The Weighted Reversing Number of a Digraph

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1. Introduction

The set of inconsistencies in a tournament T for a given ranking are those arcs $(x,y) \in A(T)$ such that x is ranked above y. That is, the cases where player x "beats" player y, but y receives a better ranking than x. A number of ranking procedures for tournaments and their associated sets of inconsistencies have been described and studied in the literature. (See, for example, Slater [1961], Moon [1968].)

Following a question posed by J.P. Barthelemy, we can change our perspective and ask the following: For a given acyclic digraph D, is $A(D)$ the set of inconsistencies for some tournament under the ranking procedure under consideration? If so, determine the size of the smallest such tournament. In Barthelemy et al. [1991], this question is examined for the ranking procedure which minimizes the size of the set of inconsistencies. In this paper we examine this question for a ranking procedure which minimizes "weighted" inconsistencies. This procedure is shown to be equivalent to ranking by non-increasing outdegrees, i.e., score sequence. We call the number of extra vertices, i.e., those vertices besides $V(D)$, in a smallest tournament T with $A(D)$ as the set of inconsistencies under a ranking based on non-increasing outdegrees in T, the weighted reversing number of D and denote it by $w(D)$. We will show that $w(D)$ can be calculated by examining all acyclic orderings of D. For example, if D contains a Hamiltonian path, i.e., $V(D)$ has a unique acyclic ordering, then there is a polynomial procedure to compute $w(D)$. We also determine $w(D)$ for certain classes of acyclic digraphs. Finally, we compute an upper bound of $\left\lfloor \frac{n}{2} \right\rfloor - 1$ on $w(D)$, for any acyclic digraph D on $n \geq 4$ vertices, if $w(D)$ is defined and D has no isolated vertices.

A *tournament* $T = (V(T), A(T))$ is a digraph with vertex set $V(T)$ such that for each pair $x, y \in V(T)$, exactly one of the arcs (x,y) or (y,x) is in $A(T)$. Note that we will use V and A for the vertex set and arc set, respectively, when there is no possibility for ambiguity. We will also use D to refer to either the digraph D, the vertex set of D, and/or the arc set of D when there is no possibility for ambiguity. We think of $V(T)$ as the players in a competitive tournament in which each pair of players meets exactly once. The arc (x,y) indicates that player x "beats" player y. A digraph is *cyclic* if it contains a set of arcs of the form: $(x_1,x_2),(x_2,x_3),\ldots,(x_{m-1},x_m),(x_m,x_1)$. Otherwise, it is *acyclic*.

An *ordering* π of digraph D is a bijection of the form $\pi : V(D) \to \{1,2,\ldots,n\}$, where $\pi(v)$ is interpreted to be the spot in the order of player v and $\pi^{-1}(i)$ indicates the player ordered i^{th}. If $(x,y) \in A(D)$ implies $\pi(y) > \pi(x)$ then π is an *acyclic ordering* of D. Also, $d_D^+(v)$ will denote the outdegree of vertex v in D, i.e., the number of arcs of D of the form (v,w). Similarly, $d_D^-(v)$ will denote the indegree of vertex v in D, i.e., the number of arcs of D of the form (w,v). Finally, we let $n = |V(T)|$, unless otherwise indicated.

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A *ranking procedure* assigns to each tournament $T = (V, A)$ a set of rankings. Each ranking σ is a bijection of the form $\sigma : V \to \{1, 2, \ldots, n\}$, where $\sigma(v)$ is interpreted to be the rank of player v and $\sigma^{-1}(i)$ indicates the player ranked i^{th}. With respect to any ranking σ of a tournament T, the *set of inconsistencies*, $I_\sigma(T)$, equals $\{(x, y) \in A(T) : \sigma(x) > \sigma(y)\} = \{(\sigma^{-1}(i), \sigma^{-1}(j)) \in A(T) : i > j\}$. It is easy to see that $I_\sigma(T)$ induces an acyclic digraph.

