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Proper and unit tolerance graphs

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Abstract

We answer a question of Golumbic, Monma and Trotter by constructing proper tolerance graphs that are not unit tolerance graphs. An infinite family of graphs that are minimal in this respect is specified.

1. Introduction

Tolerance graphs, introduced by Golumbic and Monma [2], are a generalization of interval graphs in which each vertex can be represented by an interval and a tolerance such that an edge occurs if and only if the overlap of the corresponding intervals is at least as large as the tolerance associated with one of the vertices. One can think of this as a model of conflicts for events occurring in a block of time, in which a tolerance of acceptable overlap is associated with each interval. Tolerance graphs have been examined in Refs. [2, 3, 7, 4]. Jacobson et al. [5], and Jacobson et al. [6] examine a more general scheme of tolerance intersection graphs.

A *unit* interval representation is an interval representation in which all intervals have the same length. A *proper* interval representation is one in which no interval is properly contained in another. These terms can apply to either interval graphs or tolerance graphs. It is known [8] that the classes of unit and proper interval graphs are equal. Golumbic et al. [3] asked whether this is also true for tolerance graphs. It is obvious that unit tolerance graphs are proper tolerance graphs; is the converse true? McMorris and Jacobson [4] showed equivalence between unit and proper intervals

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for sum-tolerance graphs in which an edge occurs if the overlap is larger than the sum of the tolerances.

We consider only finite graphs in this paper and show that the unit and proper tolerance graph classes are not the same. In particular, we construct an infinite family of graphs which are proper tolerance graphs, are not unit tolerance graphs, and are minimal in this respect. Thus, our family is included among the minimal forbidden subgraphs separating proper from unit tolerance graphs.

We introduce another type of tolerance graph, called a 50% tolerance graph, in which each tolerance is equal to half the length of the corresponding interval. Thus, in the conflict model, there is no edge if both intervals are free from conflict with one another at least half the time. Note that 50% tolerance representations are not necessarily proper representations. We show that the classes of 50% tolerance graphs and unit tolerance graphs are equal and use this to facilitate the proof that the graphs in our special family are not unit tolerance graphs. We can in general discuss p% tolerance graphs were the tolerance is p% of the interval length. Then, as shown in [2], 100% tolerance graphs are permutation graphs.

Formally, a tolerance graph is a graph $G = (V, E)$ which has a *tolerance representation* $(\mathscr{I}, T)$, where $\mathscr{I}$ and T are maps from the vertex set V to closed real intervals and positive real numbers, respectively. The edges are given by

$$\{x, y\} \in E \Leftrightarrow |I_x \cap I_y| \geqslant \min\{t_x, t_y\}.$$

Here I_x denotes $\mathscr{I}(x)$, t_x denotes $T(x)$ and $|I_x|$ denotes the length of interval I_x. Since we are dealing with finite graphs, we can presume that in our representations all intervals have finite lengths and that all endpoints and centers are distinct, unless otherwise noted. We will use two different descriptions of the intervals, using right and left endpoints r_x and l_x or centers and half-lengths c_x and h_x.

A representation is *bounded* if $t_x \leqslant |I_x|$ for all vertices x. A tolerance graph is bounded if it has a bounded tolerance representation. Proper and unit tolerance representations may be assumed to be bounded since the intersection of an interval with any other is less than that interval's length. So unbounded tolerances can be reduced to the interval length without affecting the representation. The 50% tolerance graphs have $h_x = t_x$ for all $x \in V$ and an edge if and only if at least one interval contains the center of the other.

Other terms from graph theory that we do not define here can be found in [1].

2. 50% tolerance graphs

We show the equivalence of unit and 50% tolerance graphs.

Theorem 1. *G is a unit tolerance graph if and only if it is a 50% tolerance graph. Moreover the sets of orderings of centers in the possible representations of the two types are identical.*

Proof. Let $(\mathscr{I}, T)$ be a unit representation of G. Assume without loss of generality that $t_v \leqslant 1$ for all vertices v. (Since the endpoints are distinct, $|I_x \cap I_y| < 1$ for all vertices x, y and so tolerances greater than one can be set to be one without affecting the representation.) Form a 50% representation $(\mathscr{I}', T')$ in which $c'_v = c_v$ and $t'_v = h'_v = 1 - t_v$ for each $v \in V$. Conversely, suppose that a 50% tolerance representation $(\mathscr{I}', T')$ is given. By scaling, we may assume that all half-lengths are at most 1. Form a unit representation $(\mathscr{I}, T)$ as follows. Let $c_v = c'_v$, $h_v = \frac{1}{2}$, and $t_v = 1 - t'_v = 1 - h'_v$ for each $v \in V$.

Then, for $c_x < c_y$, and assuming neither I'_x nor I'_y is contained in the other, we have

$$|I_x \cap I_y| = c_x + \tfrac{1}{2} - (c_y - \tfrac{1}{2}) \geqslant \min\{t_x, t_y\}$$

$$\Leftrightarrow \quad c_x - c_y + 1 \geqslant \min\{1 - t_x, 1 - t_y\} + t_x + t_y - 1$$

$$\Leftrightarrow \quad (c_x + (1 - t_x)) - (c_y - (1 - t_y)) \geqslant \min\{1 - t_x, 1 - t_y\}$$

$$\Leftrightarrow \quad |I'_x \cap I'_y| = c_x + h'_x - (c_y - h'_y) \geqslant \min\{1 - t_x, 1 - t_y\} = \min\{t'_x, t'_y\}.$$

