



Valuing investment projects with expansion options

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Abstract

Purpose – This article proposes an evaluation of capital investments that accounts for not only the initial assets, but also any potential growth options.

Design/methodology/approach – Using a piecewise linear approximation, a robust valuation technique is demonstrated for analyzing capital investment opportunities containing expansion options in a finite time horizon.

Findings – This process not only recognizes the option-like characteristics of the initial investment opportunity, but also recognizes the option-creating characteristics of the investment. This analysis shows that the value of capacity expansion options created by the initial investment has different dynamic characteristics from the assets in place. Although the growth options do not appear in the early investment premium, its impact on the investment decision is embedded in the investment threshold. When the time to expiration is short and the cost to delaying the assets in place is low, this analysis suggests that the initial investment decision might be made by ignoring the growth options.

Originality/value – This real option methodology provides a continuous solution to the optimal investment threshold and is a viable alternative to the traditional finite difference approach.

Keywords Capital, Investments, Capital growth

Paper type Technical paper

Capital investing involves more than just the purchase of physical assets. Typically, this type of investment not only results in the establishment of an initial set of assets, but it also creates an environment for growth. Therefore, the valuation of such investments must include the value of the assets in place (e.g. the initial installation of the production facility), as well as the value of any embedded growth options created (e.g. the opportunity for later capacity expansion). According to Myers (1977), the second component reflects the present value of the options created by the initial investment to make future investments, and hence has to be found jointly with the future optimal growth strategies after the initial investment is made. Regrettably, in many situations it is impossible to factor in all of the potential options created by an investment. However, one type of option created within many capital investments that can readily be included is the capacity expansion option. For example, an initial investment in a new market results in the installed or fixed capacity at the time the initial investment is made, as well as the option to add additional capacity later. Thus, both the value of the assets in place and the value of any capacity expansion options should influence the initial investment decision.

Our analysis relies on a piecewise linear approximation, an improvement over traditional finite differences. More specifically, we examine a finite-lived investment decision when product demand varies stochastically over time and show how a piecewise linear function can be used to approximate the optimal investment threshold. Our results support Myers (1977), Kester (1984), and Trigeorgis and Mason (1987), who conclude that the value of growth options could be a significant part of the value of an



investment. Furthermore, our study shows that even in the simplest case of a single firm-level investment with irreversibility and uncertainty, an investment project's value may involve multiple stochastic processes, which differ from many of the early seminal papers (such as McDonald and Siegel, 1986; Majd and Pindyck, 1987; Dixit and Pindyck, 1994), which assume that the value of an investment project follows a single stochastic process.

In as much as growth options are generally embedded in all investments and have very different dynamic characteristics from assets in place, the value of the investment generally cannot be expressed as a single stochastic process. Thus, our work continues the progression from a single stochastic framework to the consideration of an investment as a compound option. For instance, Kasanen (1993) examines the interaction between current investments and future opportunities; Trigeorgis (1993) considers the decision of sequential or staged investments; Smith and Triantis (1995) develop a real option model to examine the optimal investment policy for multiple projects that can be developed in parallel or in sequence; Kulatilaka and Perotti (1998) study the impact of strategic growth options on an initial investment decision in a single-period model; and Aguerrevere's (2003) model takes in the effects of competitive interactions on investment decisions on the dynamics of the price of a non-storable commodity.

Unfortunately, considering investments as compound options dramatically increases the complexity of investment valuation. And in many cases, compound options are specific to a particular investment problem and thus have to be solved on a case-by-case basis. In spite of this limitation, capacity expansion is the simplest class of growth options and is embedded in many initial investments. Thus, understanding the impact of the growth option on investment decision-making can help us to avoid under-investment, and is particularly important in making decisions when entering a new market.

Our decomposition of the value of the initial investment into two components follows Pindyck's (1988) and Dixit and Pindyck's (1994) investigations of firm value incremental capacity expansion[1]. However, our paper focuses on the decision to "enter" a market, not future capacity investments. The optimal capacity expansion strategy is not our direct concern, but a by-product in deriving the value of the initial investment. Thus, our contribution is to show how an initial investment decision can be made by aligning a current investment with the future growth opportunities it creates. While it is understood that an initial investment may create many different types of growth opportunities, to make an investment problem solvable we may have to ignore some of the potentialities and only take into account the dominant growth option.

It should also be stressed that in developing our piecewise linear approximation method, this paper fits within a broader literature that has explored an approximation to the early exercise boundary in an American option valuation. For example, Huang *et al.* (1996) used a step function to approximate the early exercise boundary in finding the value of an American put option; Ju (1998) approximated the early exercise boundary by a piecewise exponential function; and AitSahlia and Lai (1999/2000) proposed a piecewise linear function to approximate the optimal exercise boundary. Although Ju (1998) provides an efficient way for computing the value of an American option, the focus of his study and others using numerical methods was not the computation for early exercise boundaries. Because an accurate approximation to the

early exercise boundary may not be necessary in pricing American options, especially near the expiration date[2], both the step function and the exponential function approximations can provide efficient ways for computing the value of American options. However, they cannot be used for finding the optimal investment threshold or a continuous solution to it as well. Our analysis builds on the work of AitSahlia and Lai (1999/2000) by using a piecewise linear approximation to provide tractability and accuracy when solving for the investment threshold.

Similar to solving American options, an analytical solution to most finite-lived investment projects cannot be found. Thus, numerical methods and more specifically finite differences are commonly used for approximating the investment thresholds. Unfortunately, the finite difference methodology typically shows a very slow rate of convergence, especially for long time horizons. Our alternative, piecewise linear approximation, offers a faster rate of convergence. This is intuitive, as well as desirable, since a piecewise linear approximation can usually be estimated with only a few time points. In this paper, we show how the piecewise linear approximation method can be used to find a continuous approximation to the optimal investment threshold. Thus, our method offers a fast and robust alternative to the finite differences methodology for solving real investment problems. This application of a piecewise linear approximation to the investment threshold need not be limited to an initial investment problem. The idea of using a piecewise linear function to approximate an investment threshold (not the specific procedure we use) is general enough to be applied to many investment problems; we just provide one example of its application.

The remainder of this paper is organized as follows. The next section presents the model and its assumptions. Then we derive a stochastic representation for the value of the initial investment option. The following section examines the characteristics of the optimal investment threshold. Then we solve for the optimal investment threshold by using the piecewise linear approximation. Next we illustrate the methodology through an example. Finally, we summarize our conclusions and contributions.

Model and assumptions

To lay out a framework for our model, we consider a firm whose only asset is an option to invest in a production facility with K units of capital in place by paying a fixed investment amount, I . The option to invest is finite-lived. However, once the firm exercises the option to invest in the production facility, the facility can produce a homogeneous product forever for the market in which the firm has monopoly power. Suppose the production involves an input of labor, L , which can be chosen optimally at each instant with a cost of c per unit. Given this L and a capital amount of K , the flow of output by the firm, q , is represented by the following production function:

$$q = K^{\alpha_1} L^{\alpha_2}, \quad (1)$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ are constants. The production process is assumed to exhibit decreasing returns to scale[3], that is, $\alpha_1 + \alpha_2 < 1$.

Following Dixit and Pindyck (1994), Baldursson (1998), and many others, we assume that the firm has an isoelastic demand curve. More specifically, we assume that the firm faces the following inverse demand curve:

$$P_t = Y_t q^{-\varepsilon}, \quad (2)$$

where $-1/\varepsilon$ is the price elasticity of demand ($0 < \varepsilon < 1$), P is price, Y_t is the demand shift parameter, and q is quantity. The uncertainty of the market demand is due to both its fluctuation and stochastic shifts over time. We assume that the uncertainty of the demand is caused by the multiplicative demand shock Y_t , which follows the geometric Brownian motion:

$$dY_t = \mu Y_t dt + \sigma Y_t dw(t), \quad (3)$$

where μ and σ are constants. We assume that $\mu < r$, where r is the constant risk-free rate of interest. The randomness of the continuous demand shift parameter Y_t is attributable to a fully observable standard Brownian motion, $w(t)$, that is defined on a probability space $(\Omega, \mathcal{F}, \{F_t\}, P)$.

