[2] Supported by NSF Grant DMS-02-03720.

AMS 2000 subject classifications. Primary 60F05; secondary 60D05.

Key words and phrases. Convex hulls, maximal points, spatial birth growth processes, Gaussian limits.

$\mathcal{P}_{\lambda\rho}$ denotes a Poisson point process of intensity $\lambda\rho$ on B_d, the unit radius d-dimensional ball centered at the origin and where ρ is a continuous density on B_d. For sets $K := \{(w_1, \dots, w_d) : w_d \geq (w_1^2 + \cdots + w_{d-1}^2)^{\alpha/2}\}$, where $\alpha \in (0, 1]$ is fixed, we also establish convergence of the finite-dimensional distributions of the point measures induced by $\mathcal{M}_K(\mathcal{P}_{\lambda\rho})$, where $\mathcal{P}_{\lambda\rho}$ denotes the Poisson point process of intensity $\lambda\rho$ on $A \times \mathbb{R}_+$, where $A \subset \mathbb{R}^{d-1}$ is compact and convex and where $\rho : A \times \mathbb{R}_+$ is continuous. These results are facilitated by introducing a generalized spatial birth–growth process as a means toward obtaining explicit variance asymptotics and central limit theorems for random measures arising in convex geometry. The relevant spatial birth–growth process, possibly of independent interest, modifies the classical spatial birth–growth process introduced by Kolmogorov [20] as a model for crystal growth by allowing the possibility of *cell overlap*. As in [20], cells may grow at nonconstant growth rates.

In the context of the set of extreme points $\mathcal{V}(\mathcal{P}_{\lambda\rho})$, the approach taken here adds to the work of Reitzner [28–31] and Vu [37] in the following ways. First, the present set-up establishes convergence of the finite-dimensional distributions of the canonical point measures induced by $\mathcal{V}(\mathcal{P}_{\lambda\rho})$, whereas [28–31] and [37] deal with one-dimensional central limit theorems. Second, we establish a formula for variance and covariance asymptotics. Third, the present paper concerns the limit theory for nonuniform samples, whereas [28–31] and [37] treat uniform random samples.

In the context of the set of maximal points $\mathcal{M}_K(\mathcal{P}_{\lambda\rho})$, the present set-up establishes convergence of the finite-dimensional distributions of the canonical point measures induced by $\mathcal{M}_K(\mathcal{P}_{\lambda\rho})$, with covariances, whereas previous work [4, 16] is concerned with one dimensional central limit theorems without a formula for covariance asymptotics and/or is limited to the case when K is a cone [8].

1.1. *Terminology*, ψ*-growth processes.* Let the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the following conditions:

(Ψ1) ψ is monotone and $\lim_{l\to\infty} \psi(l) = \infty$, and
(Ψ2) there exists $\alpha > 0$ such that $\psi(l) = l^\alpha(1 + o(1))$ for l small enough.

Let $\mathbf{0}$ denote the origin of $\mathbb{R}^{d-1}$, $d \geq 2$, and let $|y|$ denote the Euclidean norm of $y \in \mathbb{R}^d$. We define $K[\mathbf{0}]$ to be the ψ-epigraph $\{(y, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h \geq \psi(|y|)\}$ and, more generally, for $\bar{x} := (x, h_x) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, we define its ψ-epigraph (or upward cone) by

$$K[\bar{x}] := \bar{x} \oplus K[\mathbf{0}] := \{(y, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h \geq h_x + \psi(|y - x|)\}. \tag{1.1}$$

Given a point set $\mathcal{X} \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$, a point $\bar{x} \in \mathcal{X}$ is called ψ*-extremal* in $\mathcal{X}$ iff $K[\bar{x}] \not\subseteq \bigcup_{\bar{y}\in\mathcal{X}\setminus\{\bar{x}\}} K[\bar{y}]$, that is to say the ψ-epigraph of $\bar{x}$ is not completely covered by the union of the ψ-epigraphs of points in $\mathcal{X} \setminus \{x\}$. Define the functional

$$\xi(\bar{x}, \mathcal{X}) := \xi(\psi; \bar{x}, \mathcal{X}) := \begin{cases} 1, & \text{if } \bar{x} \text{ is } \psi\text{-extremal in } \mathcal{X}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

With D standing for some bounded domain in $\mathbb{R}^{d-1} \times \mathbb{R}_+$, we consider the version $\xi_D(\cdot,\cdot)$ of $\xi(\cdot,\cdot)$ restricted to D, by setting $\xi_D(\bar{x}, \mathcal{X})$ to be 1 iff $K[\bar{x}] \cap D \not\subseteq \bigcup_{\bar{y} \in (\mathcal{X} \setminus \{\bar{x}\}) \cap D} K[\bar{y}]$, in which case we declare $\bar{x}$ to be ψ-extremal in $D \cap \mathcal{X}$, and otherwise we set $\xi_D(\bar{x}, \mathcal{X})$ to be zero. In case $\bar{x} \notin \mathcal{X}$ we abbreviate notation and write $\xi(\bar{x}, \mathcal{X})$ for $\xi(\bar{x}, \mathcal{X} \cup \bar{x})$ and similarly for $\xi_D(\bar{x}, \mathcal{X})$.

To provide a physical interpretation of these functionals, we regard $\mathbb{R}^{d-1} \times \mathbb{R}_+$ as d-dimensional space time, with $\mathbb{R}_+$ standing for the time coordinate, and we interpret the graph $\partial(K[\bar{x}])$, $\bar{x} := (x, t)$, as the boundary of a $(d-1)$-dimensional spherical particle born at x at time t (at which time it has initial radius zero) and growing thereupon with radial speed $v(t) := \frac{d}{dt}[\psi^{-1}(t)]$, provided the derivative exists. The particles (spheres) grow independently and do not exhibit exclusion, that is, they may overlap or penetrate one another. A particle is *extreme* iff at some time it is not completely covered by other particles. When ψ is the identity, so that the ψ graph gives a cone, we see that ψ-extremal points coincide with maximal points [8].

In the context of this representation, it should be noted that, unlike the one stated here, the classic growth process (see, e.g., [7, 13, 20, 24]) assumes that particles, upon being born at random locations $x \in \mathbb{R}^{d-1}$ at random times $h_x \in \mathbb{R}^+$, form a cell by growing radially in all directions with a possibly nonconstant speed, that is, with ψ possibly nonlinear. When one growing cell touches another, it stops growing in that direction, that is, no overlap is allowed. Furthermore, a particle born inside an existing cell is *discarded*, otherwise it is *accepted*. Letting $\hat{\xi}(\bar{x}, \mathcal{X})$ be zero or one according to whether $\bar{x}$ is accepted or not, this paper also considers such functionals $\hat{\xi}$.

The growth process giving rise to the functional ξ will henceforth be called the *ψ-growth process with overlap*, while the process corresponding to $\hat{\xi}$ will be referred to as the *ψ-growth process without overlap*. This paper will mainly concentrate on applications of the first concept and the corresponding functional ξ, but the subsequently developed general theory also treats the latter concept in the special case of linear ψ. Throughout, let A be a compact convex subset of $\mathbb{R}^{d-1}$. We shall also admit the case $A := \mathbb{R}^{d-1}$ in the sequel, in which case we assume that ρ is uniformly bounded. Consider a density function ρ on $A_+ := A \times \mathbb{R}_+$, not necessarily integrable, such that

(R1) ρ is continuous on A_+,

(R2) there exists a constant $\delta \geq 0$ and a continuous function $\rho_0 : A \to \mathbb{R}_+$ bounded away from zero such that

$$\rho(x, h) = \rho_0(x) h^\delta \big(1 + o(1)\big)$$

for h small enough and $\rho(x, h) = O(h^\delta)$ for large h uniformly in $x \in A$.

For $\lambda > 0$, we recall that $\mathcal{P}_{\lambda\rho}$ denotes the Poisson point process on A_+ with intensity measure $\lambda\rho(x, h)\, dx\, dh$. The "extreme point" empirical measures $\mu^\xi_{\lambda\rho}$

and $\mu^{\hat{\xi}}_{\lambda\rho}$ generated by $\mathcal{P}_{\lambda\rho}$ are

$$\mu^{\xi}_{\lambda\rho} := \sum_{\bar{x}\in\mathcal{P}_{\lambda\rho}} \xi(\bar{x}, \mathcal{P}_{\lambda\rho})\delta_{\bar{x}} \tag{1.3}$$

and

$$\mu^{\hat{\xi}}_{\lambda\rho} := \sum_{\bar{x}\in\mathcal{P}_{\lambda\rho}} \hat{\xi}(\bar{x}, \mathcal{P}_{\lambda\rho})\delta_{\bar{x}}, \tag{1.4}$$

with δ_x standing for the unit point mass at $x \in \mathbb{R}^d$. For any random measure σ on $\mathbb{R}^d$, we write $\bar{\sigma}$ for its centered version $\sigma - \mathbb{E}[\sigma]$, so that, for example, $\bar{\mu}^{\xi}_{\lambda\rho} := \mu^{\xi}_{\lambda\rho} - \mathbb{E}[\mu^{\xi}_{\lambda\rho}]$.

Notice that for small α the upward cones $K[\bar{x}]$ have relatively narrow apertures, making it less likely that cones having apexes with a small temporal coordinate get covered by ψ-epigraphs, that is, one expects more ψ-extreme points as α gets smaller. Also, roughly speaking, for small δ, one expects more points in $\mathcal{P}_{\lambda\rho}$ with small temporal coordinate and thus more ψ-extreme points in this case as well. One of the goals of this paper is to show (see Theorem 1.1) that the expected total mass of the extreme point empirical measures (1.3)–(1.4) is asymptotically proportional to λ^{τ}, where

$$\tau := \tau(d, \alpha, \delta) := \frac{d-1}{d-1+\alpha(1+\delta)}. \tag{1.5}$$

More general goals include establishing the variance asymptotics and the convergence of the finite-dimensional distributions of the appropriately scaled measures (1.3)–(1.4) to Gaussian distributions (see Theorems 1.2 and 1.3) and to treat the applications to extreme and maximal points described at the outset.

Notation. Given $\alpha > 0$, put

$$\psi^{(\infty)}(l) := l^{\alpha}. \tag{1.6}$$

Recalling the definition of ξ, we define the functional $\xi^{(\infty)}$ by $\xi^{(\infty)}(\cdot, \cdot) := \xi(\psi^{(\infty)}; \cdot, \cdot)$ and similarly for $\hat{\xi}^{(\infty)}$. We also let $\mathcal{P}_*$ stand for the Poisson point process in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with intensity measure $h^{\delta}\, dx\, dh$.

For all $\bar{x} := (x, h_x)$ and $\bar{y} := (y, h_y)$, let

$$m^{(\infty)}(\bar{x}) := \mathbb{E}\big[\xi^{(\infty)}(\bar{x}, \mathcal{P}_*)\big]$$

and

$$\begin{aligned} c^{(\infty)}_*(\bar{x}, \bar{y}) := {}& \mathbb{E}\big[\xi^{(\infty)}(\bar{x}, \mathcal{P}_* \cup \bar{y})\xi^{(\infty)}(\bar{y}, \mathcal{P}_* \cup \bar{x})\big] \\ & - \mathbb{E}\big[\xi^{(\infty)}(\bar{x}, \mathcal{P}_*)\big]\mathbb{E}[\xi^{(\infty)}(\bar{y}, \mathcal{P}_*)] \end{aligned}$$

respectively denote the one and two point correlation functions for the $\psi^{(\infty)}$ growth process with overlap.

For sets A and $B \subset \mathbb{R}^d$, let $d(A, B) := \inf\{|x - y| : x \in A, \ \ y \in B\}$. Let $B_d(y, r)$ denote the d-dimensional Euclidean ball centered at $y \in \mathbb{R}^d$ with radius $r \in (0, \infty)$.

Given a subset B of $\mathbb{R}^d$, let $\mathcal{C}_b(B)$ denote the bounded continuous functions on B. For any signed measure μ on A_+ and $f \in \mathcal{C}_b(A_+)$, let $\langle f, \mu \rangle := \int f \, du$. Unless otherwise specified, C denotes a generic positive constant whose value may change from line to line.

1.2. *Limit theory for Ψ-growth functionals.* For all $f \in \mathcal{C}_b(A_+)$ with $A \subset \mathbb{R}^{d-1}$ compact and convex, we define the average of the product of f and the one and two point correlation functions as follows:

$$I(f) := \int_A \int_0^\infty f(x, 0) m^{(\infty)}(\mathbf{0}, h') \rho_0^\tau(x) (h')^\delta \, dh' \, dx \tag{1.7}$$

and

$$\begin{aligned} J(f) := \int_A \int_0^\infty \int_{\mathbb{R}^{d-1}} \int_0^\infty & f(x, 0) c_*^{(\infty)}((\mathbf{0}, h'), (y', h'_y)) \\ & \times \rho_0^\tau(x) (h'_y)^\delta (h')^\delta \, dh'_y \, dy' \, dh' \, dx. \end{aligned} \tag{1.8}$$

The finiteness of $I(f)$ follows by Lemmas 3.2 and 3.3 [see the bound (3.11)], whereas the finiteness of $J(f)$ follows from Lemmas 3.4 and 3.5 [see the bound (3.22)] which imply rapid enough decay of two-point correlation functions.

The following are our main results. We state the results for $\mu_{\lambda\rho}^{\xi}$ and note that analogous results hold for $\mu_{\lambda\rho}^{\hat{\xi}}$ when ψ is linear. The first result specifies first-order behavior, whereas the second provides second-order asymptotics.

THEOREM 1.1. *We have for all $f \in \mathcal{C}_b(A_+)$*

$$\lim_{\lambda \to \infty} \lambda^{-\tau} \mathbb{E}[\langle f, \mu_{\lambda\rho}^{\xi} \rangle] = I(f). \tag{1.9}$$

THEOREM 1.2. *We have for all $f \in \mathcal{C}_b(A_+)$*

$$\lim_{\lambda \to \infty} \lambda^{-\tau} \operatorname{Var}[\langle f, \mu_{\lambda\rho}^{\xi} \rangle] = I(f^2) + J(f^2). \tag{1.10}$$

The next result establishes the convergence of the finite-dimensional distributions of $(\lambda^{-\tau/2} \overline{\mu}_{\lambda\rho}^{\xi})$.

THEOREM 1.3. *The finite-dimensional distributions $\lambda^{-\tau/2}(\langle f_1, \bar{\mu}_{\lambda}^{\xi} \rangle, \ldots, \langle f_k, \bar{\mu}_{\lambda}^{\xi} \rangle)$, $f_1, \ldots, f_k \in \mathcal{C}_b(A_+)$, of $(\lambda^{-\tau/2} \bar{\mu}_{\lambda\rho}^{\xi})$ converge as $\lambda \to \infty$ to those of a mean zero Gaussian field with covariance kernel*

$$(f, g) \mapsto I(fg) + J(fg), \qquad f, g \in \mathcal{C}_b(A_+). \tag{1.11}$$

Section 2 describes applications of ψ-growth processes with overlap, as given by the general limits of Theorems 1.1–1.3, to convex hulls and maximal points of i.i.d. samples.