If, say $I'_y \subseteq I'_x$ (the case $I'_x \subseteq I'_y$ is symmetric), then the right endpoint of I'_y is less than or equal to the right endpoint of I'_x. Therefore

$$c_x + 1 - t_x = c_x + h'_x \geqslant c_y + h'_y = c_y + 1 - t_y.$$

Then

$$|I_x \cap I_y| = c_x - c_y + 1 \geqslant 1 - t_y + t_x \geqslant t_x.$$

Thus, $\{x, y\}$ is an edge in both representations. (It is an edge in the 50% representation since the tolerances are bounded and $I'_y \subseteq I'_x$.) □

An alternative to the previous proof is to consider *parallelogram* graphs; i.e., intersection graphs of parallelograms each of which has its horizontal lines on two parallel lines and the connecting lines all have positive slope (or all negative slope). Given a bounded representation of a tolerance graph, let $l_x = c_x - h_x$ and $r_x = c_x + h_x$. Form a parallelogram with corners $(l_x, 1), (r_x - t_x, 1), (l_x + t_x, 0), (r_x, 0)$. It is not difficult to check that the graph which is the intersections graph of the parallelograms is the tolerance graph. Conversely, a parallelogram graph can be converted to a bounded tolerance representation; see Fig. 1 for example. Note that this implies that bounded tolerance graphs are a subclass of trapezoid graphs, providing a quick proof of the fact that these are co-comparability graphs (shown in [2]). Since these are co-comparability graphs, efficient algorithms for a variety of graph theoretic problems are known (see for example [1]). Efficient algorithms for the stability and chromatic number of general (not necessarily bounded) tolerance graphs are given in [7]. (Note that since tolerance graphs are perfect, the ellipsoid method

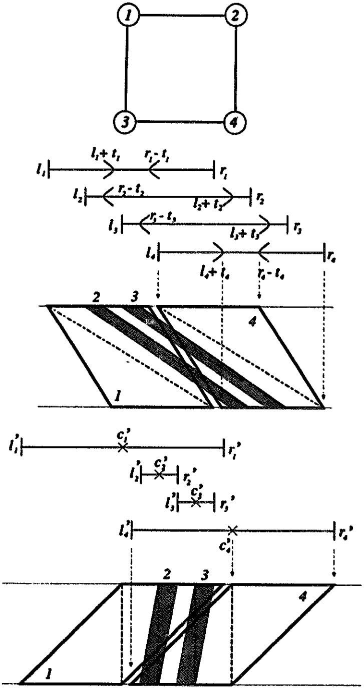

Fig. 1. (a) A four cycle and (b) a unit tolerance and corresponding parallelogram representation of the cycle and (c) a 50% and corresponding parallelogram representation. (Diagonals in the parallelograms are represented with dashed lines. The correspondance between the interval and the parallelogram for vertex four is indicated by vertical arrows.)

also provides polynomial algorithms for these parameters.) However, no efficient algorithm for recognition of any of the classes of tolerance graphs has been described.

To see the connection between unit tolerance and 50% tolerance graphs using parallelograms, note that in a unit parallelogram representation all diagonals between $(l_x, 1)$ and $(r_x, 0)$ have the same slope. As shown in Fig. 1, shift the line with second coordinate 1 to the right until all these diagonals are vertical. Intersections of trapezoids have not changed, and the new parallelograms give rise to a 50% representation.

3. Unique orderings

In order to show that it is enough to consider fixed orderings of centers when looking for a counterexample to the proper = unit question, we first examine some induced subgraphs which force certain orderings of the centers. We begin with the graph consisting of two disjoint edges (the complement of the complete bipartite graph $K_{2,2}$), making use of the following lemma which says that orienting edges in the complement of a bounded tolerance graph based on the ordering of the centers in a bounded representation produces a transitive orientation. The following proof is implied in [2], but we include it here for completeness.

Lemma 2. *Let $(\mathscr{I}, T)$ be a bounded tolerance representation of a graph G. Orient each edge $\{x, y\}$ in the complement of G from x to y if $c_x < c_y$. Then the orientation of the complement of G is transitive.*

Proof. Assume that $\{x, y\}$ and $\{y, z\}$ are edges in the complement that are oriented from x to y and from y to z. We must show that $\{x, z\}$ is an edge in the complement and it is oriented from x to z. From the orientations of $\{x, y\}$ and $\{y, z\}$, $c_x < c_y < c_z$. So it remains to show that $\{x, z\}$ is an edge in the complement. That is, $\{x, z\} \notin E$.

Note that $\{x, y\}, \{y, z\} \notin E$, the assumption that the representation is bounded and $c_x < c_y < c_z$ imply that none of the three intervals contains another. Then, since $\{x, y\} \notin E$, $t_x > |I_x \cap I_y| > |I_x \cap I_z|$. Since $\{y, z\} \notin E$, $t_z > |I_y \cap I_z| > |I_x \cap I_z|$. So $\{x, z\} \notin E$. □

Recall that proper tolerance graphs can be assumed to have bounded representations.

Lemma 3. *If the complement of $K_{2,2}$ (with edges $\{x, y\}$ and $\{z, w\}$) appears as an induced subgraph in a proper tolerance graph G, then in any proper tolerance representation, c_x and c_y are both less than or both greater than c_z and c_w.*

Proof. We cannot have c_z or c_w between c_x and c_y, or c_x or c_y between c_z and c_w in the ordering of centers, since this gives an orientation of the complement that is not transitive, contradicting Lemma 2. □

We can get a similar result for induced paths on four vertices. However, the proof requires eliminating some potential orderings where Lemma 2 is not violated. Let P_n denote a path on n vertices and C_n a cycle on n vertices.