For fixed K and Y_t , at time t , the firm's instantaneous profit (π) is given by the maximization:

$$\pi(K, Y_t) = \max \{ Y_t (K^{\alpha_1} L^{\alpha_2})^{1-\varepsilon} - cL \}. \quad (4)$$

This leads to the following profit function at time t :

$$\pi(K, Y_t) = \xi Q(K) Y_t^{1/(1-\nu)}, \quad (5)$$

where, for convenience, $Q(K) = K^{\alpha_1(1-\varepsilon)/(1-\nu)}$, $\xi = (1-\nu)(\nu/c)^{\nu/(1-\nu)}$, and $\nu = \alpha_2(1-\varepsilon)$. Equation (5) shows that the future profit cash flow is uncertain, with the uncertainty due to the fluctuation of market demand. As a simplification, we assume a risk-neutral risk preference for the firm. Hence, the cash flows are discounted at a risk-free rate[4].

Although the market demand for the firm's product is uncertain over time, it is expected to grow. As the market demand increases, the firm will reach the point where it is optimal to expand its output by increasing its capital K , rather than just its variable input L . We assume that the firm can increase its capital in place one unit at a time by incurring a sunk cost, k . Therefore, once the firm exercises the option to invest in the production facility, the firm can produce an output forever if there is a demand. The firm's value, $W(K, Y)$, can be found as:

$$W(K, Y) = \max_{\{X_t, t \geq 0\}} E \int_0^\infty e^{-rt} \{ \pi(K_t, Y) dt - k dX_t \}. \quad (6)$$

Over all non-decreasing, left-continuous and \mathcal{F}_t -adapted processes $\{X_t; t \geq 0\}$ with $X_0 = 0$, $K_0 = K$, and $Y_0 = Y$, with $\pi(\cdot, \cdot)$ defined as in equation (5). The state dynamics are given by $dY_t = \mu Y_t dt + \sigma Y_t dw(t)$ and $K_t = K + X_t$. Assuming that W is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ and is $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$ (i.e. continuous differentiability, once with respect to K and twice with respect to Y on $\mathbb{R}_+ \times \mathbb{R}_+$), the above singular stochastic control problem can be cast as a free boundary problem[5]. That is, the firm's value at time $t \geq 0$, W satisfies the following differential equation:

$$1/2\sigma^2 Y^2 W_{YY} + \mu Y W_Y - rW + \xi Q(K) Y^{[1/(1-\nu)]} = 0, \quad (7)$$

subject to the familiar boundary conditions as in Dixit and Pindyck (1994), $W_K(K, Y^*) = k$, $W_{KY}(K, Y^*) = 0$, and $W(K, 0) = 0$ where Y^* is the optimal capacity expansion threshold[6]. The solution to this free boundary problem can be found as[7]:

$$W(K, Y) = aY^{1/(1-\nu)} + bY^\beta, \tag{8}$$

where β is the positive root of the fundamental quadratic equation $1/2\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$, $\eta = r - 1/2\sigma^2\nu/(1 - \nu)^2 - \mu/(1 - \nu)$ [8], $a = \xi/\eta Q(K)$, and b is given as:

$$b = \left(\frac{\nu}{c}\right)^{\beta\nu} \left[\frac{\beta(1 - \nu) - 1}{k}\right]^{\beta(1-\nu)-1} \int_K^\infty \left[\frac{Q'(x)}{\beta\eta}\right]^{\beta(1-\nu)} dx. \tag{9}$$

We assume the convergence of the integral, i.e. $\beta(1 - \nu) - 1 > \beta\alpha_1(1 - \varepsilon)$. This requires that given each economic state, returns to capital expansion would decrease sufficiently quickly. Therefore, our assumption guarantees that there exists some optimal K for each economic state.

Equation (8) shows that the value of the initial investment has two components. The first component is the present value of the expected profit flow, as indicated by equation (5). This value is generated directly from the initial investment and therefore is the value of assets in place[9]. The second component is the value of the future capacity-expansion opportunity, or growth option. After the initial investment is made, the firm has the opportunity to expand its output when the market demand increases. For convenience, we denote the value of the assets in place as V (where $V = aY^{1/(1-\nu)}$) and the value of the growth options as G (where $G = bY^\beta$).

Equation (8) indicates that the technology used by the firm, the market demand, and its dynamics can all affect the value of the growth options and the value of assets in place. As expected, given the nature of the business, the more advanced the technology (defined by higher α_1 and α_2), the greater the growth option G and the value of the assets in place V would be. On the other hand, the higher the price elasticity of demand, the lower G and V would be. We observe that G and V are decreasing in the cost of the input variable c , and increasing in the demand shift parameter Y as well as in its expected growth rate (μ) and volatility (σ). However, the ratio of G to V increases as c decreases, and is thus an increasing function of Y , μ , and σ . We would expect that for an initial investment in a market where the cost of the variable input (say, labor) is low and the expected growth rate and the volatility are high, the growth options could be a significant part of the value of the investment. And as the market grows, the value of the growth options increases faster than the value of the assets in place.

Option to invest: a stochastic representation

The growth options and the assets in place are a direct result of the initial investment. Both are derivatives of the underlying demand shift parameter and their values change stochastically over time. Therefore, the dynamics of the growth options and the assets in place can be traced back to the demand shift parameter Y , which is a stochastic process. We can apply a dynamic programming approach to determine the value of the initial investment option and the optimal investment threshold Y^* . Let us assume that the investment opportunity expires at a finite time T . Y^* is understood to be a function of the time to expiration. Therefore, given our assumptions, the value of the initial investment option is $J(t, Y_t)$, where $J = J(t, Y)$ satisfies equations (10)-(15)[10]:

$$\frac{\partial J}{\partial t} + \mu Y \frac{\partial J}{\partial Y} + \frac{1}{2} \sigma^2 Y^2 \frac{\partial^2 J}{\partial Y^2} - rJ = 0; \quad Y < Y^*(t), \quad (10)$$

$$J > \max(aY^{1/(1-\nu)} + bY^\beta - I, 0); \quad Y < Y^*(t), \quad (11)$$

$$J = \max(aY^{1/(1-\nu)} + bY^\beta - I, 0); \quad Y \geq Y^*(t), \quad (12)$$

$$J(T, Y) = \max(aY^{1/(1-\nu)} + bY^\beta - I, 0), \quad (13)$$

$$\frac{\partial J(t, Y^*)}{\partial Y} = \left(\frac{a}{1-\nu} \right) Y^{\nu/(1-\nu)} + b\beta Y^{\beta-1}, \quad (14)$$

$$J(t, 0) = 0, \quad \text{for all } t. \quad (15)$$

Equation (11) states that at any time before the expiration, the value of the initial investment option is greater than the net value of the initial investment when Y is less than the optimal investment threshold. The term $\max(aY^{1/(1-\nu)} + bY^\beta - I, 0)$ is convex in Y , which can be easily checked, and might be considered as an obstacle for $J(t, Y)$. When Y is equal to or greater than the optimal investment threshold Y^* , $J(t, Y)$ is equal to the sum of the assets in place and the growth options minus the cost of the investment. This is the value matching condition. The firm exchanges the option with the value of the investment. Equation (13) states that at expiration (T), the value of the investment option is the net present value of the investment which is greater than or equal to 0. Equation (14) is the smooth-pasting condition. Finally, equation (15) shows that at any time, if demand is zero, i.e. $Y = 0$, the initial investment option would have no value.