Assume that a certain ranking procedure $\mathcal{R}$ is under consideration. Given an acyclic digraph $D = (V, A)$, we define the $\mathcal{R}$ *inconsistency realizing number* of D to be the number of extra vertices, i.e., $|V(T) \setminus V(D)|$, in a smallest tournament T for which there is some ranking σ assigned to T under $\mathcal{R}$ such that $D = I_\sigma(T)$. We say that such a T *realizes* D under $\mathcal{R}$. If there is no T realizing D under $\mathcal{R}$ then we define the $\mathcal{R}$ inconsistency realizing number of D to be infinite.

Given a tournament T and a ranking σ of T, let

$$R_\sigma = \sum_{\substack{(\sigma^{-1}(j), \sigma^{-1}(i)) \in T \\ i<j}} 1 = \sum_{\substack{(x,y) \in T \\ \sigma(y) < \sigma(x)}} 1.$$

Then $R_\sigma = |I_\sigma(T)|$. We call the ranking procedure which assigns to each tournament those rankings which minimize R_σ, the *minimum inconsistencies ranking procedure*. As noted in Barthelemy et al. [1991], it has been shown by a number of authors that the sets of inconsistencies under this ranking procedure are equivalent to minimum feedback arc sets, minimum transversals of the cycles of the tournament, and the complement of a maximum acyclic subdigraph. The inconsistency realizing number with respect to this procedure is examined in Barthelemy et al. [1991]. They call this the *reversing number* and denote it by $r(D)$. In this paper we examine a weighted version of the minimum inconsistencies ranking procedure.

2. Weighted Reversing Number

In a minimum inconsistencies ranking, inconsistencies in which the player ranked n^{th} beats the player ranked first and the player ranked second beats the player ranked first are considered "equally bad." It seems reasonable to consider a ranking procedure which gives some weight to the difference of the rankings of players who are ranked inconsistently. The simplest such weighting assigns a weight of $j - i$ to the inconsistency $(x, y) \in T$ if y is ranked i^{th} and x is ranked j^{th}. Note that the weights are always positive. Given a tournament T and a ranking σ of T, let

$$W_\sigma = \sum_{\substack{(\sigma^{-1}(j), \sigma^{-1}(i)) \in T \\ i<j}} j - i = \sum_{\substack{(x,y) \in T \\ \sigma(y) < \sigma(x)}} \sigma(x) - \sigma(y).$$

We define the *minimum weighted inconsistencies ranking procedure* to be the procedure which assigns to each tournament all rankings which minimize W_σ.

Let the *score ranking procedure* be the procedure which assigns to each tournament all rankings such that

$$d_T^+(\sigma^{-1}(1)) \geq d_T^+(\sigma^{-1}(2)) \geq \cdots \geq d_T^+(\sigma^{-1}(n)),$$

equivalently,

$$d_T^-(\sigma^{-1}(1)) \leq d_T^-(\sigma^{-1}(2)) \leq \cdots \leq d_T^-(\sigma^{-1}(n)).$$

Thus the score ranking procedure ranks by non-increasing outdegrees, equivalently, by non-decreasing indegrees.

We make the following observation. For completeness, we include a short proof.

Observation: *The minimum weighted inconsistencies ranking procedure and the score ranking procedure are the same procedures.*

Proof: Let σ be a fixed ranking of tournament T. We may assume that the vertices are labeled so that $\sigma^{-1}(i) = i$. For $X \subseteq V(T)$, let $in(X)$ denote the number of arcs (i,j) such that $i \notin X$ and $j \in X$, i.e., the number of arcs entering X. Hence,

$$in(X) = \left[\sum_{v \in X} d^-(v)\right] - \binom{|X|}{2}.$$

Thus,

$$\begin{aligned} W_\sigma = \sum_{\substack{(j,i)\in T \\ i<j}} j - i &= \sum_{k=1}^{n-1} \sum_{\substack{i \le k < j \\ (j,i)\in T}} 1 \\ &= \sum_{k=1}^{n-1} in(\{1,2,\ldots,k\}) \\ &= \sum_{k=1}^{n-1} \left[\sum_{j=1}^{k} d^-(j) - \binom{k}{2}\right] \\ &= -\sum_{k=1}^{n-1} \binom{k}{2} + \sum_{k=1}^{n-1} (n-k) d^-(k). \end{aligned}$$

The first term is fixed for a given tournament and the second is clearly minimized by non-decreasing indegrees. □

Let $w(D)$, the *weighted reversing number of* D, be the inconsistency realizing number with respect to the minimum weighted inconsistencies ranking procedure. Given a digraph D, let the *surplus* of v in D be given by

$$e_D(v) = \begin{cases} d_D^+(v) - d_D^-(v) & v \in V(D) \\ 0 & v \notin V(D). \end{cases}$$

The following lemma will play an important role in examing $w(D)$.