REMARKS. (i) *Applications to the ψ-growth process $\hat{\xi}$ without overlap*. The results of Theorems 1.1–1.3 for the functional $\hat{\xi}$ provide variance asymptotics and central limit theorems for the classic spatial birth–growth model in $\mathbb{R}^{d-1}$, whereby seeds are born at random locations in $\mathbb{R}^{d-1}$ and times in $\mathbb{R}_+$ according to the Poisson point process $\lambda \mathcal{P}_{\lambda\rho}$ on $\lambda^{1/d} A \times \mathbb{R}_+$ and grow linearly in time. Theorems 1.1–1.3 for $\hat{\xi}$ provide a central limit theorem for the number of seeds accepted in such models. This generalizes and extends [7, 24], which builds on work of Chiu and Quine [13, 14], Chiu [12] and Chiu and Lee [15], which do not consider convergence of finite-dimensional distributions and which often restrict to models with homogeneous temporal input.

(ii) *Scaling*. The scaling $\lambda^{-\tau}$ arises in the following way. From a conceptual and analytic point of view, it is convenient to re-scale the ψ-growth process in time and space so as to obtain an equivalent growth process on Poisson points of approximately unit intensity density on a region of volume λ. The scaling is designed to asymptotically preserve the ψ-epigraphs and the behavior of the density locally close to $h = 0$.

To achieve this, we scale A_+ in the $d-1$ spatial directions by λ^{β} and in the temporal direction by λ^{γ}. Under this temporal scaling and under (R2), the density ρ exhibits growth $(h\lambda^{\gamma})^{\delta}$ for small temporal h, and we thus require $\lambda^{\beta(d-1)+\gamma(1+\delta)} = \lambda$. This scaling maps $|x|$ and h_x to $\lambda^{\beta}|x|$ and $\lambda^{\gamma} h_x$, respectively, and therefore, it asymptotically preserves the ψ-epigraphs and condition (Ψ2), provided $(\lambda^{\beta}|x|)^{\alpha} = \lambda^{\gamma} h_x (1 + o(1))$ for (x, h_x) lying on the graph of ψ, that is, $h_x = \psi(x)$. Since $h_x = |x|^{\alpha}(1 + o(1))$ for such (x, h_x), we require $\lambda^{\beta\alpha} = \lambda^{\gamma}$. We thus require the relations

$$\beta(d-1) + \gamma(1+\delta) = 1 \quad \text{and} \quad \beta\alpha = \gamma,$$

which yields these values for the scaling exponents

$$\beta = \frac{\gamma}{\alpha} \quad \text{and} \quad \gamma = \frac{\alpha}{(d-1) + \alpha(1+\delta)}. \tag{1.12}$$

Given the re-scaled ψ-growth process on $\lambda^{\beta} A \times \mathbb{R}_+$, we expect that a point is ψ-extremal (i.e., $\xi = 1$) iff its time coordinate is small. Thus, the functional $\mu^{\xi}_{\lambda\rho}(A_+)$ should exhibit growth proportional to the Lebesgue measure of $\lambda^{\beta} A$, that is, proportional to $\lambda^{\beta(d-1)} = \lambda^{\tau}$. In the special case when $\delta = 0$ and the growth is linear ($\alpha = 1$) the ψ-epigraphs are preserved by time and space scaling by $\lambda^{1/d}$, that is, $\gamma = 1/d = \beta$. Thus, $\tau = (d-1)/d$ in this case.

(iii) *de-Poissonization.* In Section 4 we de-Poissonize Theorems 1.1–1.3 when $\alpha \in (0, 1]$. In other words, we obtain the identical limit theory when $\mathcal{P}_{\lambda\rho}$ is replaced by i.i.d. random variable $X_1, \dots, X_n$, chosen in A_+ according to the density ρ, assumed to be integrable to 1. We expect similar de-Poissonization results for $\alpha > 1$, but are unable to prove this.

(iv) We have not tried to establish a.s. convergence in (1.9), but expect that concentration inequalities should be useful in this context.

1.3. *Notation and scaling relations.* Motivated by remark (ii) above, we place the ψ-growth process on its proper scale by re-scaling as follows. With β and γ as in (1.12), for a *fixed* $x \in A$ and any generic point $\bar{y} := (y, h_y) \in A_+$, we put $\bar{y}^{(\lambda)} := \bar{y}' := (y', h'_y)$ with

$$y' := y^{(\lambda)} := \lambda^{\beta}(y - x) \quad \text{and} \quad h'_y := h_y^{(\lambda)} := \lambda^{\gamma} h_y. \tag{1.13}$$

Also, for readability, in our notation *we will not explicitly indicate the dependency of the scaling in* (1.13) *on* x. The versions of ψ, ρ, $\mathcal{P}_{\lambda\rho}$ and ξ under this re-scaling are determined by the relations

$$\psi^{(\lambda)}(l) := \lambda^{\gamma}\psi(\lambda^{-\beta} l), \tag{1.14}$$

$$\rho^{(\lambda)}(y', h'_y) := \lambda^{\delta\gamma}\rho(y, h_y), \tag{1.15}$$

$$\mathcal{P}_{\lambda\rho}^{(\lambda)} := \mathcal{P}_{\lambda\rho}^{(\lambda)}[x] := \{(y', h'_y) : (y, h_y) \in \mathcal{P}_{\lambda\rho}\} \tag{1.16}$$

and

$$\xi^{(\lambda)}((y', h'_y), \{(y'_i, h'_{y_i})\}_{i\geq 1}) := \xi((y, h_y), \{(y_i, h_{y_i})\}_{i\geq 1}) \tag{1.17}$$

and likewise for $\hat{\xi}$. Since $dy' = \lambda^{\beta(d-1)}\, dy$ and $dh'_y = \lambda^{\gamma}\, dh_y$, it follows that

$$\rho^{(\lambda)}(y', h'_y)\, dy'\, dh'_y = \lambda\rho(y, h_y)\, dy\, dh_y.$$

Note also that

$$\mathcal{P}_{\lambda\rho}^{(\lambda)} \stackrel{\mathcal{D}}{=} \mathcal{P}_{\rho^{(\lambda)}}. \tag{1.18}$$

Moreover, by (1.13) and (1.15), $\rho^{(\lambda)}(y', h'_y)(h'_y)^{-\delta} = \lambda^{\delta\gamma}\rho(y, h_y)(\lambda^{\gamma} h_y)^{-\delta}$, where $y = \lambda^{-\beta} y' + x$. Under the above re-scaling for each fixed $x \in A$ and for each (y', h'_y), we have the crucial limit

$$\lim_{\lambda\to\infty} \rho^{(\lambda)}(y', h'_y)(h'_y)^{-\delta} = \lim_{\lambda\to\infty} \rho(y, h_y)(h_y)^{-\delta} = \rho_0(x) \tag{1.19}$$

and by (Ψ2) and (1.14), for all $l \in \mathbb{R}_+$,

$$\lim_{\lambda\to\infty} \psi^{(\lambda)}(l) = l^{\alpha}. \tag{1.20}$$

It is also worth noting that $\xi^{(\lambda)}$ could alternatively be defined by following the original definition of ξ with ψ replaced there by $\psi^{(\lambda)}$; the same applies for $\hat{\xi}^{(\lambda)}$.

Observe that in fact it states approximate self-similarity of ψ-growth processes under the re-scaling given by (1.13) and (1.14). Motivated by this observation, we have already put $\psi^{(\infty)}(l) := l^\alpha$ and now we define, for all $x \in A$ and for all $(y', h'_y) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$,

$$\rho^{(\infty)}(y', h'_y) := \rho_x^{(\infty)}(y', h'_y) := \rho_0(x)(h'_y)^\delta. \tag{1.21}$$

2. Applications. We describe here applications of the main results. We limit the discussion to the following:

(i) the number of vertices in the convex hull of a Poisson sample, and
(ii) the number of maximal points in a Poisson or i.i.d. sample,

but it should be emphasized that the techniques could potentially be applied to a broader scope of examples. These include, for instance, the variance asymptotics for Johnson–Mehl growth processes [21] with nonlinear growth rates (see, e.g., Section 3.2.2 in [7] for the description of the model and the corresponding central limit theorem). Also, as observed in Section 2.3 of [6], the case $\psi(l) = l^2$ (paraboloids) may figure in the limit behavior of some point processes associated with the asymptotic solutions of Burgers equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \varepsilon \Delta v$$

in the inviscous limit $\varepsilon \to 0$. We will likewise not treat this example either.

2.1. *Number of vertices in the convex hull of an i.i.d. sample.* Recall that B_d denotes the unit radius ball centered at the origin of $\mathbb{R}^d$ and let ∂B_d denote its boundary. Let $\rho : B_d \to \mathbb{R}_+$ be a continuous density on B_d. We shall assume that $\rho(x) = \rho_0(x/|x|)(1 - |x|)^\delta(1 + o(1))$ for some $\delta \ge 0$ and that $\rho_0 : \partial B_d \to \mathbb{R}_+$ is continuous and bounded away from 0. Let $\mathcal{P}_{\lambda\rho}$ be a Poisson point process on B_d with intensity measure $\lambda\rho(x)\,dx$ and let $\operatorname{conv}(\mathcal{P}_{\lambda\rho})$ be the random polytope given by the convex hull of $\mathcal{P}_{\lambda\rho}$. Recalling that $\mathcal{V}(\mathcal{P}_{\lambda\rho})$ denotes the vertices of $\operatorname{conv}(\mathcal{P}_{\lambda\rho})$, consider the *vertex empirical point measure*

$$\mu_{\lambda\rho} := \sum_{x \in \mathcal{V}(\mathcal{P}_{\lambda\rho})} \delta_x. \tag{2.1}$$

As will be shown in Section 4, Theorems 1.1–1.3 yield the following limit theory for $\mu_{\lambda\rho}$. Let $N(0, 1)$ denote the standard normal random variable.

THEOREM 2.1. *There are constants $M := M(d, \delta)$ and $V := V(d, \delta)$ such that for all $f \in \mathcal{C}_b(B_d)$*

$$\begin{aligned}&\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d-1+2(1+\delta))}\mathbb{E}[\langle f, \mu_{\lambda\rho}\rangle] \\ &\qquad = M \int_{\partial B_d} f(s)\rho_0^{(d-1)/(d-1+2(1+\delta))}(s)\,ds\end{aligned} \tag{2.2}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d-1+2(1+\delta))} \operatorname{Var}[\langle f, \mu_{\lambda\rho}\rangle] = V \int_{\partial B_d} f^2(s)\rho_0^{(d-1)/(d-1+2(1+\delta))}(s)\,ds. \tag{2.3}$$

Moreover, the finite-dimensional distributions $\lambda^{-(d-1)/2(d-1+2(1+\delta))}(\langle f_1, \bar{\mu}_{\lambda\rho}\rangle, \dots, \langle f_k, \bar{\mu}_{\lambda\rho}\rangle)$, $f_i \in \mathcal{C}_b(B_d)$, *of* $(\lambda^{-(d-1)/2(d-1+2(1+\delta))}\bar{\mu}_{\lambda\rho})$ *converge as* $\lambda \to \infty$ *to those of a mean zero Gaussian field with covariance kernel*

$$(f, g) \mapsto V \int_{\partial B_d} f(s)g(s)\rho_0^{(d-1)/(d-1+2(1+\delta))}(s)\,ds, \qquad f, g \in \mathcal{C}_b(B_d).$$

Additionally, if $\delta = 0$, *then for all* $f \in \mathcal{C}_b(B_d)$,

$$\sup_t \left| P\left[\frac{\langle f, \bar{\mu}_{\lambda\rho}\rangle}{\sqrt{\operatorname{Var}\langle f, \bar{\mu}_{\lambda\rho}\rangle}} \le t\right] - P[N(0,1) \le t]\right| = O\big(\lambda^{-(d-1)/2(d+1)}(\log\lambda)^{3+2(d-1)}\big). \tag{2.4}$$

REMARKS. (i) Taking $f_1 \equiv 1$ (and all other $f_i \equiv 0, i = 2, \dots, k$) provides a central limit theorem for the cardinality of $\mathcal{V}(\mathcal{P}_{\lambda\rho})$.

(ii) Theorem 2.1 adds to the work of the following authors: (a) Groeneboom [18] and Cabo and Groeneboom [10], who prove a central limit theorem for the cardinality of $\mathcal{V}(\mathcal{P}_{\lambda\rho})$ when ρ is uniform and when $d = 2$, (b) Reitzner [31] who considers the one-dimensional central limit theorem and who establishes a rate of convergence $O(\lambda^{-(d-1)/2(d+1)}(\log\lambda)^{2+2/(d+1)})$ to the normal for ρ uniform (whence $\delta = 0$ in our setting), without giving asymptotics for the limiting variance and covariance, and (c) Vu [37], who proves a central limit theorem for the cardinality of $\mathcal{V}(\{X_i\}_{=1}^n)$, X_i i.i.d. uniform, but who also does not consider limiting covariances. Concerning rates, we believe that the power on the logarithm, namely, $3 + 2(d-1)$, can be reduced to $2(d-1)$, but we have not tried for this sharper rate.

(iii) As shown by Reitzner (Lemma 7 of [31]), when $\delta = 0$, the right-hand side of (2.3) is strictly positive and finite whenever f is not identically zero.

2.2. *Number of maximal points in an i.i.d. sample.* For all $\bar{w} := (w, h_w)$, we define the downward cone

$$K^{\downarrow}[\bar{w}] := \{(z, h_z) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h_z \le h_w - \psi(|z - w|)\}. \tag{2.5}$$

Consider $\psi(l) := l^\alpha$, $\alpha \in (0, 1]$, in Section 1.1 so that $K[\mathbf{0}] := \{(w_1, \dots, w_d) : w_d \ge (w_1^2 + \dots + w_{d-1}^2)^{\alpha/2}\}$. Given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$, a point $\bar{w} \in \mathcal{X}$ is called *K-maximal* iff $\bar{w}$ does not belong to any $u \oplus K[\mathbf{0}]$ for $u \in \mathcal{X}$. When $\alpha \in (0, 1]$ we have the equivalence $\bar{y} \in K[\bar{x}]$ iff $K[\bar{y}] \subseteq K[\bar{x}]$ and $\bar{x} \in K^{\downarrow}[\bar{y}]$ iff $K^{\downarrow}[\bar{x}] \subseteq K^{\downarrow}[\bar{y}]$. It thus follows that for such ψ the present notion of maximality is just a rephrasing of the maximality notion as discussed in Section 1. Indeed, we see that $\bar{w}$ is K-maximal or ψ-extremal in $\mathcal{X}$ iff $\bar{w} \oplus K^{\downarrow}[\mathbf{0}]$ contains no other points in $\mathcal{X}$. This is not the case for $\alpha > 1$, where the equivalence $\bar{y} \in K[\bar{x}]$ iff $K[\bar{y}] \subseteq K[\bar{x}]$ does not hold.