To find $J(t, Y)$, we need to solve the partial differential equation (10) subject to the boundary conditions given in equations (11)-(15)[11]. However, this problem cannot be solved analytically due to the fact that the optimal investment threshold Y^* is time-dependent and has to be found together with the value function. To avoid directly solving the partial differential equation, we seek an alternative characterization for the value of the option to invest. Kim (1990), Jacka (1991) and Carr *et al.* (1992) showed that the value of an American option can be expressed as a payoff at the expiration date plus an early exercise premium. Although the underlying value of our initial investment option consists of two components, i.e. the assets in place and the growth options, with differing dynamics, we observe that there is a one-to-one relationship between them. Therefore, instead of expressing the investment threshold in terms of the demand shift parameter Y , another approach is to consider the initial investment option in terms of the assets in place. Hence, we express the optimal investment threshold equivalently in terms of the value of the assets in place V . Although the initial investment policy can be determined by the investment threshold Y^* , it is not obvious how the payoff can be compared with the cost of the investment at the time when the initial investment is made, i.e., when $Y = Y^*$. The advantage of expressing the investment threshold in terms of the value of the assets in place is that it could help us obtain some insights into the optimal

investment rule for the initial investment. Theorem 1 provides a stochastic representation for the initial investment option.

Theorem 1

At any time $t \in [0, T]$, if the value of the option to invest in the production facility is not equal to $\max(0, V_t + G_t - I)$, then it must satisfy the following stochastic representation:

$$J(t, V_t, G_t)E_t[e^{-r(T-t)}(V_T + G_T - I)^+] + \int_t^T e^{-r(\tau-t)}E_t[(\eta V_\tau - rI)1_{\{V_\tau \geq V_\tau^*\}}] d\tau, \tag{16}$$

where E_t is the conditional expectation at time t and V_τ^* is the optimal initial investment threshold at time τ in terms of V . (See the Appendix for the proof of Theorem 1.)

Theorem 1 states that the value of the initial investment option can be expressed as the value of the option to invest at the expiration date T plus an expectation of a stochastic integral that may be considered as a gain from early investment. The growth options do not directly contribute to the gain coming from any possible early investment, as indicated in the second term of Equation (16). In addition, equation (16) shows that if the investment is made before the expiration date, the instant gain from the assets in place has to be greater than the instant cost rI . At first glance, it seems that the growth options would only play a role in the investment decision-making at the expiration date, not in determining early investment timing. However, this is not true, since the growth options do help offset the volatility effect. That is, by including the growth options, the value of the initial investment would be more likely to be greater than the cost of the investment at the expiration. Therefore, the impact of the growth options on the investment timing may not be ignored.

Optimal investment threshold

Equation (16) provides an analytical representation for the value of the initial investment option. However, as is shown, the value of the option is represented as the discounted expectation of the net payoff at the expiration date T : $(V_T + G_T - I)^+$, plus the expectation of a stochastic integral. In order to examine the characteristics of the optimal investment threshold and to facilitate implementation of the formula, it would be desirable to have an explicit expression for the expectations. Thus we first show how to evaluate these expectations, and then examine the characteristics of the optimal investment threshold. Understanding the characteristics of the optimal investment threshold is helpful for numerically solving the value of the initial investment option and its associated optimal investment strategy in a finite time horizon.

To determine the value of the initial investment option, decompose the right-hand side of the equation (16) in Theorem 1 into the following three terms:

$$J(0, V_0, G_0) = E[e^{-rT}(V_T - I)1_{\{V_T + G_T \geq I\}}] + E[e^{-rT}G_T 1_{\{V_T + G_T \geq I\}}] \\ + \int_0^T e^{-rt} E[(\eta V - rI)1_{\{V_t \geq V_t^*\}}] dt. \quad (17)$$

A further breakdown of equation (17), as shown in the Appendix, leads to the decomposition formula for the investment option which we state as Corollary 1.

Corollary 1

If X is the unique solution to the equation $V + CV^{\beta(1-\nu)} - I = 0$, then the value of the firm's initial investment option can be expressed as:

$$J(0, V_0) = V_0 e^{-\eta T} N(h) - I e^{-rT} N\left(h - \frac{\sigma}{1-\nu} \sqrt{T}\right) \\ + CV_0^{\beta(1-\nu)} N\left[h + \left(\beta - \frac{1}{1-\nu}\right) \sigma \sqrt{T}\right] + \int_0^T \eta V_0 e^{-\eta t} N(d) dt \\ - \int_0^T r I e^{-rt} N\left(d - \frac{\sigma}{1-\nu} \sqrt{t}\right) dt, \quad (18)$$

where $N(\cdot)$ represents the standard normal distribution function, and h and d are respectively defined as:

$$h = \frac{\ln(V_0/X) + \left(r - \eta + \frac{1}{2} \frac{\sigma^2}{(1-\nu)^2}\right) T}{\frac{\sigma}{1-\nu} \sqrt{T}},$$

and

$$d = \frac{\ln(V_0/V_t^*) + \left(r - \eta + \frac{1}{2} \frac{\sigma^2}{(1-\nu)^2}\right) T}{\frac{\sigma}{1-\nu} \sqrt{t}}.$$

We know when V is at the optimal investment threshold, the value of the option to invest is $J = V^* + G^* - I$. Let τ be the time to expiration: from equation (18), the optimal initial investment threshold V^* is implicitly defined by the following integral equation:

$$\begin{aligned}
 V^*(\tau) + C[V^*(\tau)]^{\beta(1-\nu)} - I &= V^*(\tau)e^{-r\tau}N(h) - Ie^{-r\tau}N\left(h - \frac{\sigma}{1-\nu}\sqrt{t}\right) \\
 &+ C[V^*(\tau)]^{\beta(1-\nu)}N\left[h + \left(\beta - \frac{1}{1-\nu}\right)\sigma\sqrt{\tau}\right] \\
 &+ \int_0^\tau \eta V^*(\tau)e^{-\eta(\tau-t)}N(d)dt \\
 &- \int_0^\tau rIe^{-r(\tau-t)}N\left(d - \frac{\sigma}{1-\nu}\sqrt{\tau-t}\right) dt, \quad (19)
 \end{aligned}$$

where

$$h = \frac{\ln\left(\frac{V^*(t)}{X}\right) + \left[r - \eta + \frac{1}{2}\frac{\sigma^2}{(1-\nu)^2}\right]\tau}{\frac{\sigma}{1-\nu}\sqrt{\tau}},$$

and

$$d = \frac{\ln\left[\frac{V^*(\tau)}{V^*(t)}\right] + \left[r - \eta + \frac{1}{2}\frac{\sigma^2}{(1-\nu)^2}\right](\tau - 1)}{\frac{\sigma}{1-\nu}\sqrt{(\tau - t)}}.$$

This leads to the following three corollaries that characterize the optimal investment threshold.

Corollary 2

Let τ be the time to expiration for the initial investment option. If the value of the initial investment is the summation of the value of assets in place V and the value of capacity expansion options $G = CV^{\beta(1-\nu)}$, the optimal initial investment threshold $V^*(\tau)$ is continuous and non-decreasing in τ , for $\tau > 0$. (See the Appendix for proof.)

Corollary 3

If the value of the initial investment is the summation of the value of assets in place V and the value of capacity expansion options $G = CV^{\beta(1-\nu)}$, then the optimal initial investment threshold is bounded above by the quantity:

$$V^*(\infty) = \frac{\beta(1-\nu)}{\beta(1-\nu) - 1} I, \quad (20)$$

and is bounded below by X . X is the unique root of function $f(V) = V + CV^{\beta(1-\nu)} - I$. (See the Appendix for proof.)

Corollary 4

If the value of the initial investment is the summation of the value of assets in place V and the value of capacity expansion options $G = CV^{\beta(1-\nu)}$, and η is defined as before, then $\lim_{\tau \rightarrow 0} V^*(\tau) = (r/\eta)I$. (See the Appendix for proof.)

Corollary 4 shows the behavior of the optimal investment threshold near the expiration date. It also suggests that in some cases the impact of the growth options on

the optimal investment threshold could decrease quickly as the time to expiration approaches zero. Because $V^*(\tau)$ is always greater than X , the numerator in h would always be positive. As the time to expiration becomes smaller and smaller, $N(h)$, $N[h - \sigma\sqrt{\tau}/(1 - \nu)]$, and $N\{h + [\beta - 1/(1 - \nu)]\sigma\sqrt{\tau}\}$ could approach unity rather quickly. The growth options appear on both sides of equation (19), and as $N\{h + [\beta - 1/(1 - \nu)]\sigma\sqrt{\tau}\}$ approaches unity, their impact on the investment threshold approaches zero. The investment threshold with the growth options converges to the investment threshold without the growth options[12].