Lemma 4. *If xyzw is an induced P_4 in a proper tolerance graph G, then in any proper tolerance representation, c_x and c_y are both less than or both greater than c_z and c_w. In particular, the possible orderings of centers are $c_x < c_y < c_z < c_w$, $c_x < c_y < c_w < c_z$, $c_y < c_x < c_z < c_w$, or the reverse of one of these orders.*

Proof. By checking all 24 orderings of the centers, it can be seen that the path must appear in one of the eight patterns (or its reverse) in Fig. 2, where we assume the centers are ordered from left to right. The graphs in Fig. 2 (e)–(h) show violations of Lemma 2. Thus, the centers of the path cannot appear in any of these patterns.

For pattern (d), label the vertices from left to right as b,a,d,c, note that $\{b,d\}\notin E$ implies $t_b > |I_b \cap I_d| > |I_b \cap I_c|$. Similarly, $\{c,a\}\notin E$ implies $t_c > |I_c \cap I_a| > |I_c \cap I_b|$. So, $t_b, t_c > |I_b \cap I_c|$, contradicting $\{b,c\} \in E$.

For pattern (c), with order of vertices a,c,b,d, $\{a,c\}\notin E$ implies $t_c, t_a > |I_a \cap I_c| > |I_a \cap I_b|$. Thus for $\{a,b\} \in E$, we have $t_b \leqslant |I_a \cap I_b|$ and so $t_c > t_b$. Similarly, $\{b,d\}\notin E$ implies $t_b, t_d > |I_d \cap I_b| > |I_d \cap I_c|$. Thus for $\{c,d\} \in E$, we have $t_c \leqslant |I_d \cap I_c|$ and so $t_b > t_c$, contradicting $t_c > t_b$.

Thus the path must appear as pattern (a) with vertex ordering a,b,c,d or as pattern (b) with vertex ordering a,b,d,c or as the reverse of pattern (b) with vertex ordering b,a,c,d. For (a), (a,b,c,d) can be either (x,y,z,w) or (w,z,y,x). For (b), (a,b,d,c) can be either (x,y,w,z) or (w,z,x,y). For the reverse of (b), (b,a,c,d) can be either (y,x,z,w) or (z,w,y,x) and the result follows. □

A similar result holds for induced cycles on four vertices.

Lemma 5. *If $xyzw$ is an induced C_4 in a proper tolerance graph G, then in any proper tolerance representation, c_x and c_z are both between c_y and c_w or c_y and c_w are both between c_x and c_z.*

Proof. By checking all 24 orderings of the centers, it can be seen that the cycle C_4 must appear in one of the three patterns in Fig. 3, where we assume the centers are ordered from left to right.

For pattern (b) in Fig. 3, label the vertices from left to right as a,b,c,d. Note that $\{a,c\}\notin E$ implies $t_a > |I_a \cap I_c| > |I_a \cap I_d|$. Similarly $\{b,d\}\notin E$ implies

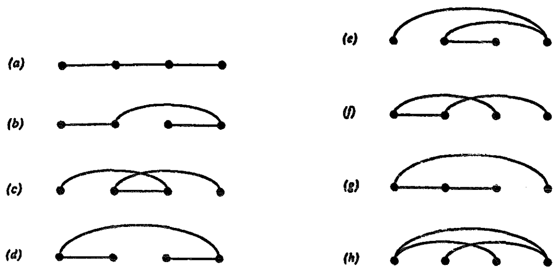

Fig. 2. Patterns for P_4.

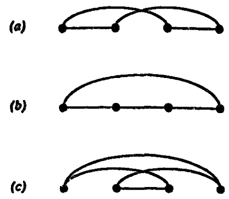

Fig. 3. Patterns for C_4.

$t_a > |I_b \cap I_d| > |I_a \cap I_d|$. So $t_a, t_d > |I_a \cap I_d|$, contradicting $\{a, d\} \in E$. The same argument shows that pattern (c) (with order of vertices a, c, b, d) cannot appear. Thus C_4 must appear as in pattern (a) with ordering of vertices a, b, d, c. By the cyclic nature of C_4, a, b, d, c can be any of (x, y, w, z), (y, z, x, w), (z, w, y, x) or (w, x, z, y) and the result follows. □

We extend the result of Lemma 4 to longer paths.

Lemma 6. *If $1, 2, 3, \ldots, n$ is an induced path P_n on $n \geqslant 4$ vertices in a proper tolerance graph G, then in any proper tolerance representation the centers of the path satisfy*

$$c_1, c_2 < c_3 < c_4 < \cdots < c_{n-2} < c_{n-1}, c_n$$

or

$$c_1, c_2 > c_3 > c_4 > \cdots > c_{n-2} > c_{n-1}, c_n.$$

Proof. By induction. The result holds for P_4 by Lemma 4. Consider P_n. By induction on $v_1 v_2 \ldots v_{n-1}$ we may assume, without loss of generality,

$$c_1, c_2 < c_3 < \cdots < c_{n-3} < c_{n-2}, c_{n-1}$$

and by induction on $v_2 v_3 \ldots v_n$, $c_{n-2} < c_{n-1}, c_n$ (since $c_{n-3} < c_{n-2}$). □

We now show how to construct counterexamples to the proper = unit question from examples which have proper but no unit representations under a specified ordering of the centers.

