Bernoulli Numbers and Ideal Classes

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An algebraic number field K is an extension of $\mathbb{Q}$ by a root of a polynomial. An important characteristic of a number field K is its class number, $h(K)$, which essentially tells us how far away from unique factorization the integers of K are. If $h(K) = 1$, then the integers of K, O_K, have unique factorization, just like the integers $\mathbb{Z}$ of $\mathbb{Q}$. One of the most important types of number fields in Algebraic Number Theory are cyclotomic extensions, i.e. $\mathbb{Q}(\zeta_p)$ where ζ_p is a pth root of unity for a prime p. It was shown about 30 years ago that $h(K) = 1$, for $K = \mathbb{Q}(\zeta_p)$ iff $p \leq 19$. This talk focuses on a 2008 paper by Kenneth Ribet, which discusses another fact about $h(K)$. The main theorem is that for an even integer k with $2 \leq k \leq p-1$, p divides the class number of K iff p divides the numerator of B_k, where we define the kth Bernoulli number, B_k, to be the the coefficient of $\frac{x_k}{k!}$ in the power series expansion of $\frac{x}{e^x-1}$. The talk gives x some historical background from Kummer that leads into this theorem of Herbrand and Ribet.