Numerical evaluation

The optimal investment threshold V^* appears in the form of a log function in Corollary 1. In order to approximate it using piecewise linear functions, we seek a time-space transformation. Let $\rho = r(1 - \nu)^2/\sigma^2$, $\alpha = \eta/r$, and $s = -[\sigma^2/(1 - \nu)^2]T$. We introduce the change of variables:

$$u = \frac{\sigma^2}{(1 - \nu)^2}(t - T), \quad (21)$$

$$z = \ln(V/X) - (\rho - \alpha\rho - 1/2)u, \quad \text{where } u \in [s, 0]. \quad (22)$$

This leads to the following corollary.

Corollary 5

In the new coordinate system, at time s the value of a firm's investment option is given in terms of the optimal investment threshold as:

$$\begin{aligned} J[s, z(s)] = & X e^{z(s) + (\rho - 1/2)s} N\left[\frac{z(s)}{\sqrt{-s}} + \sqrt{-s}\right] - I e^{\rho s} N\left[\frac{z(s)}{\sqrt{-s}}\right] \\ & + b \left(\frac{X}{a}\right)^{\beta(1-\nu)} e^{\beta(1-\nu)[z(s) + (\rho - \alpha\rho - 1/2)s]} N\left[\frac{z(s)}{\sqrt{-s}} + \beta(1 - \nu)\sqrt{-s}\right] \\ & + \int_0^{-s} \alpha\rho X e^{z(s) + (\rho + \alpha\rho - 1/2)s - \alpha\rho t} N\left[\frac{z(s) - z^*(s+t) + t}{\sqrt{t}}\right] dt \\ & - \int_0^{-s} \rho I e^{-\rho t} N\left[\frac{z(s) - z^*(s+t)}{\sqrt{t}}\right] dt, \end{aligned} \quad (23)$$

where z^* is the optimal investment threshold in the new coordinate system, and $N(\cdot)$ represents the standard normal distribution function. (See the Appendix for proof.)

Equation (23) defines a pair $\{J[s, z(s)], z^*\}$ which needs to be solved simultaneously. To solve for the pair, another equation is needed. $J[s, z(s)]$ is the value of the investment option and is equal to $J(0, V_0, G_0)$, and z^* is the optimal investment threshold which is defined on $[-\sigma^2 T, 0]$. Given a pair (u, z^*) defined on $[-\sigma^2 T, 0] \times R$, we can retrieve (t, V^*) on $[0, T] \times [0, +\infty)$ simply by mapping it back as $V^* = X e^{z^* + (\rho - \alpha\rho - 1/2)u}$ and $t = T + [(1 - \nu)^2/\sigma^2]u$.

As is shown, the optimal investment threshold is implicitly defined by equations (23) and (12), which needs to be expressed in the new coordinate system as well. When $z = z^*$ at time s , combining those two equations, we obtain:

$$\begin{aligned}
 & X e^{z^*(s) + (\rho - \alpha\rho - 1/2)s} + b \left(\frac{X}{a} \right)^{\beta(1-\nu)} e^{\beta(1-\nu)z^*(s) + \beta(1-\nu)(\rho - \alpha\rho - 1/2)s} - I \\
 &= X e^{z^*(s) + (\rho - 1/2)s} + N \left[\frac{z^*(s)}{\sqrt{-s}} + \sqrt{-s} \right] - I e^{\rho s} N \left[\frac{z^*(s)}{\sqrt{s}} \right] \\
 &+ b \left[\frac{X}{a} \right]^{\beta(1-\nu)} e^{\beta(1-\nu)z^*(s) + \beta(1-\nu)(\rho - \alpha\rho - 1/2)s} N \left[\frac{z^*(s)}{\sqrt{-s}} + \beta(1-\nu)\sqrt{-s} \right] \\
 &+ \int_0^{-s} \alpha\rho X e^{z^*(s) + (\rho - \alpha\rho - 1/2)s - \alpha\rho t} N \left[\frac{z^*(s) - z^*(s+t) + t}{\sqrt{t}} \right] dt \\
 &- \int_0^{-s} \rho I e^{-\rho t} N \left[\frac{z^*(s) - z^*(s+t)}{\sqrt{t}} \right] dt. \tag{24}
 \end{aligned}$$

The optimal investment threshold z^* solves the above integral equation. But equation (24) does not have a closed form solution. To evaluate the integrals, we start with dividing the entire time interval $[s, 0]$ into n subintervals: $s = s_n < \dots < s_0 = 0$. For any subinterval, say $[s_i, s_{i-1}]$, we could approximate $z^*(u)$ by a linear function, that is [13]:

$$z^* = \beta_i u + \gamma_i \quad \text{for } s_i < u < s_{i-1}. \tag{25}$$

Let $t_i = s_i - s$. Then:

$$z^*(s+t) = \beta_i t + \beta_i s + \gamma_i. \tag{26}$$

Substituting equation (26) into the two integrals in equation (24), we obtain the following equations:

$$\begin{aligned}
 & \int_{t_i}^{t_{i-1}} I \rho e^{-\rho t} N \left[\frac{z^*(s) - z^*(s+t)}{\sqrt{t}} \right] dt \\
 &= \int_{t_i}^{t_{i-1}} I \rho e^{-\rho t} N \{ -\beta_i t^{1/2} + [z^*(s) - \beta_i s - \gamma_i] t^{-1/2} \} dt, \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 & \int_{t_i}^{t_{i-1}} \alpha\rho X e^{z^*(s) + (\rho + \alpha\rho - 1/2)s - \alpha\rho t} N \left[\frac{z^*(s) - z^*(s+t) + t}{\sqrt{t}} \right] dt \\
 &= \int_{t_i}^{t_{i-1}} \alpha\rho X e^{z^*(s) + (\rho + \alpha\rho - 1/2)s - \alpha\rho t} N \{ (1 - \beta_i) t^{1/2} + [z^*(s) - \beta_i s - \gamma_i] t^{-1/2} \} dt. \tag{28}
 \end{aligned}$$

Using the equation obtained by Ju (1998)[14], the above two integrals can be evaluated

in terms of simple sums. The entire optimal investment threshold can then be approximated by n piecewise linear functions specified by the set of parameters $(\beta_i, \gamma_i)_{1 \leq i \leq n}$. Using piecewise linear approximation, equation (24) can be rewritten as the following equation:

$$\begin{aligned}
 & X e^{z(s) + (\rho - \alpha\rho - 1/2)s} + b \left(\frac{X}{a} \right)^{\beta(1-\nu)} e^{\beta(1-\nu)z(s) + \beta(1-\nu)(\rho - \alpha\rho - 1/2)s} - I \\
 &= X e^{z(s) + (\rho + 1/2)s} N \left[\frac{z(s)}{\sqrt{-s}} + \sqrt{-s} \right] - I e^{\rho s} N \left[\frac{z(s)}{\sqrt{-s}} \right] \\
 &+ b \left(\frac{X}{a} \right)^{\beta(1-\nu)} e^{\beta(1-\nu)z(s) + \beta(1-\nu)(\rho - \alpha\rho - 1/2)s} N \left[\frac{z(s)}{\sqrt{-s}} + \beta(1-\nu)\sqrt{-s} \right] \quad (29) \\
 &+ X e^{z(s) + (\rho - \alpha\rho - 1/2)s} \sum_{i=1}^n I_i [(1 - \beta_i), z(s) - \beta_i s - \gamma_i, \alpha\rho] \\
 &- I \sum_{i=1}^n I_i [-\beta_i, z(s) - \beta_i s - \gamma_i, \rho],
 \end{aligned}$$

where

$$\begin{aligned}
 I_i(x_1, x_2, y) &= e^{-y t_i} N(x_1 t_i^{1/2} + x_2 t_i^{-1/2}) - e^{-y t_{i-1}} N(x_1 t_{i-1}^{1/2} + x_2 t_{i-1}^{-1/2}) \\
 &+ \frac{1}{2} \left(\frac{x_1}{\sqrt{x_1^2 + 2y}} + 1 \right) e^{(\sqrt{x_1^2 + 2y} - x_1) x_2} \\
 &\times \left\{ \begin{array}{l} N(\sqrt{x_1^2 + 2y} t_{i-1}^{1/2} + x_2 t_{i-1}^{-1/2}) \\ -N(\sqrt{x_1^2 + 2y} t_i^{1/2} + x_2 t_i^{-1/2}) \end{array} \right\} \\
 &+ \frac{1}{2} \left(\frac{x_1}{\sqrt{x_1^2 + 2y}} - 1 \right) e^{-(\sqrt{x_1^2 + 2y} + x_1) x_2} \\
 &\times \left\{ \begin{array}{l} N(\sqrt{x_1^2 + 2y} t_{i-1}^{1/2} - x_2 t_{i-1}^{-1/2}) \\ -N(\sqrt{x_1^2 + 2y} t_i^{1/2} - x_2 t_i^{-1/2}) \end{array} \right\}.
 \end{aligned}$$