Recalling that $\mathcal{M}_K(\mathcal{P}_{\lambda\rho})$ denotes the collection of K-maximal points in $\mathcal{P}_{\lambda\rho}$, and with ρ and A as in Section 1.1, consider the induced *maximal point measure*

$$\mu_{\lambda\rho} := \sum_{x \in \mathcal{M}_{\mathcal{K}}(\mathcal{P}_{\lambda\rho})} \delta_x.$$

Recalling the definitions of $I(f)$ and $J(f)$ at (1.7) and (1.8), respectively, we have the following:

THEOREM 2.2. *With τ as given by* (1.5) *and $\alpha \in (0, 1]$, for all $f \in \mathcal{C}_b(A_+)$,*

$$\lim_{\lambda \to \infty} \lambda^{-\tau} \mathbb{E}[\langle f, \mu_{\lambda\rho} \rangle] = I(f) \tag{2.6}$$

and

$$\lim_{\lambda \to \infty} \lambda^{-\tau} \operatorname{Var}[\langle f, \bar{\mu}_{\lambda\rho} \rangle] = I(f^2) + J(f^2). \tag{2.7}$$

Moreover, the finite-dimensional distributions $(\langle f_1, \lambda^{-\tau/2} \bar{\mu}_{\lambda\rho} \rangle, \dots, \langle f_k, \lambda^{-\tau/2} \bar{\mu}_{\lambda\rho} \rangle)$, $f_1, \dots, f_k \in \mathcal{C}_b(A_+)$, of $\lambda^{-\tau/2} \bar{\mu}_{\lambda\rho}$ converge as $\lambda \to \infty$ to those of a mean zero Gaussian field with covariance kernel

$$(f, g) \mapsto I(fg) + J(fg), \qquad f, g \in \mathcal{C}_b(A_+).$$

Additionally, if $\delta = 0$, then for all $f \in \mathcal{C}_b(A_+)$,

$$\begin{aligned} &\sup_t \left| P\left[\frac{\langle f, \bar{\mu}_{\lambda\rho} \rangle}{\sqrt{\operatorname{Var}\langle f, \bar{\mu}_{\lambda\rho} \rangle}} \le t \right] - P[N(0, 1) \le t] \right| \\ &\qquad = O\big(\lambda^{-(d-1)/2d} (\log \lambda)^{3+2(d-1)}\big). \end{aligned} \tag{2.8}$$

Theorem 2.2 admits de-Poissonization as follows. Let $X_1, \dots, X_n$ be i.i.d. chosen in A_+ according to the density ρ, assumed to be integrable to 1, and consider the associated maximal point measure

$$\nu_n^{\xi} := \sum_{x \in \mathcal{M}_{\mathcal{K}}(\{X_i\}_{i=1}^n)} \delta_x.$$

We have then the following equivalent of Theorem 2.2 for binomial samples.

THEOREM 2.3. *With τ as given by* (1.5) *and $\alpha \in (0, 1]$, for all $f \in \mathcal{C}_b(A_+)$,*

$$\lim_{n \to \infty} n^{-\tau} \mathbb{E}[\langle f, \nu_n^{\xi} \rangle] = I(f) \tag{2.9}$$

and

$$\lim_{n \to \infty} n^{-\tau} \operatorname{Var}[\langle f, \bar{\nu}_n^{\xi} \rangle] = I(f^2) + J(f^2). \tag{2.10}$$

Moreover, the finite-dimensional distributions $(\langle f_1, n^{-\tau/2}\bar{\nu}_n^{\xi}\rangle, \ldots, \langle f_k, n^{-\tau/2}\bar{\nu}_n^{\xi}\rangle)$, $f_1, \ldots, f_k \in \mathcal{C}_b(A_+)$, *of* $n^{-\tau/2}\bar{\nu}_n^{\xi}$ *converge as* $n \to \infty$ *to those of a mean zero Gaussian field with covariance kernel*

$$(f, g) \mapsto I(fg) + J(fg), \qquad f, g \in \mathcal{C}_b(A_+).$$

REMARK. Theorems 2.2 and 2.3 extend and generalize the work of (a) Barbour and Xia [4], who establish central limit theorems for the case of homogeneous spatial temporal input, with K the positive octant in $\mathbb{R}^d$, and who consider neither convergence of finite-dimensional distributions nor convergence of variances, (b) Baryshnikov and Yukich [7], who establish convergence of finite-dimensional distributions but who restrict to homogeneous temporal input ($\delta = 0$) as well as to the case $\psi(l) = l$ (i.e., $\alpha = 1$), and (c) Baryshnikov [6], who also restricts to homogeneous temporal input and does not consider convergence of finite-dimensional distributions.

3. Proof of main results. In this section we prove Theorems 1.1–1.3. An essential component of the proofs involves introducing a notion of *localization*, which quantifies the decoupling property of the considered functional ξ over distant regions. It is straightforward to check that the proofs hold for ψ-growth without overlap when ψ is linear.

3.1. *Stabilization for* Ψ*-growth functionals.* With $B_{d-1}(y, r)$ standing as usual for the $(d-1)$-dimensional ball centered at $y \in \mathbb{R}^{d-1}$ with radius $r \in (0, \infty)$, we denote by $C_{d-1}(y, r)$ the cylinder $B_{d-1}(y, r) \times \mathbb{R}_+$. Recalling $\bar{y} := (y, h_y)$, consider for all $r > 0$ the finite range version of $\xi(\bar{y}, \mathcal{X})$, namely,

$$\xi_{[r]}(\bar{y}, \mathcal{X}) := \xi_{C_{d-1}(y,r)}(\bar{y}, \mathcal{X}),$$

that is, $\xi_{[r]}(\bar{y}, \mathcal{X})$ depends only on the local behavior of $\mathcal{X}$ with *spatial* coordinates restricted to the r-neighborhood of y. For a point process $\mathcal{P}$ (usually chosen to be Poisson in the sequel) in $\mathbb{R}^{d-1} \times \mathbb{R}_+$, the *localization radius* of ξ at $\bar{y} \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ is defined by

$$R^{\xi} := R^{\xi}[\bar{y}; \mathcal{P}] := \inf\{r \in \mathbb{R}_+ : \forall s \geq r\ \xi(\bar{y}, \mathcal{P}) = \xi_{[s]}(\bar{y}, \mathcal{P})\}. \tag{3.1}$$

In full analogy with $\xi^{(\lambda)}$ given by (1.17), we define for all $\lambda > 0$ the localization radius $R^{\xi^{(\lambda)}}[\cdot; \cdot]$ by

$$R^{\xi^{(\lambda)}} := R^{\xi^{(\lambda)}}[\bar{y}; \mathcal{P}'] := \inf\{r \in \mathbb{R}_+ : \forall s \geq r\ \xi^{(\lambda)}(\bar{y}', \mathcal{P}') = \xi^{(\lambda)}_{[s]}(\bar{y}', \mathcal{P}')\}.$$

Observe that the localization radius considered here formally differs from the stabilization radii considered in [7], [23–26], essentially defined for all $\bar{y} := (y, h)$ to be the smallest positive real r such that $\xi(\bar{y}, (\mathcal{P} \cap C_{d-1}(y, r)) \cup \mathcal{A}) = \xi(\bar{y}, (\mathcal{P} \cap C_{d-1}(y, r))$ for all finite $\mathcal{A} \subset C^c_{d-1}(y, s)$. However, the ψ-extremal functional is in general extremely sensitive to the choice of the "outside" configuration $\mathcal{A} \subset C^c_{d-1}(y,s)$, rendering the existence and use of standard stabilization radii a bit difficult. The benefit of the localization radius is that it considers only the outside configurations involving points from $\mathcal{P}$. However, since the localization radius shares many of the same properties as the stabilization radii in [7], [23–26], *we will abuse terminology and henceforth refer to the localization radius R^ξ as a stabilization radius*.

The following lemma shows that $\xi^{(\lambda)}$ given by (1.17) has a stabilization radius whose tail decays exponentially uniformly in large enough λ when $\mathcal{P}$ is $\mathcal{P}^{(\lambda)}_{\lambda\rho}$ given by (1.16) or when $\mathcal{P}$ is given by $\mathcal{P}^{(\lambda)}_{\lambda\rho} \cup \{\bar{z}'_1,\dots,\bar{z}'_k\}$, $k \ge 1$, where $\bar{z}'_i$, $i = 1,\dots,k$, are certain deterministic points (fixed atoms). This result will prove useful later in showing exponential decay of correlation functions for ψ-growth processes.

LEMMA 3.1. (i) *For A compact and convex, there exists a constant C such that, uniformly in x and λ large enough, for all $\bar{y}' \in \lambda^\beta A \times \mathbb{R}_+$ and for all collections $\{\bar{z}'_1,\dots,\bar{z}'_k\} \subseteq \lambda^\beta A \times \mathbb{R}_+$ of deterministic points, $k \ge 0$, we have for all $L > 0$*

$$P\big[R^{\xi^{(\lambda)}}[\bar{y}'; \mathcal{P}*^{(\lambda)}_{\lambda\rho}] > L\big] \le C \exp\left(-\frac{L^{\alpha+d-1}}{C}\right), \tag{3.2}$$

where $\mathcal{P}*^{(\lambda)}_{\lambda\rho} := \mathcal{P}^{(\lambda)}_{\lambda\rho} \cup \{\bar{z}_1,\dots,z_k\}$, so that, in particular, $\mathcal{P}*^{(\lambda)}_{\lambda\rho} = \mathcal{P}^{(\lambda)}_{\lambda\rho}$ for $k = 0$.

(ii) *An identical bound holds if instead $A := \mathbb{R}^{d-1}$ and $\mathcal{P}^{(\lambda)}_{\lambda\rho}$ is replaced by a homogeneous Poisson point process on $\mathbb{R}^{d-1}$.*

REMARK. In place of (3.2) we have uniformly in x and λ large enough, for all $\bar{y}' \in \lambda^\beta A \times \mathbb{R}_+$ and for all $L > 0$, the simpler bound

$$P\big[R^{\xi^{(\lambda)}}[\bar{y}'; \mathcal{P}*^{(\lambda)}_{\lambda\rho}] > L\big] \le C \exp\left(-\frac{L}{C}\right). \tag{3.3}$$

PROOF OF LEMMA 3.1. We will only prove Lemma 3.1(i) as identical arguments handle Lemma 3.1(ii). Also, since the proof relies on probability bounds for certain regions being devoid of points of the underlying point process $\mathcal{P}*^{(\lambda)}_{\lambda\rho}$, as easily noted below, we can assume without loss of generality that $k = 0$ so that $\mathcal{P}*^{(\lambda)}_{\lambda\rho} = \mathcal{P}^{(\lambda)}_{\lambda\rho}$. Moreover, to simplify the argument below, we ignore the boundary effects arising when $\bar{y}'$ is close to $\partial(\lambda^\beta A \times \mathbb{R}_+)$, noting that the absence of points of $\mathcal{P}^{(\lambda)}_{\lambda\rho}$ in the vicinity of $\bar{y}'$ can only decrease $R^{\xi^{(\lambda)}}[\bar{y}'; \mathcal{P}^{(\lambda)}_{\lambda\rho}]$. This allows us to avoid obvious but technical separate considerations for $\bar{y}'$ close to $\partial(\lambda^\beta A \times \mathbb{R}_+)$. Also, we consider x fixed but arbitrary, keeping in mind that the required uniformity in x follows by the boundedness of ρ, both from above and away from 0.

Define for fixed $\bar{y}' := (y', h'_y)$ and all $\lambda \in [0,\infty]$ the *scaled* upward cone

$$K^{(\lambda)}[\bar{y}'] := \{(v', h'_v) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h'_v \ge h'_y + \psi^{(\lambda)}(|v' - y'|)\} \tag{3.4}$$

and the *scaled* downward cone

$$K^{\downarrow}_{(\lambda)}[\bar{y}'] := \{(v', h'_v) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h'_v \le h'_y - \psi^{(\lambda)}(|v' - y'|)\}. \tag{3.5}$$

Note that $\bar{u}' \in K^{(\lambda)}[\bar{z}']$ iff $h'_u \ge h'_z + \psi(|u' - z'|)$, which is equivalent to $h'_z \le h'_u - \psi(|z' - u'|)$, and thus, the *duality* $\bar{u}' \in K^{(\lambda)}[\bar{z}']$ iff $\bar{z}' \in K^{\downarrow}_{(\lambda)}[\bar{u}']$.

To proceed, note that the event $\{R^{\xi^{(\lambda)}}[\bar{y}'; \mathcal{P}^{(\lambda)}_{\lambda\rho}] > L\}$ is equivalent to the event

$$E := \{\exists r > L : \xi^{(\lambda)}(\bar{y}', \mathcal{P}^{(\lambda)}_{\lambda\rho}) \neq \xi^{(\lambda)}_{[r]}(\bar{y}', \mathcal{P}^{(\lambda)}_{\lambda\rho})\},$$

and moreover, $E \subset E_1 \cup E_2$, where E_1 and E_2 are defined below. Roughly speaking, the event E_1 ensures that $\bar{y}'$ is extremal with respect to $\mathcal{P}^{(\lambda)}_{\lambda\rho} \cap C_{d-1}(y', r)$ for some $r > L$ but not necessarily with respect to $\mathcal{P}^{(\lambda)}_{\lambda\rho}$, whereas E_2 is just the opposite.

Event E_1: For some $r > L$, there exists a boundary point $\bar{u}' \in \partial(K^{(\lambda)}[\bar{y}']) \cap C_{d-1}(y', r)$, and such that $\bar{u}' \notin \bigcup_{\bar{z}' \in [\mathcal{P}^{(\lambda)}_{\lambda\rho} \setminus \{\bar{y}'\}] \cap C_{d-1}(y', r)} K^{(\lambda)}[\bar{z}']$ but $\bar{u}' \in \bigcup_{\bar{z}' \in \mathcal{P}^{(\lambda)}_{\lambda\rho} \cap C_{d-1}(y', r)} K^{(\lambda)}[\bar{z}']$, that is, $\xi^{(\lambda)}_{[r]}(\bar{y}', \mathcal{P}^{(\lambda)}_{\lambda\rho}) = 1$, but possibly $\xi^{(\lambda)}(\bar{y}', \mathcal{P}^{(\lambda)}_{\lambda\rho}) = 0$.

Event E_2: For some $r > L$, there exists a boundary point $\bar{u}' \in \partial(K^{(\lambda)}[\bar{y}']) \cap C^c_{d-1}(y', r)$ such that $\bar{u}' \notin \bigcup_{\bar{z}' \in \mathcal{P}^{(\lambda)}_{\lambda\rho} \setminus \{\bar{y}'\}} K^{(\lambda)}[\bar{z}']$, but $K^{(\lambda)}[\bar{y}'] \cap C_{d-1}(\bar{y}', r) \subset \bigcup_{\bar{z}' \in [\mathcal{P}^{(\lambda)}_{\lambda\rho} \setminus \{\bar{y}'\}] \cap C_{d-1}(y', r)} K^{(\lambda)}[\bar{z}']$, that is, $\xi^{(\lambda)}(\bar{y}', \mathcal{P}^{(\lambda)}_{\lambda\rho}) = 1$ but $\xi^{(\lambda)}_{[r]}(\bar{y}', \mathcal{P}^{(\lambda)}_{\lambda\rho}) = 0$.

On event E_1 writing $\bar{u}' := (u', h'_u)$, we easily check that

$$h'_u \ge \psi^{(\lambda)}\left(\frac{L}{2}\right). \tag{3.6}$$

Indeed, we have:

- either $|u' - y'| \ge r/2$ or
- $d(u', \partial B_{d-1}(y', r)) \ge r/2$ and, hence, $d(u', z') \ge r/2$ for all $\bar{z}' \in \mathcal{P}^{(\lambda)}_{\lambda\rho} \cap C^c_{d-1}(y', r)$.

In both cases, on E_1, (u', h'_u) falls into $K^{(\lambda)}[\bar{v}']$ for some $\bar{v}'$ such that $|v' - u'| \ge r/2$, either with $\bar{v}' = \bar{y}'$ or $\bar{v}' \in \mathcal{P}^{(\lambda)}_{\lambda\rho} \cap C^c_{d-1}(y', r)$. Consequently, recalling that $r > L$ and using the definition of $K^{(\lambda)}[\cdot]$, we obtain (3.6) as required.