Lemma 1 *Let T be a tournament, σ be a ranking of $V(T)$, and assume that the vertices of T are labeled so that $\sigma^{-1}(i) = i$. Let D be the digraph induced by the arcs which are inconsistent with the ranking (i.e., $D = I_\sigma(T)$). The ranking σ is assigned to T under the minimum weighted inconsistencies ranking procedure if and only if*

$$e_D(j) \le e_D(j-1) + 1, \ \ j = 2, 3, \ldots, n.$$

Proof: Note that if $I_\sigma(T) = \emptyset$, then the outdegree of vertex i equals $n - i$ since $\sigma^{-1}(i) = i$. If $I_\sigma(T) \neq \emptyset$, then the arcs which are inconsistent with σ are the arcs of D. In either case, $d_T^+(i) = d_D^+(i) - d_D^-(i) + n - i$. Therefore, since the minimum weighted inconsistencies ranking procedure and the score ranking procedure are the same procedures,

$$
\begin{array}{rcl}
d_T^+(j) & \leq & d_T^+(j-1) \\
 & \Updownarrow & \\
d_D^+(j) - d_D^-(j) + n - j & \leq & d_D^+(j-1) - d_D^-(j-1) + n - (j-1) \\
 & \Updownarrow & \\
e_D(j) + n - j & \leq & e_D(j-1) + n - (j-1) \\
 & \Updownarrow & \\
e_D(j) & \leq & e_D(j-1) + 1.
\end{array}
\qquad \square
$$

Let T be a tournament on $V(D) \cup X$ realizing D under the minimum weighted inconsistencies ranking σ. Then $D = I_\sigma(T)$ and there is no tournament T' with $|V(T')| < |V(T)|$ and $I_{\sigma'}(T') = D$ for some σ' assigned to T' under the minimum weighted inconsistencies ranking procedure. Clearly, if T realizes D under the minimum weighted inconsistencies ranking σ then $\sigma^{-1}(1), \sigma^{-1}(n) \notin X$. If $\sigma^{-1}(1) \in X$, then a contradiction is reached by removing it and hence producing a smaller tournament with D as the set of inconsistencies under the same (induced) ranking. A similar contradiction is reached if $\sigma^{-1}(n) \in X$.

The following lemma shows that no two vertices in X can appear consecutively in a minimum weighted inconsistencies ranking and that these vertices appear only under restricted conditions.

Lemma 2 *Let T be a tournament on $V(D) \cup X$ realizing D under the minimum weighted inconsistencies ranking σ. Assume that $\sigma^{-1}(i) = i$. If $x \in X$, then $x-1$ and $x+1$ are both in $V(D)$ and furthermore, $e_D(x-1) = -1$ and $e_D(x+1) = +1$.*

Proof: Consider a maximal consecutive sequence in the ranking σ containing k vertices all of which are from X. That is, assume that for some $i < j$, we have $x \in X$ for $i < x < j$, $j - i = k + 1$, and $i, j \in V(T) \setminus X = V(D)$. Let $X' = \{i+1, \ldots, j-1\}$ and $n = |V(D) \cup X|$. By Lemma 1,

$$e_D(j)+(n-j) \leq e_D(j-1)+(n-j+1) \leq \cdots \leq e_D(i+1))+(n-i-1) \leq e_D(i)+(n-i).$$

Recalling that $e_D(x) = 0$ for $x \in X$, i.e., $x \notin V(D)$,

$$e_D(j) + (n-j) \leq n - j + 1 \leq n - i - 1 \leq e_D(i) + (n-i).$$

So, $e_D(j) \leq 1$ and $e_D(i) \geq -1$. Also, $e_D(j) \leq e_D(i) + (j - i)$.