Equation (29) is non-linear. It can be solved backwards recursively by using the bisection search method[15]. We know that as the time to expiration date goes to zero, the optimal investment threshold V^* approaches $r/\eta I$. Therefore, $z^*(0)$ can be found as $\ln(I/X) - \ln(\eta/r)$ in the limit. In general, the optimal investment threshold can be well approximated by a piecewise linear function.

Illustrative example

We use an example to illustrate the impact of the growth options on the investment decision, and present some numerical results generated from our piecewise linear approximation. In choosing our numerical example, we follow McDonald and Siegel's (1986) approach for valuing their option for waiting to invest. We examine the value of initial investment option and the optimal investment threshold under varying volatilities. We assume a risk free interest rate equal to 5 percent (i.e. $r = 0.05$) and a price elasticity of demand equal to -2 (i.e. $-1/\epsilon = -2.0$). Technological parameters are chosen as 0.3 and 0.4 for α_1 and α_2 , respectively. The input cost per unit is set at 2 (i.e. $c = 2$) with a sunk cost of 2 (i.e. $k = 2$). Finally we set the initial unit of capital and the investment amount both to 1 (i.e. $K_0 = 1$ and $Y = 1$), and the cost of investment to 10 (i.e. $I = 10$), so that the present value of the investment project without taking into account the growth options is not far off from the cost of investment[16].

Table I shows the values of the assets in place (V), the growth options (G), and the initial investment option values (OV and OV_G for not taking and for taking account of the growth options, respectively) for a given set of market and technological parameters. First, Table I indicates that uncertainty affects both the value of the assets in place (V) and the value of the growth options (G), but an increase in the volatility has a much larger impact on the growth options than on the assets in place. When the volatility is low, say 10 percent (i.e. $\sigma = 0.10$), the value of the growth options only accounts for about 1.5 percent (column 6, Table I) of the value of the investment. However, when the volatility increases to 30 percent, the growth options constitute more than half of the value of the initial investment. These results support previous researchers' findings (see, for example, Aguerrevere, 2003) that growth options could account for a significant part of the value of investment in a volatile environment.

Secondly, Table I illustrates that at all levels of the volatility, taking account of the growth options would result in a much higher value of the initial investment option than if the growth options are not taken into account (column 7, Table I). When σ equals 10 percent, the value of the initial investment option is 21 percent higher than when the growth option is not considered. When the volatility increases to 30 percent, the value of the initial investment option would be 350 percent higher than without taking account of the growth options, a significant increase. Such a big increase in the value of the initial investment option is due not only to the increase in the value of the growth options, but also due to the fact that the growth options are more volatile than the assets in place. Finally, Table I shows that as volatility (σ) increases, the value of the investment option increases regardless of whether the growth options are taken

σ	V	G	OV	OV_G	$G/(V + G)\%$	$(OV_G - V)/OV\%$
0.10	9.29	0.14	0.36	0.44	1.51	21.18
0.15	9.68	0.15	0.85	1.19	4.98	40.24
0.20	10.28	1.44	1.52	2.62	12.27	72.85
0.25	11.18	4.00	2.44	5.85	26.35	139.94
0.30	12.52	14.04	3.75	16.88	52.86	349.77

Table I.
Values of the assets in place, the growth options, and the initial investment option

Note: V is the value of the assets in place and G is the value of the growth options; OV_G and OV are initial investment option values with and without taking account of the growth options, respectively

into consideration. This result is consistent with the general finding in real options literature.

Table II displays the investment thresholds now and at the expiration date for the same set of parameters. When an investment option is finitely lived, the investment threshold is non-decreasing with the time to expiration. When the volatility is low, say σ equals 10 percent, the investment threshold found without taking account of the growth options would be 13.14, which is 0.69 percent higher than 13.05, the investment threshold found with the growth options included. In this case, the difference is very small. At the expiration date, the investment rule is to invest if the value of the assets in place is not less than the cost ($I = 10$ for this example) minus the value of the growth options. Since the value of the growth options is always greater than zero, the investment threshold produced by taking into account the growth options is always lower than the one produced without taking into account the growth options. When the value of the growth options at the expiration date is significant, ignoring the growth options would result in under-investing.

As shown in Table II, when the growth options are ignored and σ equals 20 percent, the initial investment would be made if the critical value is greater than the cost (i.e. $V_0^* \geq 10.00$). This is 12.61 percent (column 7, Table II) higher than the investment threshold when the growth options are considered (i.e. $VG_0^* = 8.88$). The impact of the growth options on the investment decision-making is even more profound when volatility is increases. For instance, when σ equals 30 percent, ignoring the growth options would result in an investment threshold which is 23.92 percent higher than the investment threshold at present when taking account of the growth options and is 84.84 percent higher than that at the expiration date. Our numerical results suggest that the impact of the growth options is relatively small when the uncertainty is low, but it is much more important when the volatility is high.

Figure 1 reinforces the impact of the growth options. As is shown, for a given volatility, taking account of the growth options would result in a lower initial investment threshold (solid line in Figure 1). Therefore, the inclusion of growth options encourages investment. As time to expiration approaches zero, the two thresholds approach the same limit. As reinforced in Table II, the investment threshold is lower at the expiration date if we consider the value of the growth options as part of the value of the initial investment. On the other hand, as time to expiration approaches infinity, the growth options would have no role in our initial investment decision-making (i.e. the

σ	V_T^*	VG_T^*	V_0^*	VG_0^*	$(V_T^* - VG_T^*)/VG_T^*\%$	$(V_0^* - VG_0^*)/VG_0^*$
0.10	13.14	13.05	10.00	9.83	0.69	1.73
0.15	15.25	14.90	10.00	9.51	2.35	5.15
0.20	17.84	16.92	10.00	8.88	5.44	12.61
0.25	21.11	18.95	10.00	7.70	11.40	29.87
0.30	25.44	20.53	10.00	5.41	23.92	84.84

Note: VG_T^* and V_T^* are the critical values for the initial investment with and without taking account of the growth options, respectively, when the time to expiration is T . VG_0^* and V_0^* are the critical values at expiration date for initial investment with and without taking account of the growth options, respectively

Table II.
Initial investment
thresholds

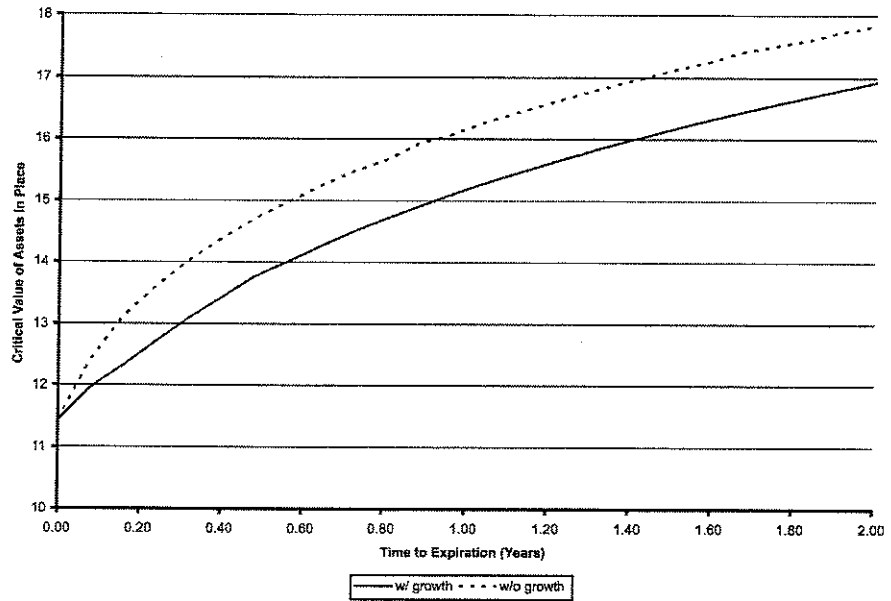


Figure 1.
Investment threshold for
the initial investment
option

Note: Figure 1 plots the investment threshold versus time to expiration for the case $\sigma = 20\%$

initial investment threshold would be the same no matter whether the growth options are taken into account).