On E_1 we have $\bar{u}' \notin \bigcup_{\bar{z}' \in [\mathcal{P}^{(\lambda)}_{\lambda\rho} \setminus \{\bar{y}'\}] \cap C_{d-1}(y', r)} K^{(\lambda)}[\bar{z}']$, implying that the downward cone $K^{\downarrow}_{(\lambda)}[\bar{u}']$ is devoid of points of $\mathcal{P}^{(\lambda)}_{\lambda\rho} \cap C_{d-1}(y', r)$. By the assumed properties of ψ and ρ, the integral of $\rho^{(\lambda)}$ over $K^{\downarrow}_{(\lambda)}[\bar{u}']$ is $\Omega(\mathrm{Vol}(K^{\downarrow}_{(\lambda)}[\bar{u}']))$, which is

$$\Omega\biggl(\int_0^{h'_u}\bigl([\psi^{(\lambda)}]^{-1}(h'_u-h')\bigr)^{d-1}\,dh'\biggr)=\Omega\biggl(\int_0^{h'_u}(h'_u-h')^{(d-1)/\alpha}\,dh'\biggr) \tag{3.7}$$
$$=\Omega\bigl((h'_u)^{(\alpha+d-1)/\alpha}\bigr),$$

with the second equality following by the definition of $[\psi^{(\lambda)}]^{-1}$, and where we use $f(\lambda)=\Omega(g(\lambda))$ to signify that $f(\lambda)/g(\lambda)$ is asymptotically bounded away from zero. Clearly, the integral of $\rho^{(\lambda)}$ over $K^{\downarrow}_{(\lambda)}[\bar{u}']\cap C_{d-1}(y',r)$ for $\bar{u}'\in C_{d-1}(y',r)$ is of the same order.

Recalling from (1.18) that the intensity measure of the Poisson process $\mathcal{P}^{(\lambda)}_{\lambda\rho}$ has its density given by $\rho^{(\lambda)}$, we thus conclude for fixed $\bar{u}'$ that the probability of the considered event $\Xi[\bar{u}']:=\{K^{\downarrow}_{(\lambda)}[\bar{u}']\cap[\mathcal{P}^{(\lambda)}_{\lambda\rho}\setminus\{\bar{y}'\}]\cap C_{d-1}(y',r)=\varnothing\}$ satisfies

$$P[\Xi[\bar{u}']]\le\exp\bigl(-\Omega\bigl((h'_u)^{(\alpha+d-1)/\alpha}\bigr)\bigr). \tag{3.8}$$

To proceed, we recall that $r>L$ and we partition $\mathbb{R}^{d-1}\times\mathbb{R}_+$ into unit volume cubes and we let $q_1,q_2,\ldots$ be an enumeration of those cubes having nonempty intersection with $\partial(K^{(\lambda)}[\bar{y}'])$. Let

$$p_i:=P\bigl[\exists\bar{u}'\in q_i:K^{\downarrow}_{(\lambda)}[\bar{u}']\cap\bigl[\mathcal{P}^{(\lambda)}_{\lambda\rho}\setminus\{\bar{y}'\}\bigr]\cap C_{d-1}(y',L)=\varnothing\bigr]$$

for all $i=1,2,\ldots$ and note that, by (3.8), we have

$$p_i\le\exp\bigl(-\Omega\bigl((h'_q)^{(\alpha+d-1)/\alpha}\bigr)\bigr),$$

where h'_{q_i} is the last coordinate of the center of the cube q_i.

We now have

$$P[E_1]\le\sum_{i=1}^{\infty}p_i\le C\int_{\psi^{(\lambda)}(L/2)}^{\infty}L^{d-2}\exp\biggl(-\frac{1}{C}(h'_u)^{(\alpha+d-1)/\alpha}\biggr)\,dh'_u$$

for some $0<C<\infty$ in view of the discussion above. Here CL^{d-2} bounds the number of cubes in the set $q_1,q_2,\ldots$ of any fixed height $h'_u\ge\psi^{(\lambda)}(L/2)$.

Recalling that $\psi^{(\lambda)}(L/2)=(1+o(1))(L/2)^{\alpha}$, it follows (using a different choice of C if necessary) that

$$P[E_1]\le C\exp\biggl(-\frac{1}{C}L^{\alpha+d-1}\biggr).$$

To estimate $P[E_2]$, note that for $\bar{u}':=(u',h'_u)\in\partial(K[\bar{y}'])$ lying in $C^c_{d-1}(y',r)$ we must have

$$h'_u\ge\psi^{(\lambda)}(r). \tag{3.9}$$

Further, since $\bar{u}'\notin\bigcup_{\bar{z}'\in\mathcal{P}^{(\lambda)}_{\lambda\rho}\setminus\{\bar{y}'\}}K^{(\lambda)}[\bar{z}']$, we have $K^{\downarrow}_{(\lambda)}[\bar{u}']\cap[\mathcal{P}^{(\lambda)}_{\lambda\rho}\setminus\{\bar{y}'\}]=\varnothing$. Denoting this event $\Xi^*[\bar{u}']:=\{K^{\downarrow}_{(\lambda)}[\bar{u}']\cap[\mathcal{P}^{(\lambda)}_{\lambda\rho}\setminus\{\bar{y}'\}]=\varnothing\}$, noting that as in (3.8) we have

$$P[\Xi^*[\bar{u}']]\le\exp\bigl(-\Omega\bigl((h'_u)^{(\alpha+d-1)/\alpha}\bigr)\bigr), \tag{3.10}$$

recalling that $r > L$ and proceeding in analogy with the case of event E_1 above, with (3.6) and (3.8) there replaced by (3.9) and (3.10) respectively and with $C^c_{d-1}(y', L)$ partitioned into unit volume cubes, we bound $P[E_2]$ by

$$P[E_2] \le C \int_{s=L}^{\infty} s^{d-2} \int_{h'_y+\psi^{(\lambda)}(s)}^{\infty} \exp\biggl(-\frac{1}{C}(h'_u)^{(\alpha+d-1)/\alpha}\biggr)\, dh'_u\, ds$$

for some $0 < C < \infty$. It follows that $P[E_2] \le C \exp(-L^{\alpha+d-1}/C)$. Since $P[R^{\xi^{(\lambda)}}[\bar{y}'; \mathcal{P}^{(\lambda)}_{\lambda\rho}] > L] = P[E] \le P[E_1] + P[E_2]$, Lemma 3.1 follows. □

Given $\bar{y} := (y', h'_y)$, we expect for large temporal h'_y, that $\bar{y}$ is ψ-extremal with small probability. Also, as previously noted in Section 1.1, we expect for small α that $\bar{y}$ is more likely to be ψ-extremal. The next lemma makes these probabilities a bit more precise and shows that the probability of having (y', h'_y) extreme in $\mathcal{P}*^{(\lambda)}_{\lambda\rho} := \mathcal{P}^{(\lambda)}_{\lambda\rho} \cup \{\bar{z}'_1, \dots, \bar{z}'_k\}, k \ge 0,$ with respect to $\psi^{(\lambda)}$ decays exponentially with h'_y uniformly in λ for λ large enough.

LEMMA 3.2. *There exists a constant C such that, uniformly in λ large enough, for all $\bar{y}' \in \lambda^{\beta} A \times \mathbb{R}_+$ and $\{\bar{z}'_1, \dots, \bar{z}'_k\}$, we have*

$$P\bigl[\xi^{(\lambda)}(\bar{y}', \mathcal{P}*^{(\lambda)}_{\lambda\rho}) = 1\bigr] \le C \exp\biggl(-\frac{1}{C}(h'_y)^{(\alpha+d-1)/\alpha}\biggr).$$

PROOF. Clearly, since adding extra points to $\mathcal{P}^{(\lambda)}_{\lambda\rho}$ decreases the probability of (y', h'_y) being extreme, we may without loss of generality choose $k = 0$ so that $\mathcal{P}*^{(\lambda)}_{\lambda\rho} = \mathcal{P}^{(\lambda)}_{\lambda\rho}$.

On the event $E := \{\xi^{(\lambda)}(\bar{y}', \mathcal{P}^{(\lambda)}_{\lambda\rho}) = 1\}$ there exists $\bar{u}' := (u', h'_u) \in \partial(K^{(\lambda)}[\bar{y}'])$ such that $\bar{u}' \notin \bigcup_{\bar{z}' \in \mathcal{P}^{(\lambda)}_{\lambda\rho} \setminus \{\bar{y}'\}} K^{(\lambda)}[\bar{z}']$, which is equivalent to $K^{\downarrow}_{(\lambda)}[\bar{u}'] \cap [\mathcal{P}^{(\lambda)}_{\lambda\rho} \setminus \{\bar{y}'\}] = \varnothing$. As in the proof of Lemma 3.1, for fixed $\bar{u}'$, the probability of the last event does not exceed

$$\exp\biggl[-\int_{K^{\downarrow}_{(\lambda)}[\bar{u}']} \rho^{(\lambda)}(v'h'_v)\, dv'\, dh'_v\biggr] \le C \exp\biggl(-\frac{1}{C}(h'_u)^{(\alpha+d-1)/\alpha}\biggr).$$

Recalling the relation $h'_u = h'_y + \psi^{(\lambda)}(|u' - y'|)$, putting $|u' - y'| = s$, and resorting again to a partition of $\mathbb{R}^{d-1} \times \mathbb{R}_+$ into unit volume cubes and summing up the respective probabilities as in the proof of Lemma 3.1, we obtain the required bound

$$\begin{aligned} P[E] &\le C \int_0^{\infty} s^{d-2} \int_{h'_y}^{\infty} \exp\biggl(-\frac{1}{C}(h'_u)^{(\alpha+d-1)/\alpha}\biggr)\, dh'_u\, ds \\ &\le C \exp\biggl(-\frac{1}{C}(h'_y)^{(\alpha+d-1)/\alpha}\biggr). \end{aligned}$$

□

3.2. *Proof of Theorem* 1.1. Recall the definition of $\mathcal{P}_{\rho_x^{(\infty)}}$ from (1.21). One benefit of stabilization is that the one point correlation function $\mathbb{E}[\xi^{(\infty)}((\mathbf{0},h'),\mathcal{P}_{\rho_x^{(\infty)}})]$ is approximated for large r by the finite range version

$$\mathbb{E}\big[\xi_{[r]}^{(\infty)}\big((\mathbf{0},h'),\mathcal{P}_{\rho_x^{(\infty)}}\big)\big]$$

and, similarly, $\mathbb{E}[\xi^{(\lambda)}((\mathbf{0},h'),\mathcal{P}_{\lambda\rho}^{(\lambda)})]$ is approximated by its finite range version $\mathbb{E}[\xi_{[r]}^{(\lambda)}((\mathbf{0},h'),\mathcal{P}_{\lambda\rho}^{(\lambda)})]$. Using the large λ weak convergence of $\mathcal{P}_{\lambda\rho}^{(\lambda)}$ to $\mathcal{P}_{\rho_x^{(\infty)}}$, one may approximate the first mentioned finite range version by the second and thus show that $\mathbb{E}[\xi^{(\lambda)}((\mathbf{0},h'),\mathcal{P}_{\lambda\rho}^{(\lambda)})]$ is asymptotically equal to $\mathbb{E}[\xi^{(\infty)}((\mathbf{0},h'),\mathcal{P}_{\rho_x^{(\infty)}})]$. This is spelled out in Lemma 3.3 below, which captures the essence of stabilization and which lies at the heart of the proof of Theorem 1.1. Note that when Lemma 3.3 is combined with Lemma 3.2, then it shows

$$\mathbb{E}\big[\xi^{(\infty)}((\mathbf{0},h'),\mathcal{P}^*)\big]\le C\exp\biggl(-\frac{1}{C}(h')^{(\alpha+d-1)/\alpha}\biggr) \tag{3.11}$$

and, therefore, $I(f)<\infty$ for $f\in\mathcal{C}_b(A_+)$. Recall from (1.17) that $\xi^{(\lambda)}$ is the rescaled version of ξ with dependency on x fixed.

LEMMA 3.3. *For all $x\in A$ and $h'\in\mathbb{R}_+$, we have*

$$\lim_{\lambda\to\infty}\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0},h'),\mathcal{P}_{\lambda\rho}^{(\lambda)}\big)\big]=\mathbb{E}\big[\xi^{(\infty)}\big((\mathbf{0},h'),\mathcal{P}_{\rho_x^{(\infty)}}\big)\big].$$

PROOF. Fix $x\in A$. Taking into account (1.19) and (1.21) and using the results of Section 3.5 in [34] [see Proposition 3.22 or Proposition 3.19 there combined with Proposition 3.6(ii) ibidem], we observe that as $\lambda\to\infty$, $\mathcal{P}_{\lambda\rho}^{(\lambda)}$ converges weakly to $\mathcal{P}_{\rho_x^{(\infty)}}$ as a point process; see ibidem. Using Theorem 5.5 in [9] with $h_\lambda:=\xi_{[r]}^{(\lambda)}((\mathbf{0},h'),\cdot)$ and $h:=\xi_{[r]}^{(\infty)}((\mathbf{0},h'),\cdot)$ there, we easily see that, by Lemmas 3.1 and 3.2, under the law of the limit process $\mathcal{P}_{\rho_x^{(\infty)}}$, the discontinuity event E ibidem [an infinitesimal move of the point configuration alters the ξ-value for $(\mathbf{0},h)$] is contained up to an event of probability 0 in the set of point configurations $\mathcal{X}$ such that either the spatial coordinates of two points in $\mathcal{X}$ coincide or such that there are at least two points $\bar{y}',\bar{y}''\in\mathcal{X}$ such that the boundaries of the upward cones $K^{(\infty)}[\bar{y}']$ and $K^{(\infty)}[\bar{y}'']$ [recall (3.4)] intersect in a point lying on the boundary of the upward cone $K^{(\infty)}[(\mathbf{0},h')]$, which clearly happens with probability 0 under the law of $\mathcal{P}_{\rho_x^{(\infty)}}$. Indeed, Lemma 3.2 states that no effects coming from $h\to\infty$ arise (no infinite range dependencies in h). A similar statement in space is provided by Lemma 3.1. Combining both these statements allows us to draw conclusions from the weak convergence of point processes as we do in the above argument; see ibidem in [34]. Thus, Theorem 5.5 in [9] yields

$$\lim_{\lambda\to\infty}\mathbb{E}\big[\xi_{[r]}^{(\lambda)}\big((\mathbf{0},h'),\mathcal{P}_{\lambda\rho}^{(\lambda)}\big)\big]=\mathbb{E}\big[\xi_{[r]}^{(\infty)}\big((\mathbf{0},h'),\mathcal{P}_{\rho_x^{(\infty)}}\big)\big]. \tag{3.12}$$

Let $R^{\xi} := R^{\xi^{(\lambda)}}[(\mathbf{0}, h'); \mathcal{P}^{(\lambda)}_{\lambda\rho}]$. We have for all $r > 0$ and all $\lambda > 0$

$$\begin{aligned}
&\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\big] \\
&\quad = \mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\mathbf{1}_{R^{\xi} \le r}\big] + \mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\mathbf{1}_{R^{\xi} > r}\big] \\
&\quad = \mathbb{E}\big[\xi^{(\lambda)}_{[r]}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\mathbf{1}_{R^{\xi} \le r}\big] + \mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\mathbf{1}_{R^{\xi} > r}\big].
\end{aligned}$$

By Lemma 3.1(i) [recall the bound (3.3)], Cauchy–Schwarz, and the boundedness of $\xi^{(\lambda)}_{[r]}$, uniformly in large λ and all $r > 0$,

$$\mathbb{E}\big[\xi^{(\lambda)}_{[r]}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\mathbf{1}_{R^{\xi} > r}\big] \le C \exp\left(-\frac{r}{C}\right)$$

for some C not depending on x. Likewise, uniformly in large λ, we have $\mathbb{E}[\xi^{(\lambda)}((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho})\mathbf{1}_{R^{\xi} > r}] \le C \exp(-r/C)$. It follows that, for large $\lambda > 0$ and all $r > 0$,

$$\big|\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\big] - \mathbb{E}\big[\xi^{(\lambda)}_{[r]}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\big]\big| \le 2C \exp\left(-\frac{r}{C}\right). \tag{3.13}$$