Consider the tournament T', with the vertices of X' deleted, under the ranking σ' consistent with σ (i.e., $\sigma^{-1}(l) = l$ for $l = 1, \ldots, i$ and $\sigma^{-1}(l) = l + (j - i - 1)$ for $l = i+1, \ldots, n-(j-i-1)$). This amounts to setting $j = i+1$, $j+1 = i+2$, …, and $n = n - (j - i - 1)$. Note that D is the set of arcs inconsistent in T' under σ'. In T' under σ', the condition of Lemma 1 holds except possibly at $j = i + 1$. If $e_D(j) < 1$ or if $e_D(i) > -1$, then

$$e_D(j) \leq e_D(i) + 1.$$

Hence, by Lemma 1, σ' is a minimum weighted inconsistencies ranking of T', contradicting the minimality of T. Thus, $e_D(i) = -1$ and $e_D(j) = 1$.

Finally, if $|X'| > 1$, then consider the tournament $T' = (T \setminus X') \cup \{i+1\}$, under the ranking σ' consistent with σ which amounts to setting $j = i+2$, $j+1 = i+3$, …, and $n = n - (j - i - 2)$. Note that D is the set of arcs inconsistent in T' under σ'. In T' under σ', the condition of Lemma 1 holds except possibly at $j = i + 2$. However,

$$e_D(j) \leq e_D(i+1) + 1$$

since $e_D(j) = 1$ and $e_D(i+1) = 0$. Hence, by Lemma 1, σ' is a minimum weighted inconsistencies ranking of T', contradicting the minimality of T. Thus, $|X'| = 1$, with $x + 1 = j$ and $x - 1 = i$. □

Theorem 3 *An acyclic digraph D has $w(D)$ finite if and only if there is an acyclic ordering π of D such that for $i = 1, \ldots, |V(D)| - 1$, either*

$$e_D(\pi^{-1}(i)) \leq e_D(\pi^{-1}(i+1)) + 1 \quad (1)$$

or

$$e_D(\pi^{-1}(i)) = 1 \text{ and } e_D(\pi^{-1}(i+1)) = -1. \quad (2)$$

Furthermore, let w_π be the number of occurrences of (2) under π. Then $w(D)$ is the minimum value of w_π under all acyclic orderings π of D.

Proof: The proof follows immediately from Lemma 2. □

Note that Theorem 3 gives a complete characterization and a polynomial procedure for determining $w(D)$ when acyclic digraph D contains a Hamiltonian path (i.e., has a unique acyclic ordering).

Corollary 4 *If acyclic digraph D contains a Hamiltonian path and $1, 2 \ldots, n$ is the unique acyclic ordering, then $w(D)$ is infinite if $e_D(i) > e_D(i+1) + 2$ for some $1 \leq i < n$ or if $e_D(i) = e_D(i+1) + 2$ with $e_D(i) \neq 1$ for some $1 \leq i < n$. Otherwise, $w(D)$ is equal to the number of i such that $e_D(i) = 1$ and $e_D(i+1) = -1$.*

3. Exact Values for $w(D)$

Theorem 3 provides an immediate characterization of $w(D)$ for certain classes of acyclic digraphs.

Corollary 5 *Let P_n be the directed path on n vertices. Then, $w(P_2) = 1$ and $w(P_n) = 0$ for $n \geq 3$.*

Proof: Let $V(P_n) = \{1, 2, \ldots, n\}$ and $A(P_n) = \{(i, i+1) \mid i = 1, \ldots, n-1\}$. For $n = 2$, P_2 is a single arc and the unique acyclic ordering $1, 2$ satisfies $e_{P_2}(1) = 1$ and $e_{P_2}(2) = -1$. So, by Theorem 3, $w(P_2) = 1$. For $n \geq 3$, the unique acyclic ordering $1, 2, \ldots, n$ satisfies $e_{P_n}(1) = 1$, $e_{P_n}(n) = -1$ and $e_{P_n}(i) = 0$ for $2 \leq i \leq n-1$. So, by Theorem 3, $w(P_n) = 0$. □