Theorem 7. *If G is a proper tolerance graph with a proper tolerance representation $(\mathcal{I}, T)$ such that the vertices are labeled with centers satisfying $c_1 < c_2 < \cdots < c_n$, then there is a proper tolerance graph G' with G as an induced subgraph such that in every proper tolerance representation of G' the centers of the vertices $1, 2, \ldots, n$ (corresponding to those of G) must satisfy $c_1 < c_2 < \cdots < c_n$ or $c_1 > c_2 > \cdots > c_n$.*

Proof. We use right and left endpoints $r_i = c_i + h_i$ and $l_i = c_i - h_i$. As noted in [3], adding some large number K to each half-length and $2K$ to each tolerance produces a new representation of the same graph for which all left endpoints are less than all right endpoints. We will assume that this is the case. We will also assume that all endpoints are distinct in the representation.

Let G and $(\mathscr{I}, T)$ be as supposed in the hypothesis of the theorem. Let $X = \{x_2, x_3, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ be disjoint from the vertex set $V(G)$ of G. Form a new graph G' with vertex set $V(G') = V(G) \cup X \cup Y$. The edges of G' will be specified by giving an interval representation. Use the presumed representation $(\mathscr{I}, T)$ of G for $V(G)$. Let g be the minimum gap between any two endpoints of the representation. Choose ε with $0 < \varepsilon < g/2$. All the new vertices will have tolerance ε.

Let L be some number smaller than the smallest left endpoint and R some number larger than the largest right endpoint in the representation of G. Then the new intervals are

$$I_{x_i} = [L - (n + 1 - i)\varepsilon, l_i - \varepsilon] \quad \text{for } i = 2, 3, \ldots, n,$$

$$I_{y_i} = [r_i + \varepsilon, R + i\varepsilon] \quad \text{for } i = 1, 2, \ldots, n - 1.$$

The interval representing x_i has its right endpoint slightly smaller than the left endpoint of I_i and its left endpoint placed so that no X interval contains another. Similarly, the y_i interval has its left endpoint slightly larger than the right endpoint of I_i and its right endpoint placed so that no Y interval contains another. The X intervals and Y intervals are disjoint. The left endpoint of each interval in X is less than the left endpoints of all intervals representing $V(G)$; the right endpoint of an interval in $V(G)$ is greater than the right endpoint of all intervals representing X. Thus the representation is proper with respect to intervals in X and $V(G)$. Similarly, it is proper with respect to Y and $V(G)$. Thus the representation of G' is proper. The intervals and tolerances representing vertices of G are unchanged in the representation of G'. So G is an induced subgraph of G'.

Note that X is a clique (complete subgraph) in G' as all the x_i's have tolerance ε and their intervals contain $[L - \varepsilon, L]$. Similarly, Y is a clique. Furthermore, since intervals representing vertices of X are disjoint from intervals representing vertices of Y, there are no edges between an X vertex and a Y vertex. Thus, $X \cup Y$ induces a complete bipartite subgraph in G'.

Fix any proper representation of G'. In any transitive orientation of the complement of G', either all edges are oriented from X to Y or vice versa, since these vertices induce a complete bipartite subgraph. Thus, by Lemma 2, either all of the centers for the x_i interval are less than all of the centers for the y_i intervals, or conversely. Assume for definiteness that the x_i centers are all less than the y_i centers. Then, for $i = 1, 2, \ldots, n - 1$, the vertices $x_{i+1}, y_i, i, (i + 1)$ form either a complement of $K_{2,2}$ (with edges $\{i, x_{i+1}\}$ and $\{(i + 1), y_i\}$) or a P_4 $x_{i+1} i (i + 1) y_i$ depending on the adjacency of i and $i + 1$ in G (and thus in G'). By Lemmas 3 and 4 and the assumption that the x centers are all less than the y centers we have $c_i < c_{i+1}$ for $i = 1, 2, \ldots, n - 1$. □

Although the previous theorem is not strictly necessary in light of Section 5, it provides an immediate proof that the examples in Section 4 are counterexamples to the unit = proper conjecture. Additionally, Theorem 7 should be useful for working on the problem of determining all forbidden subgraphs separating proper tolerance graphs from unit tolerance graphs.

4. Counterexamples

We are now ready to construct counterexamples to the proper = unit conjecture. Using Theorem 7 as a guide, we present a family of graphs and show that for a certain ordering of the centers they have a proper representation but no unit representation. This provides a counterexample. In Section 5, we show that the ordering of centers can be forced using two extra vertices instead of the $2n-2$ used in the proof of Theorem 7.

We first consider a (counter) example on nine vertices.

Example. Let G^2 be the graph in Fig. 4. We will show that G^2 has no 50% tolerance representation in which the ordering of the centers satisfies.

$$c_1 < c_2 < \cdots < c_7 < c_8 \quad \text{and} \quad c_x < c_5. \tag{1}$$

Consider 1, 2, 3, 4. These vertices form a subgraph with edges $\{1,3\}$ and $\{2,3\}$. With the specified ordering we have $c_3 > c_2 > c_1 + h_1$ since $\{1,2\} \notin E$. Thus for $\{1,3\} \in E$, it must be true that $c_3 - h_3 < c_1$. Since $\{3,4\} \notin E$, $c_3 + h_3 < c_4$. Therefore

$$c_4 - c_3 > h_3 > c_3 - c_1 > c_2 - c_1.$$

Similarly,

$$c_6 - c_5 > h_5 > c_5 - c_3 > c_4 - c_3$$

and

$$c_8 - c_7 > h_7 > c_7 - c_5 > c_6 - c_5.$$

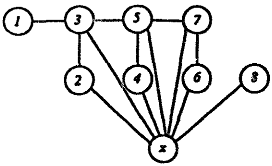

Fig. 4. G^2.