Similarly, Lemma 3.1(ii) gives, for all $r > 0$,

$$\big|\mathbb{E}\big[\xi^{(\infty)}\big((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}}\big)\big] - \mathbb{E}\big[\xi^{(\infty)}_{[r]}\big((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}}\big)\big]\big| \le 2C \exp\left(-\frac{r}{C}\right).$$

Write

$$\begin{aligned}
&\big|\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\big] - \mathbb{E}\big[\xi^{(\infty)}\big((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}}\big)\big]\big| \\
&\quad \le \big|\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\big] - \mathbb{E}\big[\xi^{(\lambda)}_{[r]}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\big]\big| \\
&\qquad + \big|\mathbb{E}\big[\xi^{(\lambda)}_{[r]}\big((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\big] - \mathbb{E}\big[\xi^{(\infty)}_{[r]}\big((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}}\big)\big]\big| \\
&\qquad + \big|\mathbb{E}\big[\xi^{(\infty)}_{[r]}\big((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}}\big)\big] - \mathbb{E}\big[\xi^{(\infty)}\big((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}}\big)\big]\big|.
\end{aligned} \tag{3.14}$$

For fixed r, the second term on the right-hand side of (3.14) goes to zero as $\lambda \to \infty$ by (3.12). The first and third terms are bounded above by $2C \exp(-r/C)$. Letting $r \to \infty$ completes the proof of Lemma 3.3. □

Given Lemmas 3.2 and 3.3, we now prove Theorem 1.1 as follows. We have

$$\mathbb{E}[\langle f, \mu^{\xi}_{\lambda\rho}\rangle] = \int_A \int_0^{\infty} f(x, h_x)\mathbb{E}[\xi((x, h_x), \mathcal{P}_{\lambda\rho})]\lambda\rho(x, h_x)\, dh_x\, dx.$$

By (1.17), we have $\xi((x, h_x), \mathcal{P}_{\lambda\rho}) = \xi^{(\lambda)}((\mathbf{0}, h'_x), \mathcal{P}^{(\lambda)}_{\lambda\rho})$ and by (1.15), we have $\rho(x, h_x) = \lambda^{-\gamma\delta}\rho^{(\lambda)}(\mathbf{0}, h'_x)$. Thus, putting $h'_x := \lambda^{\gamma} h_x$ and recalling $1 - \gamma(\delta + 1) = \tau$ [see (1.5) and (1.12)], we obtain

$$\mathbb{E}[\langle f, \mu^{\xi}_{\lambda\rho}\rangle] = \int_A \int_0^{\infty} f(x, h'_x\lambda^{-\gamma})\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'_x), \mathcal{P}^{(\lambda)}_{\lambda\rho}\big)\big]\lambda^{\tau}\rho^{(\lambda)}(\mathbf{0}, h'_x)\, dh'_x\, dx$$

or, simply,

$$\lambda^{-\tau}\mathbb{E}[\langle f, \mu^{\xi}_{\lambda\rho}\rangle] = \int_A \int_0^\infty f(x, h'_x\lambda^{-\gamma})\mathbb{E}[\xi^{(\lambda)}((\mathbf{0}, h'_x), \mathcal{P}^{(\lambda)}_{\lambda\rho})]\rho^{(\lambda)}(\mathbf{0}, h'_x)\, dh'_x\, dx.$$

We put

$$g_\lambda(x, h'_x) := \mathbb{E}[\xi^{(\lambda)}((\mathbf{0}, h'_x), \mathcal{P}^{(\lambda)}_{\lambda\rho})]\rho^{(\lambda)}(\mathbf{0}, h'_x).$$

For all $x \in A$ and $h'_x \in \mathbb{R}_+$, we have by Lemma 3.3 and (1.19)

$$\lim_{\lambda\to\infty} g_\lambda(x, h'_x) = \mathbb{E}[\xi^{(\infty)}((\mathbf{0}, h'_x), \mathcal{P}_{\rho_x^{(\infty)}})]\rho_0(x){h'}^{\delta}_x$$

and moreover, by Lemma 3.2 for all $(x, h) \in A_+$, $g_\lambda(x, h'_x)$ is bounded uniformly in λ by the function $(x, h') \mapsto C'(h')^\delta \exp(-h'/C)$, which is integrable on A_+. Consequently, the dominated convergence theorem yields

$$\begin{aligned}&\lim_{\lambda\to\infty} \lambda^{-\tau}\mathbb{E}[\langle f, \mu^{\xi}_{\lambda\rho}\rangle]\\ &\quad= \int_A \int_0^\infty f(x, 0)\mathbb{E}[\xi^{(\infty)}((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}} \cup \{(\mathbf{0}, h')\})]\rho_0(x)(h')^\delta\, dh'\, dx.\end{aligned} \tag{3.15}$$

Using the scaling relations (1.13), (1.14), (1.6) and (1.21), we see that

$$\begin{aligned}&\xi^{(\infty)}((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}} \cup \{(\mathbf{0}, h')\})\\ &\quad\stackrel{\mathcal{D}}{=} \xi^{(\infty)}((\mathbf{0}, [\rho_0(x)]^\gamma h'), \mathcal{P}_* \cup \{(\mathbf{0}, [\rho_0(x)]^\gamma h')\}),\end{aligned} \tag{3.16}$$

with $\stackrel{\mathcal{D}}{=}$ standing for equality in law. Theorem 1.1 follows by using (3.16), changing variables $h'' := [\rho_0(x)]^\gamma h'$ in the integral in (3.15) and recalling that $\tau = 1 - \gamma(\delta + 1)$.

3.3. *Proof of Theorem* 1.2. Fix $x \in A$ and recall from (1.16) that $\mathcal{P}^{(\lambda)}_{\lambda\rho} := \mathcal{P}^{(\lambda)}_{\lambda\rho}[x]$. For all $\lambda > 0$, $h' \in \mathbb{R}_+$, and $(y', h'_y) \in \lambda^\beta A \times \mathbb{R}_+$, consider the pair correlation function for the re-scaled growth process:

$$\begin{aligned}&c^{(\lambda)}((\mathbf{0}, h'), (y', h'_y))\\ &\quad:= c^{(\lambda)}_x((\mathbf{0}, h'), (y', h'_y))\\ &\quad:= \mathbb{E}[\xi^{(\lambda)}((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho} \cup (y', h'_y))\xi^{(\lambda)}((y', h'_y), \mathcal{P}^{(\lambda)}_{\lambda\rho} \cup (\mathbf{0}, h'))]\\ &\qquad- \mathbb{E}\xi^{(\lambda)}((\mathbf{0}, h'), \mathcal{P}^{(\lambda)}_{\lambda\rho})\mathbb{E}\xi^{(\lambda)}((y', h'_y), \mathcal{P}^{(\lambda)}_{\lambda\rho}).\end{aligned} \tag{3.17}$$

Consider also the pair correlation function for the limit growth process $\xi^{(\infty)}$:

$$\begin{aligned}&c^{(\infty)}_x((\mathbf{0}, h'), (y', h'_y))\\ &\quad:= \mathbb{E}[\xi^{(\infty)}((\mathbf{0}, h), \mathcal{P}_{\rho_x^{(\infty)}} \cup (y', h'_y))\xi^{(\infty)}((y', h'_y), \mathcal{P}_{\rho_x^{(\infty)}} \cup (\mathbf{0}, h'))]\\ &\qquad- \mathbb{E}\xi^{(\infty)}((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}})\mathbb{E}\xi^{(\infty)}((y', h'_y), \mathcal{P}_{\rho_x^{(\infty)}}).\end{aligned}$$

A second benefit of stabilization, as shown by the next lemma, is that it facilitates convergence of pair correlation functions and thus leads to variance asymptotics. The next lemma is the second-order counterpart to Lemma 3.3.

LEMMA 3.4 (Convergence of two point correlation function). *For all* $(x, h_x) := (x, h) \in A_+$, *and* $(y', h'_y) \in \lambda^\beta A \times \mathbb{R}_+$, *we have*

$$\lim_{\lambda\to\infty} c_x^{(\lambda)}((\mathbf{0}, h'), (y', h'_y)) = c_x^{(\infty)}((\mathbf{0}, h'), (y', h'_y)).$$

PROOF. In view of Lemma 3.3, it will suffice to show

$$\begin{aligned}(3.18)\qquad &\lim_{\lambda\to\infty} \mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)\big)\xi^{(\lambda)}\big((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')\big)\big] \\ &\quad = \mathbb{E}\big[\xi^{(\infty)}\big((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}} \cup (y', h'_y)\big)\xi^{(\infty)}\big((y', h'_y), \mathcal{P}_{\rho_x^{(\infty)}} \cup (\mathbf{0}, h')\big)\big].\end{aligned}$$

Let $R^\xi := R^{\xi^{(\lambda)}}[(\mathbf{0}, h'); \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)]$ and let $R^\xi_{y'} := R^{\xi^{(\lambda)}}[(y', h'_y); \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')]$. For all $r > 0$, we let $E_r := \{R^\xi_{y'} \le r,\ R^\xi \le r\}$. We split the left-hand side of (3.18) as

$$\begin{aligned}&\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)\big)\xi^{(\lambda)}\big((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')\big)\mathbf{1}_{E_r}\big] \\ &\quad + \mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)\big)\xi^{(\lambda)}\big((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')\big)\mathbf{1}_{E_r^c}\big].\end{aligned}$$

The second expectation is bounded by $C\exp(-r/C)$ for some C not depending on x by Lemma 3.1(i) and by Cauchy–Schwarz. By the definition of the stabilization radius, the first is simply

$$\mathbb{E}\big[\xi_{[r]}^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)\big)\xi_{[r]}^{(\lambda)}\big((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')\big)\mathbf{1}_{E_r}\big].$$

Again, by Lemma 3.1(i) and by Cauchy–Schwarz, for all $r > 0$, this is within $C\exp(-r/C)$ of

$$\mathbb{E}\big[\xi_{[r]}^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)\big)\xi_{[r]}^{(\lambda)}\big((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')\big)\big],$$

hence,

$$\begin{aligned}(3.19)\qquad &\big|\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)\big)\xi^{(\lambda)}\big((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')\big)\big] \\ &\quad - \mathbb{E}\big[\xi_{[r]}^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)\big)\xi_{[r]}^{(\lambda)}\big((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')\big)\big]\big| \\ &\quad \le 2C\exp\left(\frac{-r}{C}\right)\end{aligned}$$

uniformly in x. Now, in analogy with (3.12), we have

$$\begin{aligned}(3.20)\qquad &\lim_{\lambda\to\infty} \mathbb{E}\big[\xi_{[r]}^{(\lambda)}\big((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y)\big)\xi_{[r]}^{(\lambda)}\big((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h')\big)\big] \\ &\quad = \mathbb{E}\big[\xi_{[r]}^{(\infty)}\big((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}} \cup (y', h'_y)\big)\xi_{[r]}^{(\infty)}\big((y', h'_y), \mathcal{P}_{\rho_x^{(\infty)}} \cup (\mathbf{0}, h')\big)\big].\end{aligned}$$

By Lemma 3.1(ii), we have for all $r > 0$

$$\begin{aligned}&\big|\mathbb{E}[\xi^{(\infty)}((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}} \cup (y', h'_y))\xi^{(\infty)}((y', h'_y), \mathcal{P}_{\rho_x^{(\infty)}} \cup (\mathbf{0}, h'))]\\ &\quad - \mathbb{E}[\xi_{[r]}^{(\infty)}((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}} \cup (y', h'_y))\xi_{[r]}^{(\infty)}((y', h'_y), \mathcal{P}_{\rho_x^{(\infty)}} \cup (\mathbf{0}, h'))]\big|\\ &\quad \leq 2C \exp\left(-\frac{r}{C}\right)\end{aligned} \tag{3.21}$$

as in (3.19). Again, note that C does not depend on x since $\rho_0(x)$ is bounded away from zero. Combining (3.19), (3.20) and (3.21) yields

$$\begin{aligned}\limsup_{\lambda\to\infty}&\big|\mathbb{E}[\xi^{(\lambda)}((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y))\xi^{(\lambda)}((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h'))]\\ &\quad - \mathbb{E}[\xi^{(\infty)}((\mathbf{0}, h'), \mathcal{P}_{\rho_x^{(\infty)}} \cup (y', h'_y))\xi^{(\infty)}((y', h'_y), \mathcal{P}_{\rho_x^{(\infty)}} \cup (\mathbf{0}, h'))]\big|\\ &\quad \leq 4C \exp\left(-\frac{r}{C}\right)\end{aligned}$$

for all $r > 0$. We conclude the proof of Lemma 3.4 by letting $r \to \infty$. □

Lemma 3.4 is not enough to establish second-order asymptotics. We will also need that $c_x^{(\lambda)}$ is bounded by an integrable function on $A_+ \times \lambda^\beta A \times \mathbb{R}_+$, that is, we will need to establish the exponential decay of the correlation function (3.17). This is done in the following lemma, which combined with Lemma 3.4, shows that

$$\big|c_x^{(\infty)}((\mathbf{0}, h'), (y', h'_y))\big| \leq C \exp\left(-\frac{1}{C} \max\left(\frac{|y'|}{2}, h'_y, h'\right)\right) \tag{3.22}$$

and, therefore, $J(f) < \infty$ for all $f \in \mathcal{C}_b(A_+)$.

LEMMA 3.5. *There exists a constant C such that, for all $\lambda > 0$, $(x, h_x) := (x, h) \in A_+$, and $(y', h'_y) \in \lambda^\beta A \times \mathbb{R}_+$, we have*

$$\big|c_x^{(\lambda)}((\mathbf{0}, h'), (y', h'_y))\big| \leq C \exp\left(-\frac{1}{C} \max\left(\frac{|y'|}{2}, h'_y, h'\right)\right).$$

PROOF. Let $r \leq |y'|/2$ and note that, by definition of $\xi_{[r]}^{(\lambda)}$, we have

$$\begin{aligned}&\mathbb{E}[\xi_{[r]}^{(\lambda)}((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (y', h'_y))\xi_{[r]}^{(\lambda)}((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)} \cup (\mathbf{0}, h'))]\\ &\quad = \mathbb{E}[\xi_{[r]}^{(\lambda)}((\mathbf{0}, h'), \mathcal{P}_{\lambda\rho}^{(\lambda)})]\mathbb{E}[\xi_{[r]}^{(\lambda)}((y', h'_y), \mathcal{P}_{\lambda\rho}^{(\lambda)})].\end{aligned}$$

Recalling (3.13) and (3.19), we see that

$$\big|c_x^{(\lambda)}((\mathbf{0}, h'), (y', h'_y))\big| \leq 4C \exp\left(-\frac{r}{C}\right)$$

for all $r \le |y'|/2$. In other words, putting $r = |y'|/2$ yields for all $(x, h) \in A_+$ and $(y', h'_y) \in \lambda^\beta A \times \mathbb{R}_+$

$$\left|c_x^{(\lambda)}((\mathbf{0}, h'), (y', h'_y))\right| \le C \exp\left(-\frac{|y'|}{2C}\right).$$

Appealing to Lemma 3.2 shows

$$\left|c_x^{(\lambda)}((\mathbf{0}, h'), (y', h'_y))\right| \le 2C \exp\left(-\frac{1}{C}\max(h'_y, h')\right).$$

Combining the previous two displays concludes the proof of Lemma 3.5. □

Given Lemmas 3.4 and 3.5, we now prove Theorem 1.2 as follows. By the Palm theory for Poisson processes (see, e.g., Theorem 1.6 of [22]), we express $\mathrm{Var}[\langle f, \mu^\xi_{\lambda\rho}\rangle]$ as

$$\begin{aligned}(3.23)\qquad &\lambda \int_{A+} f^2(\bar{x})\mathbb{E}[\xi(\bar{x}, \mathcal{P}_{\lambda\rho})]\rho(\bar{x})\,d\bar{x}\\ &+ \lambda^2 \int_{A_+}\int_{A_+} f(\bar{x})f(\bar{y})c_x^{(1)}\big((\mathbf{0}, h_x), (y - x, h_y)\big)\rho(\bar{x})\rho(\bar{y})\,d\bar{x}\,d\bar{y},\end{aligned}$$

where $\bar{x} := (x, h_x)$ and $\bar{y} := (y, h_y)$.