Corollary 6 *Let T_n be the acyclic tournament on n vertices. Then, $w(T_2) = 1$ and $w(T_n)$ is infinite for $n \geq 3$.*

Proof: $T_2 = P_2$. Hence, $w(T_2) = 1$ by Corollary 5. For $n \geq 3$, let T_n be labeled so that $(i,j) \in T_n$ if and only if $i < j$. Then, the unique acyclic ordering of T_n is $1, 2, \ldots, n$. Hence, $e_D(1) = n - 1 > 1$ and $e_D(2) = n - 3$. So, by Theorem 3, $w(T_n)$ is infinite. □

An *alternating path* AP_n is a digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set either

$$\{(i, i-1), (i, i+1) : i \text{ is odd, and both vertices are in } V\}$$

or

$$\{(i, i-1), (i, i+1) : i \text{ is even, and both vertices are in } V\}.$$

Corollary 7 *Let AP_n be an alternating path on $n \geq 2$ vertices. Then $w(AP_n) = 1$ if n is even and $w(AP_n)$ is infinite if n is odd.*

Proof: When $n = 3$, the set of e_{AP_n} values is either $\{-2, +1\}$ or $\{-1, +2\}$. When n is odd, $n \geq 5$, the set of e_{AP_n} values is either $\{-2, +1, +2\}$ or $\{-2, -1, +2\}$. Clearly, there is no acyclic ordering satisfying the conditions of Theorem 3.

If $n = 2$, then $AP_2 = P_2$ and $w(P_2) = 1$ by Corollary 5.

For n even, $n \geq 4$, we may assume that AP_n is labeled so that the arc set of AP_n is $\{(i, i-1), (i, i+1) : i \text{ odd}, 3 \leq i \leq n-1\} \cup \{(1,2)\}$. Hence, $e_{AP_n}(1) = +1$, $e_{AP_n}(n) = -1$, $e_{AP_n}(i) = +2$ for i odd, $3 \leq i \leq n-1$ and $e_{AP_n}(i) = -2$ for i even, $2 \leq i \leq n-2$. Thus, the set of e_{AP_n} values is $\{+2, +1, -1, -2\}$. For any acyclic ordering π of AP_n satisfying Theorem 3, it is easy to check that (2) in Theorem 3 must occur at least once, i.e., $w(AP_n) \geq 1$. Let π be any ordering of $V(AP_n)$ with $\pi(1) = n/2$, $\pi(n) = n/2 + 1$, $\pi(i) < n/2$ for i odd, and $\pi(i) > n/2 + 1$ for i even. It is not difficult to check that π is an acyclic ordering of AP_n. Note that either (1) or (2) in Theorem 3 holds for $i = 1, 2, \ldots, n$. That is, $w(AP_n)$ is finite. Also, (2) occurs exactly once. So, by Theorem 3, $w(AP_n) = 1$. □

A connected graph which is regular of degree two is called a *circuit graph*. As a special case, the next corollary applies to acyclic orientations of circuit graphs.

Corollary 8 *Let R_n be an acyclic digraph on n vertices whose underlying graph is r-regular. Then*

$$w(R_n) = \begin{cases} 1 & r = 1 \\ \textit{infinite} & \textit{else.} \end{cases}$$

Proof: If R_n is 1−regular, then n must be even and the set of e_{R_n} values is $\{-1, +1\}$. So, $w(R_n) \geq 1$ by Theorem 3. We finish the proof of this case by showing that $w(R_n) \leq 1$. Without loss of generality, let $A(R_n) = \{(1,2), (3,4), \ldots, (n-1, n)\}$. Note that $1, 3, \ldots, n-1, 2, 4, \ldots, n$ is an acyclic ordering of R_n and by Theorem 3, $w(R_n) \leq 1$.

Suppose that $r \geq 2$. Note that for any acyclic ordering π of R_n, $e_{R_n}(\pi^{-1}(1)) = r$ and $e_{R_n}(\pi^{-1}(n)) = -r$. Also, the absolute difference between any two e_{R_n} values is even. So, since the e_{R_n} values are a subset of $\{-r, -r+2, \ldots, r-2, r\}$, there is no acyclic ordering satisfying the conditions of Theorem 3. Thus, $w(R_n)$ is infinite for $r \geq 2$. □

Next, we consider unions of disjoint acyclic digraphs.