Thus

$$c_8 - c_7 > c_7 - c_5 > c_5 - c_3 > c_3 - c_1 .$$

So the sum of the first two terms is greater than the sum of the second two, i.e.,

$$c_8 - c_5 > c_5 - c_1 . \tag{2}$$

It is easy to see that $c_x < c_1 < c_2$ along with $\{1,x\}, \{1,2\} \notin E$ and $\{2,x\} \in E$ produces a contradiction. So, $c_x > c_1$. Then, from $\{1,x\} \notin E$,

$$c_x - h_x > c_1 . \tag{3}$$

From $\{7,8\} \notin E$, $c_8 - h_8 > c_7 > c_x$. So for $\{x,8\} \in E$ we must have

$$c_x + h_x > c_8 . \tag{4}$$

Combining Eqs. (2)–(4) yields $2c_x > 2c_5$. This contradicts the assumption about the ordering of the centers. Similarly, the centers cannot satisfy

$$c_1 > c_2 > \cdots > c_7 > c_8 \quad \text{and} \quad c_x > c_5 . \tag{5}$$

The following is a proper tolerance representation of G^2 for which the centers satisfy (1). Then, by Theorem 7 there is a graph G' with G^2 as an induced subgraph such that G' has a proper representation and in every such representation the centers of G^2 must satisfy (1) or (5). So G' is a proper tolerance graph but not a unit tolerance graph:

$$I_1 = [0,2], \quad t_1 = 2 - \delta, \qquad I_2 = [\delta + \varepsilon, 2 + \delta + \varepsilon], \quad t_2 = 2,$$

$$I_3 = [2\delta, 2 + 2\delta], \quad t_3 = 2 - 2\delta, \qquad I_4 = [4\delta + \varepsilon, 2 + 4\delta + \varepsilon], \quad t_4 = 2,$$

$$I_5 = [1 + 2\delta, 2 + 5\delta], \quad t_5 = 1 - \gamma,$$

$$I_6 = [5\delta + 1 + \varepsilon + \gamma, 5\delta + 2 + \varepsilon + \gamma], \quad t_6 = 1,$$

$$I_7 = [5\delta + 1 + 2\gamma, 5\delta + 2 + 2\gamma], \quad t_7 = 1 - 2\gamma,$$

$$I_8 = [5\delta + 1 + \varepsilon + 4\gamma, 5\delta + 2 + \varepsilon + 4\gamma], \quad t_8 = 1,$$

and

$$I_x = [1 - \varepsilon + 2\delta, 5\delta + 2 - \varepsilon], \qquad t_x = |I_2 \cap I_x| = 1 + 2\varepsilon - \delta .$$

As long as $t_x \leqslant 1 - 2\varepsilon - 4\gamma = |I_x \cap I_8|$, the adjacencies for x are correct. Selecting $0 < \varepsilon \ll \gamma \ll \delta < 1$ will allow this and insure that $\{2,5\} \notin E$. The remaining adjacencies and non-adjacencies can easily be checked.

In general, consider the following family of graphs. The first is G^2 of the above example. For integer $m \geqslant 2$ let G^m have vertex set $\{x, 1, 2, \ldots, 4m\}$ and edge set

$$\begin{aligned} E(G^m) = &\{\{2i-1, 2i+1\} \mid i = 1,2,\ldots,2m-1\} \\ &\cup \{\{2i, 2i+1\} \mid i = 1,2,\ldots,2m-1\} \\ &\cup \{\{x, j\} \mid j = 2,3,\ldots,4m\} . \end{aligned}$$

See Fig. 5 for example when $m = 4$.

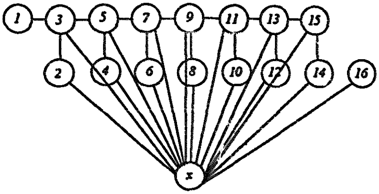

Fig. 5. G^4.

Lemma 8. *For $m \geqslant 2$, the graph G^m has no 50% tolerance representation in which the ordering of the centers satisfies*

$$c_1 < c_2 < \cdots < c_{4m} \quad \text{and} \quad c_x < c_{2m+1}$$

or

$$c_1 > c_2 > \cdots > c_{4m} \quad \text{and} \quad c_x > c_{2m+1}.$$

Proof. Recall that two vertices are adjacent in a 50% representation if and only if one of the corresponding intervals contains the center of the other. We will assume that $c_1 < c_2 < \cdots < c_{4m}$ and $c_x < c_{2m+1}$ holds and obtain a contradiction. The proof for $c_1 > c_2 > \cdots > c_{4m}$ and $c_x > c_{2m+1}$ is symmetric.

Note that for $i = 1, 2, \ldots, 2m-1$ the vertices $(2i-1), 2i, (2i+1), (2i+2)$ form a subgraph with edges $\{2i-1, 2i+1\}$ and $\{2i, 2i+1\}$. As in the example for G^2 with vertices $1, 2, 3, 4$, we get

$$c_{2i+2} - c_{2i+1} > h_{2i+1} > c_{2i+1} - c_{2i-1} > c_{2i} - c_{2i-1}. \tag{6}$$

Thus

$$c_{4m-1} - c_{4m-3} > c_{4m-3} - c_{4m-5} > \cdots > c_5 - c_3 > c_3 - c_1. \tag{7}$$