Following verbatim the proof of Theorem 1.1 shows that after normalization by λ^τ, the first integral converges as $\lambda \to \infty$ to

$$\int_A \int_0^\infty f^2(x, 0)\mathbb{E}\big[\xi^{(\infty)}((\mathbf{0}, h'_x), \mathcal{P}_{\rho_x^{(\infty)}})\big]\rho_0(x)(h'_x)^\delta\,dh'_x\,dx,$$

which by the definition of $m^{(\infty)}$ and the scaling relation (3.16) equals

$$\int_A \int_0^\infty f^2(x, 0)m^{(\infty)}(\mathbf{0}, h'_x)\rho_0^\tau(x)(h'_x)^\delta\,dh'_x\,dx.$$

Making again the usual substitutions $y' = \lambda^\beta(y - x)$, $h'_x = \lambda^\gamma h_x$, and $h'_y = \lambda^\gamma h_y$ and recalling $\rho(x, h_x) = \lambda^{-\gamma\delta}\rho^{(\lambda)}(\mathbf{0}, h'_x)$, the second integral in (3.23) becomes

$$\begin{aligned}\lambda^{2-2\gamma-2\gamma\delta-\beta(d-1)} \int_A \int_{\lambda^\beta A}\int_0^\infty\int_0^\infty &f(x, h'_x\lambda^{-\gamma})f(\lambda^{-\beta}y' + x, h'_y\lambda^{-\gamma})\\ &\times c_x^{(\lambda)}((\mathbf{0}, h'_x), (y', h'_y))\\ &\times \rho^{(\lambda)}(\mathbf{0}, h'_x)\rho^{(\lambda)}(y', h'_y)\,dh'_x\,dh'_y\,dy'\,dx.\end{aligned}$$

Recalling from (1.12) that $\beta(d - 1) + \gamma(1 + \delta) = 1$, we have by definition of τ [see (1.5)] that $2 - 2\gamma - 2\gamma\delta - \beta(d - 1) = 1 - \gamma(1 + \delta) = \tau$. After normalization by λ^τ, the above integral equals

$$\int_A \int_{\lambda^\beta A} \int_0^\infty \int_0^\infty f(x, h'_x \lambda^{-\gamma}) f(\lambda^{-\beta} y' + x, h'_y \lambda^{-\gamma}) \times g_\lambda(x, h'_x, y', h'_y)\, dh'_x\, dh'_y\, dy'\, dx,$$

where we put

$$g_\lambda(x, h'_x, y', h'_y) := c_x^{(\lambda)}((\mathbf{0}, h'_x), (y', h'_y)) \rho^{(\lambda)}(\mathbf{0}, h'_x) \rho^{(\lambda)}(y', h'_y).$$

Clearly, $f(x, h'_x \lambda^{-\gamma}) f(\lambda^{-\beta} y' + x, h'_y \lambda^{-\gamma})$ converges to $f^2(x, 0)$ as $\lambda \to \infty$. Lemma 3.4 implies for all $(x, h'_x, y', h'_y) \in A_+ \times \lambda^\beta A \times \mathbb{R}_+$ that the product $g_\lambda(x, h'_x, y', h'_y)(h'_x)^{-\delta}(h'_y)^{-\delta}$ converges to

$$c_x^{(\infty)}((\mathbf{0}, h'_x), (y', h'_y)) \rho_0^2(x)$$

as $\lambda \to \infty$. Since, by Lemma 3.5 and (R2), $g_\lambda(x, h'_x, y', h'_y)(h'_x)^\delta (h'_y)^\delta$ is dominated in absolute value by the integrable function

$$(x, h'_x, y', h'_y) \mapsto C'(h'_x)^\delta (h'_y)^\delta \exp\left(-\frac{1}{C} \max\left(\frac{|y'|}{2}, h'_x, h'_y\right)\right)$$

on $A_+ \times \mathbb{R}^{d-1} \times \mathbb{R}_+$, the dominated convergence theorem combined with relation (3.16) produces the desired limit (1.10).

3.4. *Proof of Theorem* 1.3. Given Theorems 1.1 and 1.2, one may prove Theorem 1.3 either by the method of cumulants [7] or by the Stein method [26]. The first approach shows that the Fourier transform of $\lambda^{-\tau/2}\langle f, \bar{\mu}^\xi_{\lambda\rho}\rangle$, namely,

$$\mathbb{E} \exp[i \lambda^{-\tau/2} \langle f, \bar{\mu}^\xi_{\lambda\rho}\rangle],$$

converges as $\lambda \to \infty$ to the Fourier transform of a normal mean zero random variable with variance $\sigma_f^2 := I(f^2) + J(f^2)$. Even though we use a formally different version of stabilization, this is accomplished by following [7] nearly verbatim. Indeed, recall that Lemma 3.5 shows the exponential decay of the two point correlation function $c_x^{(\lambda)}((\mathbf{0}, h'), (y', h'_y))$. In a similar way we may establish the exponential decay of k-point correlation functions, and, more generally, that the k-point correlation functions cluster exponentially, as shown in Lemma 5.2 of [7]. In this way we show (as in Lemma 5.3 of [7]) that for all $k = 3, 4, \dots$ and $f \in \mathcal{C}_b(A_+)$ that

$$\lim_{\lambda \to \infty} \lambda^{-\tau k/2} \langle f^{\otimes k}, c_\lambda^k \rangle = 0, \tag{3.24}$$

where c_λ^k denotes the kth cumulant figuring in the logarithm of the Laplace transform (their existence follows by Lemma 3.2). This consequently shows that

$\lambda^{-\tau/2}\langle f, \bar{\mu}^{\xi}_{\lambda\rho}\rangle$ converges to a mean zero normal random variable with variance σ_f^2. The convergence of the finite-dimensional distributions follows from the Cramér–Wold device and is standard (see, e.g., page 251 of [7] or [23]).

Alternatively, we may also use the Stein method [23, 26]. This is a bit simpler and has the advantage of yielding rates of convergence when $\sigma_f^2 > 0$, as would be the case when $\delta = 0$ and $\alpha = 2$ (Lemma 7 of [31] combined with Section 4 below) or when $\alpha = 1$ (Theorem 2.2 of [8]). (When $\sigma_f^2 = 0$, then $\lambda^{\tau/2}\langle f, \bar{\mu}^{\xi}_{\lambda\rho}\rangle$ converges to a unit point mass.) Our proof is based closely on [26], which uses a formally different version of stabilization. For simplicity, we assume $A = [0, 1]^{d-1}$.

Recalling that $\bar{x} := (x, h_x)$, we have

$$\langle f, \mu^{\xi}_{\lambda\rho}\rangle = \sum_{\bar{x}\in\mathcal{P}_{\lambda\rho}} \xi(\bar{x}, \mathcal{P}_{\lambda\rho}) f(\bar{x}) = \sum_{\bar{x}\in\mathcal{P}_{\lambda\rho}} \xi^{(\lambda)}\big((\mathbf{0}, h'_x), \mathcal{P}^{(\lambda)}_{\lambda\rho}[x]\big) f((x, h'_x\lambda^{-\gamma})).$$

For all $L > 0$, let

$$T_\lambda := T_\lambda(L) := \sum_{\bar{x}\in\mathcal{P}_{\lambda\rho}\cap([0,1]^{d-1}\times[0,L\lambda^{-\gamma}\log\lambda])} \xi^{(\lambda)}\big((\mathbf{0}, h'_x), \mathcal{P}^{(\lambda)}_{\lambda\rho}[x]\big) f((x, h'_x\lambda^{-\gamma}))$$

$$= \sum_{\bar{x}\in\mathcal{P}_{\lambda\rho}:\, h'_x\le L\lambda^{-\gamma}\log\lambda} \xi^{(\lambda)}\big((\mathbf{0}, h'_x), \mathcal{P}^{(\lambda)}_{\lambda\rho}[x]\big) f((x, h'_x\lambda^{-\gamma})).$$

By Lemma 3.2, given arbitrarily large $\kappa > 0$, if L is large enough, then $\langle f, \mu^{\xi}_{\lambda\rho}\rangle$ and T_λ coincide except on a set with probability $O(\lambda^{-\kappa})$ in λ. Thus, T_λ has the same asymptotic distribution as $\langle f, \mu^{\xi}_{\lambda\rho}\rangle$ and it suffices to find a rate of convergence to the standard normal for $(T_\lambda - \mathbb{E}T_\lambda)/\sqrt{\operatorname{Var} T_\lambda}$.

Subdivide $[0, 1]^{d-1}$ into $V(\lambda) := \lambda^{\beta(d-1)}(\rho_\lambda)^{-(d-1)}$ sub-cubes C_i^λ of edge length $\lambda^{-\beta}\rho_\lambda$ and of volume $\lambda^{-\beta(d-1)}(\rho_\lambda)^{d-1}$, where $\rho_\lambda := M\log\lambda$ for some large M, exactly as in Section 4 of [26].

Enumerate $\mathcal{P}_{\lambda\rho}\cap(C_i^\lambda \times L\lambda^{-\gamma}\log\lambda)$ by $\{\bar{X}_{i,j}\}_{j=1}^{N_i}$ where $\bar{X}_{i,j} := (x_{ij}, h_{ij})$. Rewrite T_λ as

$$T_\lambda = \sum_{i=1}^{V(\lambda)}\sum_{j=1}^{N_i} \xi^{(\lambda)}\big((\mathbf{0}, h'_{ij}), \mathcal{P}^{(\lambda)}_{\lambda\rho}[x_{ij}]\big) f((x_{ij}, h'_{ij}\lambda^{-\gamma})).$$

This is the analog of T_λ in [26].

For any random variable X and any $p > 0$, let $\|X\|_p := (\mathbb{E}[|X|^p])^{1/p}$. For all $1 \le i \le V(\lambda)$, we have $\sum_{j=1}^{N_i} \xi^{(\lambda)}((\mathbf{0}, h'_{ij}), \mathcal{P}^{(\lambda)}_{\lambda\rho}[x_{ij}]) \le N_i$, where N_i is Poisson with mean

$$\lambda\int_{C_i^\lambda\times[0,L\lambda^{-\gamma}\log\lambda]} \rho(u)\,du = O([\log\lambda]^{1+\delta}).$$

It follows by the boundedness of f that

$$\left\|\sum_{j=1}^{N_i} \xi^{(\lambda)}\big((\mathbf{0}, h'_{ij}), \mathcal{P}^{(\lambda)}_{\lambda\rho}[x_{ij}]\big) f((x_{ij}, h'_{ij}\lambda^{-\gamma}))\right\|_3 \le C\|f\| L^{1+\delta}(\log\lambda)^{1+\delta}(\rho_\lambda)^{d-1},$$

where $\|f\|$ denotes the essential supremum of f. This is the analog of Lemma 4.3 in [26] (putting $q = 3$ there) with an extra logarithmic factor.

For all $1 \le i \le V(\lambda)$ and $j = 1, 2, \ldots$, let $R_{i,j}$ denote the radius of stabilization for $\xi^{(\lambda)}$ at $X_{i,j}$ for $\mathcal{P}_{\lambda\rho}^{(\lambda)}$ if $1 \le j \le N_i$ and let $R_{i,j}$ be zero otherwise.

As in [26], put $E_i := \bigcap_{j=1}^{\infty}\{R_{i,j} \le \rho_\lambda\}$ and let $E_\lambda := \bigcap_{i=1}^{V(\lambda)} E_i$. Then by Lemma 3.1(i), we have $P[E_\lambda^c] \le \lambda^{-\kappa}$ for κ arbitrarily large if M is large enough. This is the analog of (4.11) of [26].

Next, recalling $\rho_\lambda = M \log \lambda$, we define the analog of T'_λ in [26]:

$$T'_\lambda := \sum_{i=1}^{V(\lambda)} \sum_{j=1}^{N_i} \xi_{[\rho_\lambda]}^{(\lambda)}\big((\mathbf{0}, h'_{ij}), \mathcal{P}_{\lambda\rho}^{(\lambda)}[x_{ij}]\big) f((x_{ij}, h'_{ij}\lambda^{-\gamma})).$$

Then we define, for all $1 \le i \le V(\lambda)$,

$$S_i := S_{Q_i} := (\operatorname{Var} T'_\lambda)^{-1/2} \sum_{j=1}^{N_i} \xi_{[\rho_\lambda]}^{(\lambda)}\big((\mathbf{0}, h'_{ij}), \mathcal{P}_{\lambda\rho}^{(\lambda)}[x_{ij}]\big)\ f((x_{ij}, h'_{ij}\lambda^{-\gamma})).$$

We define $S_\lambda := \sum_{i=1}^{V(\lambda)}(S_i - \mathbb{E}S_i)$, noting that it is the analog of S in [26].

Notice that T'_λ is a close approximation to T_λ and that, by definition of E_i, $1 \le i \le V(\lambda)$, it has a high amount of independence between summands. In fact, by the independence property of Poisson point processes, it follows that S_i and S_k are independent whenever $d(C_i^\lambda, C_k^\lambda) > 2\lambda^{-\beta}\rho_\lambda$.

Next we define a graph $G_\lambda := (\mathcal{V}_\lambda, \mathcal{E}_\lambda)$ as follows. The set $\mathcal{V}_\lambda$ consists of the sub-cubes $C_1^\lambda, \ldots, C_{V(\lambda)}^\lambda$ and the edges $(C_i^\lambda, C_j^\lambda)$ belong to $\mathcal{E}_\lambda$ if $d(C_i^\lambda, C_j^\lambda) \le 2\lambda^{-\beta}\rho_\lambda$. Since S_i and S_k are independent whenever $d(C_i^\lambda, C_k^\lambda) > 2\lambda^{-\beta}\rho_\lambda$, it follows that G_λ is a dependency graph for $\{S_i\}_{i=1}^{V(\lambda)}$.

Now proceed exactly as in [26], noting that:

(i) $V(\lambda) = \lambda^{\beta(d-1)}(\rho_\lambda)^{-(d-1)}$,

(ii) the maximum degree of G_λ is bounded by 5^d,

(iii) for all $1 \le i \le V(\lambda)$, we have $\|S_i\|_3 \le K(\operatorname{Var}(T'_\lambda))^{-1/2}(\log\lambda)^{1+\delta} \times (\rho_\lambda)^{d-1} =: \theta[\lambda]$.