Corollary 9 *Let T_p be the acyclic tournament on $p \geq 2$ vertices. Then,*

$$w(T_n \cup T_m) = \begin{cases} 1 & m = n = 2 \\ 0 & |m-n| = 1 \\ \textit{infinite} & \textit{else.} \end{cases}$$

Proof: Let T_n be labeled so that $(x_i, x_j) \in T$ if and only if $i < j$. Next, let T_m be labeled so that $(y_i, y_j) \in T$ if and only if $i < j$. Then, the unique acyclic orderings of T_n and T_m are $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$, respectively.

Without loss of generality, we may assume that $m \leq n$. If $m = n = 2$, then, by Corollary 8, $w(T_2 \cup T_2) = 1$. If $m = n \neq 2$, then, by Corollary 8, $w(T_n \cup T_n)$ is infinite.

If $m = n-1$, then $x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1}, x_n$ is an acyclic ordering of $T_n \cup T_{n-1}$. This ordering satisfies (1) and (2) of Theorem 3 and has no occurences of (2). Thus, $w(T_n \cup T_{n-1}) = 0$.

If $m \leq n-2$, note that $e_{T_n \cup T_m}(x_1) = n - 1 > 1$. Since $m \leq n-2$, no other vertex has surplus greater than $n-3$. Thus, no acyclic ordering of $T_n \cup T_m$ satisfies the conditions of Theorem 3. □

4. Bounds on $w(D)$

In this section we give upper and lower bounds on $w(D)$ (when it is finite).

Theorem 10 *Let D be an acyclic digraph with $w(D)$ finite, containing no isolated vertices, and $n = |V(D)|$. Then,*

$$w(D) = 1 \text{ if } n = 2$$

$$w(D) = 0 \text{ if } n = 3$$

$$0 \leq w(D) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ if } n \geq 4$$

and both bounds are attained.

Proof: If $n = 2$ then $D = P_2$. By Corollary 5, $w(P_2) = 1$, i.e., $w(D) = 1$. If $n = 3$, then the only acyclic digraphs on three vertices with no isolated vertices are T_3, P_3, or AP_3. By Corollaries 6 and 7, $w(D)$ is infinite except when $D = P_3$ in which case $w(D) = 0$ by Corollary 5.

For $n \geq 4$, $w(P_n) = 0$ by Corollary 5, so the lower bound is always attained.

Consider $n \geq 4$. Let D be an acyclic digraph on $|V(D)| = n$ vertices with $w(D)$ finite so that $w(D) \geq w(D')$ for all other acyclic digraphs D' on n vertices with $w(D')$ finite. Assume that tournament T realizes D under the minimum weighted inconsistencies ranking σ. Let $V(T) = V(D) \cup X$. Without loss of generality we may assume that $\sigma^{-1}(i) = i$. Let $m = |V(D)| + |X|$. We have already noted that $1, m \notin X$.

We next show that if $w(D) \geq 2$, then $2, m-1 \notin X$. If $2 \in X$, then, by Lemma 2, $3 \in V(D)$ and $e_D(3) = 1$ and $e_D(1) = -1$. Since $e_D(3) = 1$, there must be at least one arc of D of the form $(3, i)$ with $3 > i$, since the arcs of D are inconsistent with the ordering σ. Since $2 \notin V(D)$, the only such arc is $(3,1)$ and thus there is no arc of the form $(i, 3)$, $i > 3$ in D. Also, since $e_D(1) = -1$ and since there can be no

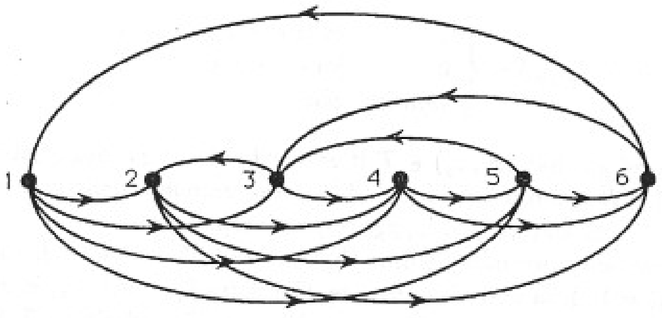

Figure 1.