The gaps between the centers of the vertices with odd labels increase as the labels increase. From (6) with $i = 2m-1$, $c_{4m} - c_{4m-1} > c_{4m-1} - c_{4m-3}$. Then, from (7),

$$c_{4m} - c_{4m-1} > c_{2m+1} - c_{2m-1}. \tag{8}$$

Also, from (7),

$$c_{2m+2i+1} - c_{2m+2i-1} > c_{2i+1} - c_{2i-1} \quad \text{for } i = 1, 2, \ldots, m-1. \tag{9}$$

Combining (8) and (9) for $i = 1, 2, \ldots, m-1$ gives

$$c_{4m} - c_{2m+1} > c_{2m+1} - c_1. \tag{10}$$

As in the example for G^2, it is easy to see that $c_x < c_1 < c_2$ along with $\{1,x\},\{1,2\} \notin E$ and $\{2,x\} \in E$ produces a contradiction. So, $c_x > c_1$. Then, from $\{1,x\} \notin E$,

$$c_x - h_x > c_1. \tag{11}$$

Since $\{4m, 4m-1\} \notin E$, $c_{4m} - h_{4m} > c_{4m-1} > c_x$. So to get $\{x, 4m\} \in E$, we need

$$c_x + h_x > c_{4m}. \tag{12}$$

However, combining Eqs. (10)–(12) yields $2c_x > 2c_{2m+1}$, a contradiction. □

By Theorem 1, there is no unit representation of G^m with the order of centers given in the statement of Lemma 8. While, in the general case the representation problem is more delicate, the representation for G^2 is generalized in the following.

Lemma 9. *For $m \geqslant 2$, the graph G^m is a proper tolerance graph. Furthermore, G^m has a proper tolerance representation in which the centers satisfy $c_1 < c_2 < \cdots < c_{4m}$ and $c_{2m} < c_x < c_{2m+1}$.*

Proof. We present such a proper tolerance representation of G^m. Let $0 < \varepsilon \ll \gamma \ll \delta < 1$:

$$I_{2i-1} = \left[\left(\binom{i+1}{2} - 1\right)\delta, 2 + \left(\binom{i+1}{2} - 1\right)\delta\right]$$

for $i = 1, 2, \ldots, m$ with $t_{2i-1} = 2 - i\delta$;

$$I_{2i} = \left[\varepsilon + \left(\binom{i+2}{2} - 2\right)\delta, 2 + \varepsilon + \left(\binom{i+2}{2} - 2\right)\delta\right]$$

for $i = 1, 2, \ldots, m$ with $t_{2i} = 2$. Let $K = (\binom{m+2}{2} - 1)\,\delta$:

$$I_{2m+2i-1} = \left[K + 1 + \left(\binom{i+1}{2} - 1\right)\gamma, K + 2 + \left(\binom{i+1}{2} - 1\right)\gamma\right]$$

for $i = 2, 3, \ldots, m$ with $t_{2m+2i-1} = 1 - i\gamma$;

$$I_{2m+2i} = \left[K + 1 + \varepsilon + \left(\binom{i+2}{2} - 2\right)\gamma, K + 2 + \varepsilon + \left(\binom{i+2}{2} - 2\right)\gamma\right]$$

for $i = 1, 2, \ldots, m$ with $t_{2m+2i} = 1$;

$$I_{2m+1} = \left[1 + \left(\binom{m+1}{2} - 1\right)\delta, 2 + K\right]$$

with $t_{2m+1} = 1 - \gamma$;

$$I_x = \left[1 - \varepsilon + \left(\binom{m+1}{2} - 1\right)\delta, K + 2 - \varepsilon\right]$$

with

$$|I_1 \cap I_x| = 1 + \varepsilon - \left(\binom{m+1}{2} - 1\right)\delta < t_x < 1 - 2\varepsilon - \left(\binom{m+2}{2} - 2\right)\gamma = |I_x \cap I_{4m}|.$$

This choice of t_x is justified by setting ε small relative to γ and γ small relative to δ. Then, setting $t_x = |I_2 \cap I_x|$, x will have the necessary adjacencies. A check of the remaining adjacencies and non-adjacencies shows that we have a valid representation of G^m. □

Corollary 10. *The class of proper tolerance graphs properly contains the class of unit tolerance graphs.*

Proof. By Lemmas 9 and 8 and Theorems 7 and 1. □

5. Minimality

In this section we add two vertices to G^m to obtain a graph H^m that has a proper tolerance representation but no unit tolerance representation. Furthermore, H^m is minimal in the sense that if any vertex of H^m is deleted, the reduced graph is a unit tolerance graph.

Let y and z denote the vertices added to G^m to get H^m. We take y adjacent to $4m$, $4m-1$ and z and z adjacent to $2m+1, 2m+2, \ldots, 4m$ and y. All other edges of H^m are those of G^m.

Lemma 11. *The centers in every proper tolerance representation of H^m must satisfy*

$$c_1 < c_2 < \cdots < c_{4m} \quad \textit{and} \quad c_x < c_{2m+1}$$

or

$$c_1 > c_2 > \cdots > c_{4m} \quad \textit{and} \quad c_x > c_{2m+1}.$$

Proof. By applying Lemma 6 to the induced path $1, 3, 5, \ldots, 2i-1, \ldots, 4m-1, y, 4m$ we can assume for definiteness that

$$c_1, c_3 < c_5 < \cdots < c_{2i-1} < c_{2i+1} < \cdots < c_{4m-1} < c_y, c_{4m}.$$

Then Lemma 4 applied to the paths $2i, 2i+1, 2i+3, 2i+2$ ($i = 1, 2, \ldots, 2m-2$) and $4m-2, 4m-1, y, 4m$ and $1, 3, 5, 4$ gives

$$\begin{aligned} &c_1, c_2, c_3 < c_4, c_5 < \cdots < c_{2i}, c_{2i+1} < c_{2i+2}, c_{2i+3} < \cdots \\ &\qquad < c_{4m-2}, c_{4m-1} < c_y, c_{4m}. \end{aligned} \tag{13}$$

Path $2m-1, x, 4m-1, z$ with $c_{2m-1} < c_{4m-1}$ (via (13)) yields $c_x < c_{4m-1}$. Also, $c_{4m-1} < c_{4m}, c_y$ from (13). Then, since $x, 4m, y, 4m-1$ is a cycle on four vertices, the only ordering of the cycle satisfying the above conditions and Lemma 5 is $c_x < c_{4m-1} < c_{4m} < c_y$.