These bounds correspond to the analogous bounds (i), (ii) and (iii) on pages 54–55 of [26]. Moreover, provided $\sigma_f^2 > 0$, then the counterpart of (v) of [26] holds, namely,

$$\operatorname{Var}[T'_\lambda] = \Theta(\operatorname{Var}[T_\lambda]) = \Theta(\lambda^\tau).$$

Putting $q = 3$ in (4.1) and (4.18) of [26] gives a rate of convergence for both S_λ and $(T_\lambda - \mathbb{E}T_\lambda)/\sqrt{\operatorname{Var} T_\lambda}$ to the standard normal. This rate is

$$O(V(\lambda)\theta[\lambda]^3) = O\big(\lambda^{\beta(d-1)}(\rho_\lambda)^{-(d-1)}(\lambda^\tau)^{-3/2}(\log\lambda)^{3(1+\delta)}\rho_\lambda^{3(d-1)}\big).$$

Recalling that $\tau = \beta(d-1)$, we rewrite this as

$$O\big(\lambda^{-\tau/2} \log \lambda^{3(1+\delta)+2(d-1)}\big). \tag{3.25}$$

This completes the proof of Theorem 1.3. □

4. Proofs of applications. The purpose of the present section is to derive Theorems 2.1 and 2.2 from our general theorems of Section 1.

PROOF OF THEOREM 2.1. To derive Theorem 2.1 from our general theory, we translate the convex hull problem into the language of ψ-growth processes with overlap. To this end, recall first that for a compact convex body $C \subseteq \mathbb{R}^d$ we define its support function $h_C : S_{d-1} \to \mathbb{R}$ by

$$h_C(u) := \sup_{\bar{x} \in C} \langle \bar{x}, u \rangle, \qquad u \in S_{d-1},$$

with now $\langle \cdot, \cdot \rangle$ standing for the usual scalar product in $\mathbb{R}^d$; see Section 1.7 in [35]. An easily verified and yet crucial feature of the support functional $h_{\cdot}(\cdot)$ is that

$$h_{\mathrm{conv}\{\bar{x}_1,\dots,\bar{x}_k\}}(u) = \max_{1 \le i \le k} h_{\{\bar{x}_i\}}(u), \qquad u \in S_{d-1}, \tag{4.1}$$

for each collection $\{\bar{x}_1, \dots, \bar{x}_k\}$ of points in $\mathbb{R}^d$. Moreover, by definition, it is clear that, for all $u \in S_{d-1}$, we have $h_{\{\bar{x}\}}(u) = \langle \bar{x}, u \rangle$, $u \in S_{d-1}$.

This leads to the following way of describing $\mathcal{V}(\mathcal{P}_{\lambda\rho})$ considered in Theorem 2.1. For a particular realization $\{\bar{x}_1, \dots, \bar{x}_k\}$ of $\mathcal{P}_{\lambda\rho}$ in B_d, we consider the collection $H[\bar{x}_1], \dots, H[\bar{x}_k]$ of *support epigraphs* given by

$$H[\bar{x}] := \big\{(y, h_y) \in S_{d-1} \times \mathbb{R}_+ : h_y \ge 1 - h_{\{\bar{x}\}}(y)\big\}, \tag{4.2}$$

where h_y stands for the distance between $\bar{y}$ and the boundary $S_{d-1} = \partial B_d$. A compact convex body is uniquely determined by its support functional (cf. Section 1.7 in [35]), and in view of (4.1), the set $\mathrm{conv}(\{\bar{x}_1, \dots, \bar{x}_k\})$ is in one-to-one correspondence with the union $\bigcup_{i=1}^k H[\bar{x}_i]$. Further, the number of vertices in the convex hull is easily seen to coincide with the number of those $\bar{x}_i$, $i = 1, \dots, k$, for which $H[\bar{x}_i]$ is not completely contained in the union $\bigcup_{j \ne i} H[\bar{x}_j]$.

Next we shall also write $r_y := 1 - h_y$ for the distance between $\bar{y}$ and the origin of $\mathbb{R}^d$. Note now that the intensity measure $\rho(\bar{x})\,d\bar{x}$, $\bar{x} \in B_d$, coincides with $\rho((x,r)) r^{d-1}\,dr\,dx = \rho((x,r))(1-h)^{(d-1)}\,dh\,dx$, where $\bar{x} := (x, r)$, with $r \in [0,1]$ denoting the distance between $\bar{x}$ and the origin of $\mathbb{R}^d$, with $h := 1 - r$ and with $x \in S_{d-1}$ being the radial projection of $\bar{x}$ onto $\partial B_d = S_{d-1}$. Observe also that the support epigraph $H[\bar{x}]$ as given in (4.2) can be represented by

$$H[(x,r)] = \{(y, h_y) \in S_{d-1} \times \mathbb{R}_+ : h_y \ge 1 - r\cos(\mathrm{dist}_{S_{d-1}}(x,y))\}$$

with $\mathrm{dist}_{S_{d-1}}(x,y) := \cos^{-1}\langle x, y \rangle$ denoting the geodesic distance in S_{d-1} between x and y. Now put

$$\psi(l) := 1 - \cos(l).$$

Writing the inequality $h_y \geq 1 - r\cos(\mathrm{dist}_{S_{d-1}}(x, y))$ as $h_y \geq 1 - r + r\psi(\mathrm{dist}_{S_{d-1}}(x, y))$, we have

$$H[(x,r)] = \{(y, h_y) \in S_{d-1} \times \mathbb{R}_+ : h_y \geq h + r\psi(\mathrm{dist}_{S_{d-1}}(x, y))\}, \tag{4.3}$$

in other words, the support epigraphs are remarkably similar to the upward cones (1.1) described at the outset.

The above observations naturally suggest *identifying the cardinality of the studied set* $\mathcal{V}(\mathcal{P}_{\lambda\rho})$ *with the number of extreme points in the* $r\psi$*-growth process with overlap in the sense of Section* 1 *with the underlying point density* $\rho((x,r))r^{d-1} = \rho((x,r))(1-h)^{d-1}$. Likewise, the vertex empirical measure $\mu_{\lambda\rho}$ in (2.1) corresponds to the empirical measure $\mu^{\xi}_{\lambda\rho}$, $\xi := \xi(\psi; \cdot)$; see (1.2).

This identification is valid modulo the following issues though:

(1) the "spatial" coordinate x of a point $\bar{x} := (x,r) \in B_d$ falls into S_{d-1} rather than into a subset A of $\mathbb{R}^{d-1}$, as required in Section 1,

(2) ψ as given above is monotone only in a neighborhood of 0, and moreover, we do not have $\lim_{l\to\infty} \psi(l) = \infty$, which violates ($\Psi$1),

(3) the support epigraph $H[(x,r)]$ coincides with the (x,h)-shifted ψ-epigraph $K[x,h]$ given by (1.1) only when $r = 1$ and, hence, only when $h = 0$; in general, for $0 \leq r \leq 1$, the set $H[(x,r)]$ is an (x,h)-shifted $r\psi$-epigraph.

We claim, however, that the above three restrictions can be neglected in the asymptotic regime $\lambda \to \infty$, thus rendering the theory of Section 1 applicable. Indeed, first note that the sphere S_{d-1}, unlike the boundary of a general smooth convex body, has a spatially homogeneous structure and so the behavior of ψ is independent of x, exactly as in Section 1. Moreover, the sphere S_{d-1}, being a smooth manifold, has a local geometry coinciding with that of $\mathbb{R}^{d-1}$, which takes care of issue (1). Concerning issues (2) and (3), for each $r \in (0,1)$, the convex hull $\mathrm{conv}(\mathcal{P}_{\lambda\rho})$ coincides with $\mathrm{conv}(\mathcal{P}_{\lambda\rho} \cap (B_d \setminus B_d(0,r)))$ with overwhelming probability, that is, the probability of the complement event goes to zero exponentially fast in λ; see the discussion in [19] and the references therein. This allows us to focus on the geometry of $\mathrm{conv}(\mathcal{P}_{\lambda\rho})$ in a thin shell $B_d \setminus B_d(0,r)$ within a distance $1-r$ from the boundary S_{d-1}.

Consequently, only the behavior of ψ in a neighborhood of 0 matters. Recalling that the standard re-scaling of Section 1.3 involves scaling in the spatial directions by λ^{β}, it follows that for a given $\bar{x} := (x,r)$ and support epigraph $H[\bar{x}]$, the contribution of points distant from x by more than $O(\lambda^{-\beta})$ is negligible in view of the argument in Lemma 3.1(i) and no distortions from the local Euclidean geometry have to be taken into account in the limit under this re-scaling. Likewise, we only have to control the geometry of $H[\bar{x}]$, $\bar{x} := (x,r)$, for r arbitrarily close to 1. This allows us to rewrite the proofs of Theorems 1.1–1.3 for the thus modified r-dependent ψ. Indeed, the stabilization Lemma 3.1, as well as Lemma 3.2, do not require any modifications in their proofs and neither does Lemma 3.4 nor Lemma

3.5. Consequently, the arguments leading to the central limit theorem in Section 3.4 do not require modification either. In this context we note that the proof of Lemma 3.1 would break down if the sphere S_{d-1} were replaced by a nonconvex set allowing for long-range dependencies between extreme points.

It only remains to show the limit arguments in Sections 3.3 and 3.2 remain valid for the modified ψ. To see that this is indeed the case, we note that the arguments rely on two main ingredients: on stabilization which holds with no changes as stated above, and on re-scaling relations discussed in Section 1.3. However, it is easily seen that the re-scaling relations and their proofs can be readily rewritten for the modified ψ, the only essential modification being to add one extra argument ($h := 1 - r$) to the ψ-function, which anyway vanishes in the scaling limit of Section 1.3 with $h = 1 - r$ tending to 0 as discussed above (whereas the contribution coming from smaller h is negligible in view of Lemma 3.2). This discussion takes care of issues (2) and (3) above.

Thus, we can now conclude that the considered convex hull process falls into the range of applicability of the general theory of Section 1, with $\alpha = 2$ in (Ψ2) and δ in (R2) coinciding with that in the statement of Theorem 2.1. Thus, we obtain the required Theorem 2.1 as a consequence of the general Theorems 1.1–1.3. The rate of convergence follows from (3.25) by putting $\delta = 0$ and $\alpha = 2$. □

PROOFS OF THEOREMS 2.2 AND 2.3. Theorem 2.2 follows directly by the general theory in Section 1 (Theorems 1.1–1.3 with $\alpha \in (0, 1]$). The rate (2.8) follows from (3.25) by putting $\delta = 0$ and $\alpha = 1$. We thus focus attention on establishing Theorem 2.3. The first lemma yields (2.9).

LEMMA 4.1. *For all* $f \in \mathcal{C}_b(A_+)$, *we have*

$$|\mathbb{E}[\langle f, \nu_n^{\xi}\rangle] - \mathbb{E}[\langle f, \mu_{n\rho}^{\xi}\rangle]| = O(n^{-\tau'}). \tag{4.4}$$

PROOF. For all $\bar{w} \in A_+$, let $p(\bar{w}) := \int_{K^{\downarrow}[\bar{w}]} \rho(u)\,du$, where $K^{\downarrow}[\bar{w}]$ is as in (2.5) with $\psi(l) = l^{\alpha}$. Note that in our current setting for all $w \in A_+$ we have $p(w) \in [0, 1]$ since ρ is a probability density. Also, note that $\psi^{(n)} \equiv \psi$ and $K^{(n)} \equiv K$ with $K^{(n)} := \{(y^{(n)}, h_y^{(n)}) : (y, h_y) \in K\}$, that is, the self-similarity under the re-scaling is immediate rather than emerging as $n \to \infty$. For all $s \in [0, 1]$ and $f \in \mathcal{C}_b(A_+)$, let $B_f(s) := \int_{p(\bar{w}) \le s} f(\bar{w})\rho(\bar{w})\,d\bar{w}$. Recalling that for $\alpha \in (0, 1]$ the ψ-extremality of a point w in a given sample is equivalent to having no other sample points in $K^{\downarrow}[w]$ (see the discussion at the beginning of Section 2.2), we have

$$\begin{aligned}\mathbb{E}[\langle f, \nu_n^{\xi}\rangle] &= n\int_{A_+} (1 - p(w))^{n-1} f(w)\rho(w)\,dw \\ &= n\int_0^1 \int_{p(w)=s} (1-s)^{n-1} f(w)\rho(w)\,dw\,ds = n\int_0^1 (1-s)^{n-1}\,dB_f(s)\end{aligned}$$

by Fubini's theorem. Similarly,

$$\mathbb{E}[\langle f, \mu_{n\rho}^{\xi}\rangle] = n \int_0^1 e^{-ns}\, dB_f(s) \sim C_f n^{\tau}, \tag{4.5}$$

where the asymptotics are given by Theorem 1.1. B_f is monotone, nondecreasing and Karamata's Tauberian theorem (e.g., Theorem 2.3 in [36]) gives $B_f(s) \sim C_f s^{\tau'}$ as $s \to 0^+$. Notice

$$\begin{aligned}
|\mathbb{E}[\langle f, \mu_{n\rho}^{\xi}\rangle] - \mathbb{E}[\langle f, \nu_n^{\xi}\rangle]| &= n \int_0^1 \bigl(e^{-ns} - (1-s)^{n-1}\bigr)\, dB_f(s) \\
&\le n \int_0^1 \bigl(e^{-ns} - e^{n \ln(1-s)}\bigr)\, dB_f(s) \\
&\le Cn^2 \int_0^1 e^{-ns} s^2\, dB_f(s) \\
&= Cn^2 \int_0^{1/n} e^{-ns} s^2\, dB_f(s) + Cn^2 \int_{1/n}^1 e^{-ns} s^2\, dB_f(s).
\end{aligned}$$

The first integral behaves like $Cn^{-\tau'}$ since $B_f(s) \sim C_f s^{\tau'}$, whereas the second behaves like $\frac{C}{n} \int_1^n u^2 e^{-u}\, dB_f(u/n) \le C/n$, since B_f is bounded by $B_f(1)$. This gives (4.4). □

We now establish the remainder of Theorem 2.3. Recall $\bar{u}' := (u', h_u')$. For all $\lambda > 0$, define

$$A'(\lambda) := \left\{ \bar{y}' \in \lambda^{\beta} A \times \mathbb{R}_+ : \int_{K^{\downarrow}[\bar{y}']} \rho^{(\lambda)}(u)\, du \le C \log \lambda \right\}.$$

Let $A(\lambda) := \{\bar{y} \in A_+ : \bar{y}' \in A'(\lambda)\}$ and put $a_\lambda := \int_{A(\lambda)} \rho(w)\, dw$. Note that by Lemma 3.2 the probability that a sample point from $\bar{\mathcal{X}}_n := \{X_i\}_{i=1}^n$ in $A_+ \setminus A(\lambda)$ is ψ-extremal is at most

$$C \exp\left(-\frac{n \log \lambda}{C\lambda}\right) \tag{4.6}$$

and the same holds for $\bar{\mathcal{X}}_n$ replaced by the Poisson sample with intensity $n\rho$. Indeed, although Lemma 3.2 was originally established for Poisson samples, it is easily seen that the same proof works also for binomial samples, as it essentially relies on exponentially decaying upper bounds for probabilities of certain sets in A_+ being devoid of points of the underlying point process. Thus, the ψ-extremal points are predominantly concentrated in $A(\lambda)$, a fact which we will use to show (2.10). First we find growth bounds for a_λ.

LEMMA 4.2. *We have* $a_\lambda \le C(\log \lambda)^{\alpha(1+\delta)/(\alpha+d-1)} \lambda^{-\alpha(1+\delta)/(d-1+\alpha(1+\delta))}$.