To compare our piecewise linear approach with the commonly used finite difference methods[17], we can apply the finite difference methods to the firm's investment problem with the growth options. We use the same set of parameters as above to generate the investment thresholds. As shown in Figure 2, the curve with long dotted lines is the investment threshold generated by using about 300 time steps under the finite difference approach; the curve with short dotted lines is the investment threshold generated by using about 1,300 time steps under the finite difference approach; and the curve with the solid line is our piecewise linear approach. Figure 2 shows that for a fixed time to maturity, as the number of time steps increases to about 1,300, the investment threshold generated from the finite difference method is almost lying on the investment threshold generated from our piecewise linear approach. We do not report the value of investment project generated from two different methods here due to the focus of this research being on the investment threshold, and the fact that the accuracy of pricing in general can be achieved without the need of very good approximation for the investment threshold.

Although we present only one set of market and technological parameters, the results can be generalized to other scenarios. Our simulation using different sets of parameters suggests that the impact of the growth options on the initial investment decision varies but, in all cases, the initial investment threshold increases as the volatility increases, holding the other factors constant. Therefore, as long as the opportunity cost of delaying the initial investment cannot be ignored, the impact of the growth options on the initial investment would be significant in risky environments.

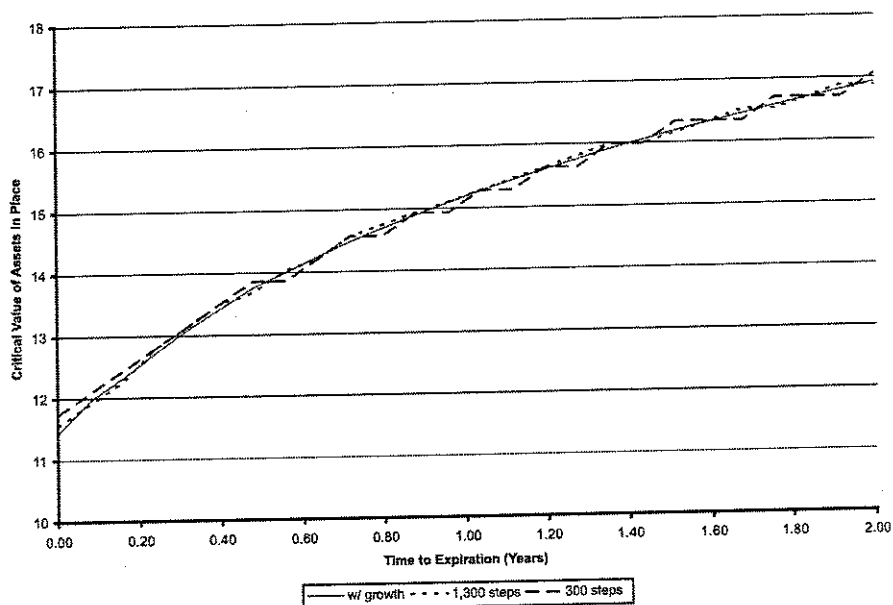


Figure 2.
Comparison graphs for
investment threshold
functions

Our analysis could also be expanded outside the manufacturing sector where most of this type of research has been undertaken. Other potential applications include exploration, research and development, financial services, and new product development. For instance, within oil exploration, an initial field with uncertain reserves may not present a positive investment decision without also taking into consideration the prospect of additional fields, also with uncertain reserves and/or of a lesser quantity or quality than this initial field. Another complication within the exploration decision is that additional inferior fields typically would only be explored if prices for the underlying natural resources increase or if the cost of exploration decreases due to the expertise acquired from exploration of the initial field. This type of expansion option could have a major impact on the initial decision.

An R&D example can be found within the pharmaceutical industry. A major expense for most drug companies is research and development, because if a firm is unsuccessful in producing a steady stream of profitable new drugs, it will soon be out of business. Development of a new drug relies on stages of investments. At each stage of development, the decision to progress further is dependent upon the probability of success and/or other potential outcomes (i.e. related drugs, alternative treatments, or clinical trials which can also be considered as expansion options). Of course, our single-phased capacity expansion analysis would have to undergo significant modifications to incorporate the multi-phased options found within pharmacology (i.e. Phase I, II, III, NDA approval, etc.). Each phase would need its own set of parameters and be linked to the success of the proceeding phase.

The financial services sector also relies on expansion options. Commercial banks typically must explore entry into new markets to grow. For instance, suppose a bank decides it wishes to explore the possibility of expanding into a new market (i.e. a new territory). It has been successful in its current market, but feels the need to expand. The

expansion plans call for a phased entry into the new market, i.e. the bank plans to test the new market with a single branch. If the demand exists, the bank plans additional branches (i.e. an expansion option). This is a common strategy for banks as well as franchise operations. Other examples where an investment would not have been undertaken if expansion options were not also considered include personal computers, high-speed internet connections, local phone services through cable, high-definition TVs, DVD recorders, and picture phones.

Conclusion

Many investments create growth opportunities. Therefore, the value of an investment should consist of the value of both the assets in place and any embedded growth options. In this paper, we examine an initial investment in a finite-time horizon and develop a process that takes into account the growth options it creates. Our process not only recognizes the option-like characteristics of the initial investment opportunity, but also recognizes the option-creating characteristics of the investment. Our analysis shows that the value of capacity expansion options created by the initial investment has different dynamic characteristics from the assets in place. Although the growth options do not appear in the early investment premium, its impact on the investment decision is embedded in the investment threshold. When the time to expiration is short and the cost to delaying the assets in place is low, our analysis suggests that the initial investment decision might be made by ignoring the growth options.

Our decomposition of the payoff from an initial investment yields insights into how the value of the initial investment can be generated from different sources, and indicates that growth options could be a significant part of the value of the investment. Since growth options may not yield any future cash flows, applying a simple NPV rule can result in under-investment. To capture the value of growth options, one must evaluate the current investment by aligning it with the future investment opportunities it creates. This paper shows how to evaluate an initial investment by aligning it with one kind of growth option – the future capacity expansion option in an uncertain market.

Although we developed a piecewise linear approximation for solving the value of the initial investment opportunity, as well as the investment threshold in a finite-time horizon, this method is applicable to many other investment problems as well. One of our key contributions is that by expressing the initial investment threshold in transformed coordinates, we are able to approximate the investment threshold by a linear spline with relatively few time points. The optimal initial investment threshold is accurately computed by using a bisection search algorithm. The value of the initial investment option can hence be efficiently computed. Our piecewise linear approximation provides a continuous solution to the investment threshold.

Notes

1. In fact, the value of an initial investment is the value of the “new” firm in the market.
2. We have conducted extensive empirical testing for American option pricing. We found that when the early exercise boundary for an American option is steep near the expiration, the accuracy of the approximation near the expiration has very little impact on computing the option price. This partially explains why Ju’s (1998) piecewise exponential approximations with $n = 3$ can result in very accurate prices for American options in most cases without being able to approximate the optimal exercise boundary well.