If $c_{4m-1} < c_{4m-2}$ then the centers of path $4m-2, 4m-1\,y, 4m$ appear as $c_{4m-1} < c_{4m-2} < c_{4m} < c_y$, which is forbidden ordering (d) in the proof of Lemma 4. So $c_{4m-2} < c_{4m-1}$.

We show that $c_{2i} < c_{2i+1}$ for $i = 1, 2, \ldots, 2m-1$. This true for $i = 2m-1$ from the previous paragraph. Suppose to the contrary that $c_{2i+1} < c_{2i}$ for some $i = 1, 2, \ldots, 2m-2$. Then $c_{2i+3} < c_{2i+2}$ since the ordering $c_{2i+1} < c_{2i} < c_{2i+2} < c_{2i+3}$ is forbidden ordering (d) in the proof of Lemma 4. Repeating this argument for the paths $2j, 2j+1, 2j+3, 2j+2$ ($j = i+1, \ldots, 2m-2$) yields $c_{4m-1} < c_{4m-2}$, a contradiction. Hence $c_{2i} < c_{2i+1}$ for $i = 1, 2, \ldots, 2m-1$. Thus it remains to show that $c_1 < c_2$ and $c_x < c_{2m+1}$.

From path $2, x, 2m+1, z$ with $c_2 < c_{2m+1}$ and Lemma 4, we get $c_x < c_{2m+1}$. From path $1, 3, x, 4m$ with $c_3 < c_{4m}$ and Lemma 4, we get $c_3 < c_x$. We have $c_1, c_2 < c_3 < c_x$. The ordering $c_2 < c_1 < c_x$ violates Lemma 2, so $c_1 < c_2$. □

Corollary 12. *For $m \geqslant 2$, H^m is a proper tolerance graph but not a unit tolerance graph.*

Proof. By Lemmas 8 and 11 and Theorem 1, H^m has no unit representation. Use the representation of G^m in the proof of Lemma 9 as a basis. Also, let

$$K = \left(\binom{m+2}{2} - 1\right)\delta$$

as in the proof of Lemma 9 and let

$$R = K + 2 + \varepsilon + \left(\binom{m+2}{2} - 1\right)\delta.$$

Then let $t_y = t_z = \varepsilon/2$ and

$$I_z = \left[K + 2 + \frac{\varepsilon}{2}, R + (2m+1)\frac{\varepsilon}{2}\right]$$

and

$$I_y = \left[K + 2 + \varepsilon + \left(\binom{m+1}{2} - 2\right)\gamma + \frac{\varepsilon}{2}, R + (4m-1)\frac{\varepsilon}{2}\right].$$

It can be directly checked that y and z have the proper adjacencies. Alternatively, note that y and z are the Y vertices from the proof of Theorem 7 corresponding to $4m-2$ and x. (When applying Theorem 7 in this case, set the ε of Theorem 7 to $\varepsilon/2$ and use the ordering $1, 2, \ldots, 2m, x, 2m+1, \ldots, 4m-1, 4m$ for the ordering of the centers.) □

We say that a non-unit tolerance graph is *minimal* if removing any vertex leaves a graph with a unit (equivalently 50%) tolerance representation.

Theorem 13. *For $m \geqslant 2$, H^m is a minimal non-unit tolerance graph.*

Proof. Using Theorem 1, we give 50% representations for the graphs obtained by deleting a vertex from H^m, We will consider y to be $4m+1$. The intervals are given as either by centers and half-lengths or as intervals, whichever is more convenient.

The following intervals will be used, with some slight modification, as a basis for the individual cases. Let $V = \{1, 2, \ldots, 4m+1, x, z\}$ (with y identified with $4m+1$). Let $\binom{r}{s} = 0$ if $r < s$ and let $0 < \varepsilon \ll \zeta, \eta < 1$. The values of ζ and η will be determined more precisely when necessary:

$$c'_1 = 0 \quad \text{and} \quad h'_1 = 0;$$

$$c'_{2i+1} = i\zeta + \binom{i}{2}\eta, \quad h'_{2i+1} = \zeta + (i-1)\eta \qquad \text{for } i = 1, \ldots, 2m;$$

$$c'_{2i} = c'_{2i+1} - \varepsilon = i\zeta + \binom{i}{2}\eta - \varepsilon, \quad h'_{2i} = 0 \qquad \text{for } i = 1, 2, \ldots, 2m;$$

$$c'_x = c'_{2m+1} - \frac{\varepsilon}{2} = m\zeta + \binom{m}{2}\eta - \frac{\varepsilon}{2}, \qquad h'_x = c'_{4m} - c'_x = m\zeta + \frac{m(3m-1)}{2}\eta - \frac{\varepsilon}{2};$$

$$c'_z = m\zeta + \binom{m}{2}\eta + B, \quad h'_z = B \quad \text{for some } B > c'_{4m+1} + h'_{4m+1} - c'_{2m+1}.$$

Case 1: Delete x. Let $I_j = I'_j$ for $j \in V \setminus \{x\}$. Note that the centers satisfy $c_1 < c_2 < \cdots < c_{4m} < c_{4m+1} = c_y$. I_z overlaps all centers greater than or equal to $m\zeta + \binom{m}{2}\eta = c_{2m+1}$ and no others. By the choice of B no intervals overlap c_z. So z has the correct adjacencies. Each even interval I_{2i} is a point interval with only interval I_{2i+1} containing it. So the adjacencies of the even intervals are correct. Finally, the left endpoint of interval I_{2i+1} is c_{2i-1}. So this interval contains c_{2i} and c_{2i-1} and no other centers and the only interval containing c_{2i+1} is I_{2i+3}. So the adjacencies of the odd intervals are correct.