arc of the form $(1,i)$, $i > 1$ in D, $(3,1)$ is the only arc incident to 1. Thus, the arc $(3,1)$ is disconnected from the rest of D. Note then that there are no arcs of the form $(4,i)$, $i \leq 4$. So, $e_D(4) \leq -1$. If $w(D) \geq 2$, then there is some $k > 3$ such that $k \in X$, $k-1, k+1 \in V(D)$ and $e_D(k+1) = 1$ and $e_D(k-1) = -1$. Then, consider the ordering δ on $(V(D) \cup X) \setminus \{2\}$ obtained from σ by letting $\delta^{-1}(1) = 1$, $\delta^{-1}(i-2) = i$ for $4 \leq i \leq k$, $\delta^{-1}(k-1) = 3$, and $\delta^{-1}(i-1) = i$ for $k+1 \leq i \leq m$. It is easily checked that δ is a minimum weighted inconsistencies ranking, and that D is the set of arcs inconsistent with δ. This contradicts the assumption that T realizes D under the minimum weighted inconsistencies ranking procedure.

A symmetric argument shows that if $w(D) \geq 2$, then $m - 1 \notin X$.

Now, we have either $w(D) \leq 1$ or $w(D) \geq 2$ and $2, m-1 \notin X$. In the second case, $1, 2, (m-1), m \notin X$. Note that if $i \in X$, then by Lemma 2, $i+1 \in V(D)$ and $e_D(i+1) = +1$. So, $e_D(i+1) \neq -1$ and again by Lemma 2, $i+2 \notin X$. So, if $\{i,j\} \subseteq X$, with say $j > i$, then $j - i > 2$. Thus, $|X| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

If $n = 4$ or 5 and if $w(D) \geq 2$, then by the above bound, $w(D) \leq 1$, a contradiction. So $w(D) \leq 1$ in these cases and the examples below show that $w(D) = 1$ can be attained.

We now construct acyclic digraphs D_n on $n \geq 4$ vertices such that $w(D_n) = \left\lfloor \frac{n}{2} \right\rfloor - 1$, completing the proof. For $n = 4$, the alternating path AP_4 has $w(AP_4) = 1$ by Corollary 7. For $n = 5$, consider the acyclic digraph D' with $V(D') = \{1,2,3,5,6\}$ and $A(D') = \{(6,3),(6,1),(5,3),(3,2)\}$. Let T be the tournament with $V(T) = \{1,2,3,4,5,6\}$ and arcs (j,i) for $j < i$ except for the arcs of D'. Then T has $A(D')$ as the set of inconsistent arcs with the ranking $\sigma^{-1}(i) = i$. (See Figure 1.) Note that

$$e_{D'}(1) = -1,\ e_{D'}(2) = -1,\ e_{D'}(3) = -1,\ e_{D'}(4) = 0,\ e_{D'}(5) = 1, \text{ and } e_{D'}(6) = 2.$$

So, by Lemma 1, σ is assigned to T under the minimum weighted inconsistencies ranking procedure. From the surplus values of the vertices of D', it is clear that any acyclic ordering of D' satisfying Theorem 3 has at least one "jump" from $e_{D'} = +1$ to $e_{D'} = -1$. That is, for any acyclic ordering π of D' satisfying Theorem 3, there is at least one occurence of $e_{D'}(\pi^{-1}(i)) = +1$ and $e_{D'}(\pi^{-1}(i+1)) = +1$. Hence, by Theorem 3, $w(D') \geq 1$ and from the tournament T we see that $w(D') = 1$.