3. This assumption is not really necessary given the assumption of the convergence of the integral in equation (9).
4. A common practice in early real options research was to assume that there exists a tradable perfect hedge portfolio. This assumption would allow us to use the risk-neutral valuation method for calculating option values. But, as many researchers point out, such a portfolio, in general, does not exist. Without assuming that there exists a perfect spanning asset for the project value, we do not have risk-neutral probability and a risk-adjusted interest rate is needed. In this paper, we follow some researchers, for example, Grenadier (2002), to directly assume a risk-neutral preference for the firm. In reality, firms often use their own risk-adjusted interest rates to discount expected cash flows. Risk-neutral results are often used to serve as a benchmark. However, our valuation approach would still be valid if we used a risk-adjusted interest rate instead of a risk-free rate. For the discussion of valuation without a perfect spanning asset and a risk-neutral preference, we refer readers to Henderson (2002, 2004) and Musiela and Zariphopoulou (2004a, b).
5. See Fleming and Lions (1988), Fleming and Soner (1993), and Karatzas (1983) for the discussion of stochastic control problems.
6. Note that $W_K(K, Y^*) = k$ and $W_{KY}(K, Y^*) = 0$ are the smooth pasting and super contact conditions derived in Dumas (1991).
7. The capacity expansion threshold can be found as $Y^*(K) = (c/\nu)^\nu (k\beta\eta / \{[\beta(1-\nu) - 1]Q'(K)\}^{1-\nu})$. Since the focus of this paper is the optimal policy for the initial investment, not the future capacity expansion, we do not discuss it. See Dixit and Pindyck (1994) for a discussion of incremental capacity expansion strategy. Grenadier (2002) examines incremental expansion in a strategic setting.
8. Note that $\beta > 1$, which follows from $\mu < r$, and η represents the opportunity cost of delaying the initial investment.
9. See Myers (1977) and Pindyck (1988) for the discussion of such decomposition.
10. When $T = \infty$, an analytic solution can be found. Therefore, we will not address the case when $T = \infty$ in this paper.
11. The above free boundary problem has not been analyzed for a mathematical proof of the existence of solutions. A rigorous mathematical proof is beyond the scope of this paper. Our focus here, however, is on the numerical approach and its application.
12. If the firm ignores the growth options, X in h would be replaced by I .
13. Note that although Ju's (1998) approach can be used to solve the value of the option very efficiently, the piecewise exponentials are not suitable to be used for approximating the optimal investment threshold, which is relatively flat in the most part. In addition, in most situations, volatility in real investments is relatively low. Therefore, near the expiration, the gradually dropping optimal investment threshold could be reasonably well approximated by a piecewise linear function. Our later numerical example shows an approximation to an optimal threshold when volatility is 20 percent.
14. Equation (5) in Ju (1998).
15. A simple bisection search method can be used for determining the parameters (β_i, γ_i) $1 \leq i \leq n$.
16. Note that the payoff representing the growth options is more convex than the payoff representing the assets in place: it should be the case that the impact of volatility of the demand on the growth options is greater than the assets in place.
17. The implicit finite difference method is very robust and always converges, but is more complicated. Explicit finite difference methods are computationally more efficient, but we need to take some special precautions to ensure convergence (see, for example, Hull and White, 1990).

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Appendix

Proof of Theorem 1

For simplicity, we assume at the present time that $t = 0$. Notice that $V = aY^{1/(1-\nu)}$ and $G = bY^\beta$, and both are continuous and monotonic increasing functions on $[0, +\infty)$. Therefore, there is a one-to-one relationship between V and Y , and a one-to-one relationship between G and Y . $J(t, V_t, G_t)$ can be expressed as $J(t, Y_t)$ on $[0, T] \times [0, +\infty)$. Let $e^{-rt}J(t, Y_t)$ be the discounted value of the investment option, defined in the region $D = \{(t, Y) : t \in [0, T], Y \in [0, +\infty)\}$. An application of Ito's lemma to the process $e^{-rt}J(t, Y_t)$ yields:

$$e^{-rT}J(T, Y_T) - J(0, Y_0) - \int_0^T A[e^{-rt}J(t, Y_t)] dt = \sigma \int_0^T e^{-rt} Y_t J_Y'(t, Y_t) dw(t), \quad (A1)$$

where A is a differential operator, defined as:

$$A = \frac{\partial}{\partial t} + \mu Y \frac{\partial}{\partial Y} + \frac{1}{2} \sigma^2 Y^2 \frac{\partial}{\partial Y^2}.$$

Since $e^{-rt}Y_t J_Y'(t, Y_t)$ satisfies a polynomial growth condition on the region D , the right-hand side of equation (A1) is a martingale. By taking expectations we obtain:

$$J(0, Y_0) = E[e^{-rT}J(T, Y_T)] - E\left\{\int_0^T A[e^{-rt}J(t, Y_t)] dt\right\}. \quad (A2)$$

To find the value for second term of the above right-hand-side equation, we separate the value of the investment option into two regions, $J(t, Y_t) = J(t, Y_t)1_{\{Y_t < Y_t^*\}} + J(t, Y_t)1_{\{Y_t \geq Y_t^*\}}$. We then apply the operator A to the process $e^{-rt}J(t, Y_t)$ to obtain:

$$A[e^{-rT}J(t, Y_t)] = A[e^{-rT}J(t, Y_t)1_{\{Y_t < Y_t^*\}}] + A[e^{-rT}J(t, Y_t)1_{\{Y_t \geq Y_t^*\}}] \quad (A3)$$

From equation (10), we know that when $Y_t < Y_t^*$, $A[e^{-rT}J(t, Y_t)] = 0$, and when $Y_t \geq Y_t^*$, from equation (12) we have:

$$\begin{aligned} A[e^{-rT}J(t, Y_t)] &= e^{-rT} \left\{ \left[\frac{1}{2} \sigma^2 \beta(\beta - 1) + \mu\beta - r \right] bY_t^\beta + \left[\frac{1}{2} \sigma^2 \frac{\nu}{(1-\nu)^2} + \frac{\mu}{1-\nu} - r \right] aY_t^{1/(1-\nu)} + rI \right\} \\ &= e^{-rT} [rI - \eta aY_t^{1/(1-\nu)}], \end{aligned} \quad (A4)$$

where the last term uses the definition of η and the fact that β is the positive root of the fundamental quadratic equation. Finally, the proof is completed by recognizing that $V = aY_t^{1/(1-\nu)}$ and $V^* = a(Y_t^*)^{1/(1-\nu)}$, in conjunction with condition (13).

Decomposition of equation (17) into Corollary 1

For convenience, denote $C = b/a^{\beta(1-\nu)}$. Let us consider function $f(V) = V + CV^{\beta(1-\nu)} - I$ on $[0, +\infty)$. We have $f(0) = -I < 0$, $f(+\infty) = +\infty > 0$, and $f'(V) = 1 + \beta(1-\nu)CV^{\beta(1-\nu)-1} > 0$. Thus, function $f(V)$ is continuous and monotonically increasing in V , so that equation $f(V) = 0$ has a unique positive solution. Let X be that unique solution, i.e. $f(X) = 0$; then the events $V_T + G_T \geq I$ and $V_T \geq X$ are equivalent. Thus, the first two terms of equation (17) can be expressed as:

$$\begin{aligned} &E[e^{-rT}(V_T - I)1_{\{V_T + G_T \geq I\}}] + E[e^{-rT}G_T 1_{\{V_T + G_T \geq I\}}] \\ &= E[e^{-rT}(V_T - I)1_{\{V_T \geq X\}}] + E[e^{-rT}G_T 1_{\{V_T \geq X\}}]. \end{aligned} \quad (A5)$$

Applying the Black-Scholes formula to the first term on the right-hand side of equation (A5), we obtain:

$$E[e^{-rT}(V_T - I)1_{\{V_T \geq X\}}] = V_0 e^{-\eta T} N(h) - I e^{-rT} N\left(h - \frac{\sigma}{1-\nu} \sqrt{T}\right), \quad (A6)$$

where

$$h = \frac{\ln(V_0/X) + \left(r - \eta + \frac{1}{2} \frac{\sigma^2}{(1-\nu)^2}\right) T}{\frac{\sigma}{1-\nu} \sqrt{T}}.$$