PROOF. If $M(\lambda) := \sup\{h_y : h_y \in A(\lambda)\}$, then note that $\alpha(\lambda)$ grows like $\int_0^{M(\lambda)} h_y^\delta \, dh_y = C(M(\lambda))^{1+\delta}$. We now find $M(\lambda)$.

If $\bar{y}' := (y', h'_y) \in A'(\lambda)$, then, by (3.7), we have $h'^{(\alpha+d-1)/\alpha}_y \le C \log \lambda$. Since $h'_y = \lambda^\gamma h_y$ and since $\gamma = \beta\alpha$, it follows that

$$h_y^{(\alpha+d-1)/\alpha} \lambda^{\beta(\alpha+d-1)} \le C \log \lambda.$$

Since $\gamma(d-1)/\alpha = \tau$, we have

$$h_y^{(\alpha+d-1)/\alpha} \lambda^{\gamma+\tau} \le C \log \lambda \quad \text{or} \quad h_y \le (\log \lambda)^{\alpha/(\alpha+d-1)} \lambda^{-\alpha(\gamma+\tau)/(\alpha+d-1)},$$

that is,

$$M(\lambda) \le (\log \lambda)^{\alpha/(\alpha+d-1)} \lambda^{-\alpha(\gamma+\tau)/(\alpha+d-1)}.$$

Recall that $\gamma + \tau = (\alpha + d - 1)/(d - 1 + \alpha(1 + \delta))$ to get the result. □

The next lemma yields (2.10). The proof borrows heavily from [8] and, for the sake of completeness, we provide the details.

LEMMA 4.3. *For all* $f \in \mathcal{C}_b(A_+)$, *we have* $\lim_{n\to\infty} n^{-\tau} \operatorname{Var}[\langle f, \nu_n^\xi \rangle] = \lim_{n\to\infty} n^{-\tau} \operatorname{Var}[\langle f, \mu_{n\rho}^\xi \rangle]$.

PROOF. Recall $\bar{\mathcal{X}}_n := \{X_i\}_{i=1}^n$. Let $N_n := \operatorname{card}\{\bar{\mathcal{X}}_n \cap A(n)\}$ and $N'_n := \operatorname{card}\{\mathcal{P}_{n\rho} \cap A(n)\}$. For all $r = 1, 2, \ldots,$ denote by $e(r) := e_f(r)$ the expected value of the functional $\langle f \cdot \mathbf{1}(A(n)), \mu_n^\xi \rangle$ *conditioned on* $\{N(n) = r\}$, and by $v(r)$ the variance of this functional conditioned on $\{N(n) = r\}$. Let $\nu_n^A := \nu_n^{\xi, A}$ denote the point measure induced by the ψ-extremal points in $\{\bar{\mathcal{X}}_n \cap A(n)\}$. Similarly, let $\mu_{n\rho}^A := \mu_{n\rho}^{\xi, A}$ denote the point measure induced by the ψ-extremal points in $\{\mathcal{P}_{n\rho} \cap A(n)\}$.

By the bound (4.6) on the probability of a given point outside $A(n)$ being extremal, ν_n^A coincides with ν_n and $\mu_{n\rho}^{\xi,A}$ coincides with $\mu_{n\rho}^\xi$ except on a set with probability at most $nC \exp(-C \log n) = Cn^{-C+1}$. Since C can be chosen arbitrarily large, it suffices to show that

$$\lim_{n\to\infty} n^{-\tau} \operatorname{Var}[\langle f, \nu_n^A \rangle] = \lim_{n\to\infty} n^{-\tau} \operatorname{Var}[\langle f, \mu_{n\rho}^A \rangle]. \tag{4.7}$$

The conditional variance formula implies that

$$\operatorname{Var}[\langle f, \nu_n^A \rangle] = \operatorname{Var}[e(N_n)] + \mathbb{E}[v(N_n)] \quad \text{and}$$

$$\operatorname{Var}[\langle f, \mu_{n\rho}^A \rangle] = \operatorname{Var}[e(N'_n)] + \mathbb{E}[v(N'_n)].$$

We prove (4.7) by showing that:

(i) the terms $\mathbb{E}[v(N_n)]$ and $\mathbb{E}[v(N'_n)]$ are dominant and that their ratio tends to one as $n \to \infty$, and

(ii) $\mathrm{Var}[e(N_n)]$ and $\mathrm{Var}[e(N_n')]$ are both $o(n^\tau)$.

We will first show (ii) as follows. For all $s > 0$, recall that $B_f(s) := \int_{p(\bar{w})\le s} f(\bar{w})\rho(\bar{w})\,d\bar{w}$. By Fubini's theorem, for all $r = 1, 2, \dots$ and with $a_n = \int_{A(n)} \rho(w)\,dw$, we obtain

$$e(r) = \frac{r}{a_n}\int_{A(n)}\left(1 - \frac{p(w)}{a_n}\right)^{r-1} f(w)\rho(w)\,dw = \frac{r}{a_n}\int_0^{a_n}\left(1 - \frac{s}{a_n}\right)^{r-1} dB_f(s).$$

Letting Δ_r denote the difference $e(r+1) - e(r)$, we obtain

$$\Delta_r = \frac{1}{a_n}\int_0^{a_n}\left(1 - \frac{s}{a_n}\right)^r - \frac{rs}{a_n}\left(1 - \frac{s}{a_n}\right)^{r-1} dB_f(s).$$

Setting $u = rs/a_n$ and applying $B_f(s) \sim C_f s^{\tau'}$, we see that ($\tau = 1 - \tau'$)

$$|\Delta_r| \le \frac{C_f}{r}\int_0^r \left|\left(1 - \frac{u}{r}\right)^r - u\left(1 - \frac{u}{r}\right)^{r-1}\right| \left(\frac{ua_n}{r}\right)^{-\tau} du.$$

Since $\sup_{r>0}\int_0^r |(1-\frac{u}{r})^r - u(1-\frac{u}{r})^{r-1}| u^{-\tau}\,du \le C$, it follows that $|\Delta_r| \le C(\frac{a_n}{r})^{-\tau}$.

When $r \in I_n := (na_n - C(\log n)(na_n)^{1/2}, na_n + C(\log n)(na_n)^{1/2})$, then, by Lemma 4.2, for n large,

$$\begin{aligned}|\Delta_r| &\le C(na_n)^{-1}n^\tau = Ca_n^{-1}n^{-\tau'}\\ &= Cn^{\alpha(1+\delta)/(d-1+\alpha(1+\delta))}n^{-\tau'}(\log n)^{-\alpha(1+\delta)/(\alpha+d-1)}.\end{aligned}$$

Recalling that $\tau' = (1+\delta)\alpha/(d-1+\alpha(1+\delta))$, we see that for $r \in I_n$ we have

$$|\Delta_r| \le \Delta(n) := C(\log n)^{-\alpha(1+\delta)/(\alpha+d-1)}.$$

Write $e(N_n) = e(1) + \sum_{j=2}^{N_n}(e(j) - e(j-1))$ and observe that $e(N_n)$ differs from the constant $e(1) + \sum_{j=2}^{E[N_n]}(e(j) - e(j-1))$ by at most

$$\sum_{j\in J_n}\big(e(j) - e(j-1)\big),$$

where $J_n := (\min(\mathbb{E}[N_n], N_n),\ \max(\mathbb{E}[N_n], N_n))$. Thus,

$$\begin{aligned}\mathrm{Var}[e(N_n)] &\le \mathbb{E}\left[\sum_{j\in J_n}\big(e(j) - e(j-1)\big)\right]^2\\ &\le \mathbb{E}\left[\sum_{j\in J_n}\big(e(j) - e(j-1)\big)\mathbf{1}_{N_n\in I_n}\right]^2 + o(1),\end{aligned}$$

by Cauchy–Schwarz and since (by increasing C in the definition of I_n) standard concentration inequalities (see, e.g., Proposition A.2.3(ii), (iii) and Proposition

A.2.5(ii), (iii) in [3]) show that $P[N_n \in I_n^c]$ can be made smaller than any negative power of n.

For $j \in J_n$ and $N_n \in I_n$, we have $j \in I_n$ and so $(e(j) - e(j-1)) \le \Delta(n)$. Since the length of J_n is bounded by $|N_n - \mathbb{E}N_n|$, it follows that $\mathrm{Var}[e(N_n)] \le \mathrm{Var}[N_n](\Delta(n))^2 + o(1)$. Note that $\mathrm{Var}[N_n] \le Cn^\tau (\log n)^{\alpha(1+\delta)/(\alpha+d-1)}$. It follows that $\mathrm{Var}[e(N_n)] \le Cn^\tau (\log n)^{-\alpha(1+\delta)/(\alpha+d-1)} + o(1)$, that is, $\mathrm{Var}[e(N_n)] = o(n^\tau)$. Similarly, $\mathrm{Var}[e(N_n')] = o(n^\tau)$ and so condition (ii) holds.

We now show condition (i) by showing that the ratio $\mathbb{E}[v(N_n)]/\mathbb{E}[v(N_n')]$ is asymptotically one, as $n \to \infty$. Let $p_{n,r} := P[N_n = r]$ and $p'_{n,r} := P[N'_n = r]$. Stirling's formula implies that, for $|r - a_n n| \le n^\beta$, where $0 < \beta < 1/2$,

$$\lim_{n\to\infty} \frac{p_{n,r}}{p'_{n,r}} = 1 \tag{4.8}$$

uniformly. Now, for $|r - a_n n| > n^\beta$, where $\beta \in (0, 1/2)$ is chosen so that $n^{2\beta}/na_n$ grows faster than some (small) power of n, we have that both $p_{n,r}$ and $p'_{n,r}$ are bounded by $C\exp(-n^\delta/C)$ for some $C,\ \delta > 0$ (see, e.g., Proposition A.2.3(i) and Proposition A.2.5(i) in [3]). Write

$$\mathbb{E}[v(N_n)] = \sum_{|r-a_n n|\le n^\beta} v(r) p_{n,r} + \sum_{|r-a_n n|> n^\beta} v(r) p_{n,r}.$$

The second sum is negligible since $0 < v(r) < r^2$ and $p_{n,r}$ is exponentially small. Consider the terms in the first sum. By (4.8), we have $p_{n,r} = p'_{n,r}(1 + o(1))$ uniformly for all $|r - a_n| \le n^\beta$ and since the terms in the first sum are positive, it follows that

$$\lim_{n\to\infty} \frac{\mathbb{E}[v(N_n)]}{\mathbb{E}[v(N_n')]} = 1. \tag{4.9}$$

Now from before we know $\mathrm{Var}[\langle f, \mu^A_{n\rho}\rangle]$ has asymptotic growth $Cn^\tau, C > 0$. It follows that $\mathbb{E}[v(N_n')]$ has the same growth, since $\mathrm{Var}[e(N_n')] = o(n^\tau)$. Thus, by (4.9) and the growth bounds $\mathrm{Var}[e(N_n)] = o(n^\tau)$ and $\mathrm{Var}[e(N_n')] = o(n^\tau)$, the desired identity (4.7) follows, completing the proof of Lemma 4.3. □

We conclude the proof of Theorem 2.3 by showing for all $f \in \mathcal{C}_b(A_+)$

$$\lim_{n\to\infty} d_{\mathrm{TV}}(n^{-\tau/2}\langle f, \bar{\nu}^\xi_n\rangle,\ n^{-\tau/2}\langle f, \bar{\mu}^\xi_{n\rho}\rangle) = 0,$$

where the total variation distance between two measures m_1 and m_2 is $d_{\mathrm{TV}}(m_1, m_2) := \sup_B |m_1(B) - m_2(B)|$, where the sup runs over all Borel subsets in $\mathbb{R}^d$. Since $n^{-\tau/2}|\mathbb{E}[\langle f, \nu^\xi_n\rangle] - \mathbb{E}[\langle f, \mu^\xi_{n\rho}\rangle]| \to 0$ by (4.4) and since $n^{-\tau/2}\langle f, \bar{\mu}^\xi_{n\rho}\rangle$ converges in law to an appropriate Gaussian distribution, recalling that $a_n = o(1)$ (see Lemma 4.2), Theorem 2.3 follows at once from the following:

LEMMA 4.4. *For all* $f \in \mathcal{C}_b(A_+)$, *we have*

$$d_{\mathrm{TV}}(\langle f, \nu_n^{\xi}\rangle, \langle f, \mu_{n\rho}^{\xi}\rangle) = O(a_n). \tag{4.10}$$

PROOF. We follow the proof of Lemma 7.1 in [8]. Recall that ν_n^A is the measure induced by the maximal points in $\{(X_i, h_i)\}_{i=1}^n \cap A(n)$ and, similarly, let $\mu_{n\rho}^{\xi,A}$ be the measure induced by the maximal points in $\mathcal{P}_{n\rho} \cap A(n)$. If C is large enough in the definition of $A(n)$, then the probability that points in $A_+ \setminus A(n)$ contribute to ν_n^{ξ} or $\mu_{n\rho}^{\xi}$ is $O(n^{-2})$. It follows that, for all $f \in \mathcal{C}_b(A_+)$

$$d_{\mathrm{TV}}(\langle f, \mu_{n\rho}^{\xi}\rangle, \langle f, \mu_{n\rho}^{\xi,A}\rangle) = O(n^{-2}) = o(a_n)$$

and

$$d_{\mathrm{TV}}(\langle f, \nu_n^{\xi}\rangle, \langle f, \mu_n^{\xi,A}\rangle) = O(n^{-2}) = o(a_n).$$

Thus, we only need to show $d_{\mathrm{TV}}(\langle f, \nu_n^A\rangle, \langle f, \nu_{n\rho}^A\rangle) = O(a_n)$.

Recall that N_n is the number of points from $\bar{\mathcal{X}}_n$ belonging to $A(n)$. Conditional on $N = r$, $\langle f, \nu_n^A\rangle$ is distributed as $\langle f, \tilde{\nu}_r^A\rangle$, where $\tilde{\nu}_r^A$ is the point measure induced by considering the maximal points among r points placed randomly according to the restriction of ρ to $A(n)$. The same is true for $\langle f, \mu_{n\rho}^{\xi,A}\rangle$ conditional on the cardinality of $\{\mathcal{P}_{n\rho} \cap A(n)\}$ taking the value r.

Hence, with $Bi(n, p)$ standing for a binomial random variable with parameters n and p and $Po(\alpha)$ standing for a Poisson random variable with parameter α, we have for all $f \in \mathcal{C}_b(A_+)$

$$d_{\mathrm{TV}}(\langle f, \nu_n^A\rangle, \langle f, \mu_{n\rho}^A\rangle) \le C d_{\mathrm{TV}}(Bi(n, a_n),\ Po(na_n)) \le C\frac{1}{na_n}\sum_{i=1}^{n}(a_n)^2 \le Ca_n,$$

where the penultimate inequality follows by standard Poisson approximation bounds (see, e.g., (1.23) of Barbour, Holst and Janson [3]). This is the desired estimate (4.10). □

Acknowledgments. The authors gratefully acknowledge helpful discussions with Yuliy Baryshnikov, who, in particular, encouraged developing theory under the general condition (R2) and who also contributed to the proofs of Lemmas 4.1 and 4.3. The authors also thank an anonymous referee for comments leading to an improved exposition.

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
NICHOLAS COPERNICUS UNIVERSITY
TORUŃ
POLAND
E-MAIL: tomeks@mat.uni.torun.pl

DEPARTMENT OF MATHEMATICS
LEHIGH UNIVERSITY
BETHLEHEM, PENNSYLVANIA 18015
USA
E-MAIL: joseph.yukich@lehigh.edu