For n even, $n \geq 6$, construct the digraph D_n on n vertices as follows. Let $u = \left\lfloor \frac{n}{2} \right\rfloor - 1 = \frac{n-2}{2}$ and $m = n + u = 3u + 2$. Let $V(D_n) = \{3i+1, 3i+2 : i = 0, 1, \ldots, u\}$

and

$$\begin{aligned} A(D_n) &= \{(m,m-3),(m,m-6),(4,1)\}\cup \\ &\quad \{(3i+1,3i-1): i=1,\ldots,u\}\cup \\ &\quad \{(3i-1,3i-2): i=1,\ldots,u\}\cup \\ &\quad \{(3i+1,3i-4): i=2,\ldots,u-1\}. \end{aligned}$$

(In the case $n=6$, the last set is empty.) Let T be the tournament with $V(T)=\{1,\ldots,m\}=V(D_n)\cup X=V(D_n)\cup\{3i: i=1,\ldots,u\}$ and arcs (i,j) for $i<j$ except for the arcs of D_n. Thus, $A(D_n)$ is the set of inconsistent arcs under the ordering σ where $\sigma^{-1}(i)=i$. Also, it can be easily checked that the surplus values for the vertices in T are:

$$e_{D_n}(3i)=0 \text{ for } i=1,\ldots,u,$$

$$e_{D_n}(3i+1)=+1 \text{ for } i=1,\ldots,u,$$

$$e_{D_n}(3i+2)=-1 \text{ for } i=0,\ldots,u-1,$$

$$e_{D_n}(1)=-2 \text{ and } e_D(m)=+2.$$

Again, it is easily checked that the ordering σ of $V(T)$ is a minimum weighted inconsistencies ordering by Lemma 1. Finally, note that D_n contains the directed path $3u+1, 3u-1, 3(u-1)+1, 3(u-1)-1, \ldots, 4, 2$. Then the sequence of surplus values along the path is $+1$, -1, $+1$, ..., $+1$, -1 with u "jumps." Since the remaining two vertices in $V(D_n)$, namely, 1 and m, have surplus values of -2 and 2, respectively, any acyclic ordering of D_n has at least u "jumps." Thus, by Theorem 3, $w(D_n)\geq u$ and from the tournament T we see that $w(D_n)=u$.

Similarly, for n odd, $n\geq 7$, construct the digraph D_n on n vertices as follows. Let $u=\left\lfloor\frac{n}{2}\right\rfloor-1=\frac{n-3}{2}$ and $m=n+u=3(u+1)$. Let $V(D_n)=\{3i+2, 3i+3: i=0,1,\ldots,u\}\cup\{1\}$ and

$$\begin{aligned} A(D_n) &= \{(m,m-3),(m,m-6),(5,1)\}\cup \\ &\quad \{(3i+2,3i): i=1,\ldots,u\}\cup \\ &\quad \{(3i,3i-1): i=1,\ldots,u\}\cup \\ &\quad \{(3i+2,3i-3): i=2,\ldots,u-1\}. \end{aligned}$$

(In the case $n=7$, the last set is empty.) Let T be the tournament with $V(T)=\{1,\ldots,m\}=V(D_n)\cup X=V(D_n)\cup\{3i+1: i=1,\ldots,u\}$ and arcs (i,j) for $i<j$ except for the arcs of D_n. Thus, $A(D_n)$ is the set of inconsistent arcs under the ordering σ where $\sigma^{-1}(i)=i$. Also, it can be easily checked that the surplus values for the vertices in T are:

$$e_{D_n}(3i+1)=0 \text{ for } i=1,\ldots,u,$$

$$e_{D_n}(3i)=-1 \text{ for } i=1,\ldots,u,$$

$$e_{D_n}(3i+2)=+1 \text{ for } i=1,\ldots,u,$$

$$e_{D_n}(1)=e_D(2)=-1 \text{ and } e_D(m)=+2.$$

Again, it is easily checked that the ordering σ of $V(T)$ is a minimum weighted inconsistencies ordering by Lemma 1. Finally, note that D_n contains the directed path $3u+2, 3u, 3(u-1)+2, 3(u-1), \ldots, 5, 3$. Then the sequence of surplus values along the path is $+1$, -1, $+1$, ..., $+1$, -1 with u "jumps." Since the remaining three vertices in $V(D_n)$, namely, 1, 2 and m, have surplus values of -1, -1 and 2, respectively, any acyclic ordering of D_n has at least u "jumps." Thus, by Theorem 3, $w(D_n) \geq u$ and from the tournament T we see that $w(D_n) = u$. □

5. Bibliography

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