For the second term in the right-hand side of (A5), we know at time t , $G_t = G_0 e^{(r-\beta^2\sigma^2/2)t + \beta\sigma\omega_t}$. So, by defining an equivalent probability measure, P_Q (see Duffie, 1998) and using the Girsanov theorem (Karatzas and Shreve, 1991), we can express the second term in the right-hand side of equation (A5) as:

$$\begin{aligned} E[e^{-rT}G_T 1_{\{V_T \geq X\}}] &= E[e^{-rT}G_0 e^{(r-\beta^2\sigma^2/2)T + \beta\sigma\omega_T} 1_{\{V_T \geq X\}}] = E_Q[G_0 1_{\{V_T \geq X\}}] \\ &= CV_0^{\beta(1-\nu)} N\left[h + \left(\beta - \frac{1}{1-\nu}\right) \sigma\sqrt{T}\right], \end{aligned} \quad (A7)$$

where the last line is obtained by using $G_0 = CV_0^{\beta(1-\nu)}$, and taking the expectation with respect to:

$$dV_t = \left(r - \eta + \frac{\beta\sigma^2}{1-\nu}\right) V_t dt + \frac{\sigma}{1-\nu} V_t d\omega_t^Q \quad (A8)$$

Finally, by applying the Black-Scholes formula to the integrand of the third term in equation (17), we have:

$$\int_0^T e^{-rt} E \left[(\eta V_t - rI) 1_{(V_t \geq V_t^*)} \right] dt = \int_0^T \left[\eta V_0 e^{-\eta t} N(d) - rI e^{-rt} N \left(d - \frac{\sigma}{1-\nu} \sqrt{t} \right) \right] dt, \quad (\text{A9})$$

where:

$$d = \frac{\ln(V_0/V_t^*) + \left[r - \eta + \frac{1}{2} \frac{\sigma^2}{(1-\nu)^2} \right] t}{\frac{\sigma}{1-\nu} \sqrt{t}}.$$

Therefore, we obtain a decomposition formula for the investment option as stated in Corollary 1.

Proof of Corollary 2

See Van Morebeke (1976) for the proof of continuity. Intuitively, continuity of the sample paths and the definition of the optimal investment threshold imply continuity of $V^*(\cdot)$. If $V^*(\cdot)$ were discontinuous, it would be possible for the value of $V(\cdot)$ to be strictly above $V^*(\cdot)$ without crossing it. This contradicts the definition of the optimal investment threshold. Now consider the second part of the proposition. Suppose there are two investment options $J(\tau_1, V)$ and $J(\tau_2, V)$, and the only difference between them is the time to expiration, τ_1 and τ_2 . Let $\tau_2 > \tau_1$, and we have $J(\tau_2, V) \geq J(\tau_1, V)$. Suppose that it is optimal to exercise $J(\tau_2, V)$, but not $J(\tau_1, V)$; then, $J(\tau_2, V) = V + CV^{\beta(1-\nu)} - I$ and $J(\tau_1, V) > V + CV^{\beta(1-\nu)} - I$. Thus $J(\tau_1, V) > J(\tau_2, V)$, which contradicts our proposed relationship. Therefore, the investment option $J(\tau_1, V)$ cannot be kept alive when $J(\tau_2, V)$ is optimally exercised. In other words, we must have $V^*(\tau_2) \geq V^*(\tau_1)$. This implies that the optimal investment threshold is non-decreasing in τ .

Proof of Corollary 3

Since the optimal investment threshold is non-decreasing in τ (the time to expiration), the maximum value is $V^*(\infty)$, which can be found from equation (19) by letting $\tau = \infty$. However, noticing that $V^*(\infty)$ is a time-invariant constant, we can find it by directly evaluating the initial investment option in an infinite-horizon setting. It is straightforward to show that when the option to invest has no time constraint, the optimal investment threshold is given by equation (20). Therefore $V^*(\infty)$ represents the upper bound for the optimal investment threshold. It is obvious that the optimal initial investment threshold should not be less than X , since X is the value of assets in place at which the value of the investment equals the cost of the investment.

Proof of Corollary 4

In order to investigate the behavior of V^* near the expiration, we rearrange equation (19) into:

$$\begin{aligned} \frac{V^*(\tau)}{I} &= \left[1 - e^{-r\tau} N \left(h - \frac{\sigma}{1-\nu} \sqrt{\tau} \right) - \int_0^\tau r e^{-r(\tau-t)} N \left(d - \frac{\sigma}{1-\nu} \sqrt{\tau-t} \right) dt \right] \\ &\times \left(C [V^*(\tau)]^{\beta(1-\nu)-1} \left\{ 1 - N \left[h + \left(\beta - \frac{1}{1-\nu} \right) \sigma \sqrt{\tau} \right] \right\} + 1 - e^{-\eta\tau} N(h) - \int_0^\tau \eta e^{-\eta(\tau-t)} N(d) dt \right)^{-1}. \end{aligned} \quad (\text{A10})$$

Let X be the unique solution to equation $f(V) = V + CV^{\beta(1-\nu)} - I = 0$. We have $I > X$. Using l'Hopital rule, we can evaluate the limit of the right-hand side of equation (20) as $\lim_{\tau \rightarrow 0} [V^*(\tau)/I] = r/\eta$. Since $\eta < r$, Corollary 4 shows that at the limit $V^*(0)$ is greater than I . This creates the discontinuity for the investment threshold at the expiration date. At the

expiration date, the investment would be made if $V(0) \geq X$, i.e. the value of the assets in place plus the growth options is no less than the investment cost I .

Proof of Corollary 5

Let $V_0 = Xe^{z(s)+(\rho-\alpha\rho-1/2)s}$ and $V^* = Xe^{z^*(u)+(\rho-\alpha\rho-1/2)u}$. By equation (21), in the new coordinate system the first three terms in Corollary 1 can be expressed as:

$$V_0 e^{-\eta T} N(h) = X e^{z(s)+(\rho-1/2)s} N\left[\frac{z(s)}{\sqrt{-s}} + \sqrt{-s}\right], \tag{A11}$$

$$I e^{-rT} N\left(h - \frac{\sigma}{1-\nu} \sqrt{T}\right) = I e^{\rho s} N\left[\frac{z(s)}{\sqrt{-s}}\right], \tag{A12}$$

$$G_0 N\left[h + \left(\beta - \frac{1}{1-\nu}\right) \sigma \sqrt{T}\right] = b \left(\frac{X}{a}\right)^{\beta(1-\nu)} e^{\beta(1-\nu)[z(s)+(\rho-\alpha\rho-1/2)s]} N\left[\frac{z(s)}{\sqrt{-s}} + \beta(1-\nu)\sqrt{-s}\right]. \tag{A13}$$

And the first integral in Corollary 1 is:

$$\begin{aligned} \eta V_0 \int_0^T e^{-\eta t} N(d) dt &= \frac{\eta(1-\nu)^2}{\sigma^2} X \int_s^0 e^{z(s)+(\rho-\alpha\rho-1/2)s-\alpha\rho(u-s)} N\left[\frac{z(s)-z^*(s)+u-s}{\sqrt{u-s}}\right] du \\ &= \int_0^{-s} \alpha\rho X e^{z(s)+(\rho-\alpha\rho-1/2)s-\alpha\rho t} N\left[\frac{z(s)-z^*(s+t)+t}{\sqrt{t}}\right] dt. \end{aligned} \tag{A14}$$

Finally, the last integral from Corollary 1 is:

$$\begin{aligned} rI \int_0^T e^{-rt} N\left(d - \frac{\sigma}{1-\nu} \sqrt{t}\right) dt &= \frac{r(1-\nu)^2}{\sigma^2} I \int_s^0 e^{-\rho(u-s)} N\left[\frac{z(s)-z^*(s)}{\sqrt{u-s}}\right] du \\ &= \rho I \int_0^{-s} e^{-\rho t} N\left[\frac{z(s)-z^*(s+t)}{\sqrt{t}}\right] dt. \end{aligned} \tag{A15}$$

Combining equations (A11), (A12), (A13), (A14) and (A15) yields equation (23) in Corollary 5. This completes the proof.