Case 2: Delete $y = 4m+1$. Let $I_j = I'_j$ for $j \in V \setminus \{x, 4m-1, 4m, 4m+1\}$. Set $c_{4m} = c'_{4m-2} + \varepsilon$ with $h_{4m} = 0$. Set $I_{4m-1} = [c_{4m-3}, c_{4m-2}]$ and $c_x = c'_x$ with $h_x = c_{4m} - c_x$. (Here we use the new c_{4m} not c'_{4m}.) All adjacencies other than the changed intervals are correct from case 1. The choice of I_{4m-1} and I_{4m} puts $4m$ adjacent only to x and z and puts $4m-1$ adjacent only to $4m-3$, $4m-2$, z and x.

Note that the left endpoint of I_x is

$$\begin{aligned} c_x - h_x &= 2c_x - c_{4m} \\ &= 2m\zeta + 2\binom{m}{2}\eta - \varepsilon - (2m-1)\zeta - \binom{2m-1}{2}\eta \\ &= \zeta - \varepsilon - (m-1)^2\eta. \end{aligned}$$

By an appropriate choice of ζ, η and ε this left endpoint will be greater than $c_1 = 0$ and less than $c_2 = \zeta - \varepsilon$. Then I_x contains every center except c_1 and c_z, as needed. Also, c_x is not contained in I_z or I_1.

Case 3: Delete z. Let $I_j = I_j'$ for $j \in V \setminus \{x, z\}$. Set $I_x = [c_2, c_{4m}]$. Then I_x contains every center except c_1 and c_y. The rest of the adjacencies are as above.

Case 4: Delete 1. Let $I_j = I_j'$ for $j \in V \setminus \{1\}$. Then I_x contains every center except c_z and c_y. The rest of the adjacencies are as above.

Case 5: Delete 2. Let $I_j = I_j'$ for $j \in V \setminus \{1, 2\}$. Let $I_1 = [-C, -C]$ for some large number C. The rest of the adjacencies are as above.

Case 6: Delete $2i + 1$ for $i = 1, 2, \ldots, m - 1$. Let $I_j = I_j'$ for $j \in V \setminus \{1, 2, \ldots, 2i + 1\}$. Shift the primed intervals for $1, 2, \ldots, 2i$ to the left until $c_x - c_2 = c_{4m} - c_x$. That is, for $j = 1, 2, \ldots, 2i$, let $h_j = h_j'$ and let

$$c_j = c_j' - (c_{4m}' - c_x') + (c_x' - c_2') = c_j' - (m^2\eta + \zeta - \varepsilon).$$

Thus, x has the correct adjacencies as it overlaps all centers except c_1, c_y, c_z. There are no adjacencies between $1, 2, \ldots, 2i$ and $2i + 2, 2i + 3, \ldots, 4m + 1, z$ and these non-adjacencies are preserved by the shift. The rest of the adjacencies are as above.

Case 7: Delete $2i$ for $i = 2, 3, \ldots, m$. Let $I_j = I_j'$ for $j \in V \setminus \{1, 2, \ldots, 2i\}$. As in case 6 shift the intervals $1, 2, \ldots, 2i - 2$ to the left until $c_x - c_2 = c_{4m} - c_x$. That is, for $j = 1, 2, \ldots, 2i$, let $h_j = h_j'$ and let

$$c_j = c_j' - (c_{4m}' - c_x') + (c_x' - c_2') = c_j' - (m^2\eta + \zeta - \varepsilon).$$

Let $I_{2i-1} = [c_{2i-3}, c_{2i+1}]$. As above, adjacencies between intervals unchanged relative to each other are the same, and non-adjacencies between $\{1, 2, \ldots, 2i - 2\}$ and $\{2i + 1, \ldots, 4m + 1, z\}$ are maintained. The choice of I_{2i-1} puts it adjacent to $2i - 3, 2i + 1$, and x, as needed.

Case 8: Delete $2i + 1$ for $i = m, m + 1, \ldots, 2m - 1$. Let $I_j = I_j'$ for $j \in V \setminus \{2i + 1, 2i + 2, \ldots, 4m + 1, x\}$. Shift the remaining intervals to the left as follows. For $j = 2i + 2$ and $j = 2i + 4, 2i + 5, \ldots, 4m + 1$ let $h_j = h_j'$. Let $h_{2i+3} = \varepsilon$. For $j = 2i + 2, 2i + 3, \ldots, 4m + 1$ let

$$c_j = c_j' - \zeta.$$

Let $c_x = c_x'$ and $h_x = c_{4m} - c_x$. (Here we use the new c_{4m} not c_{4m}'.) The shifts maintain non-adjacencies between $\{1, 2, \ldots, 2i\}$ and $\{2i + 2, 2i + 3 \ldots, 4m + 1\}$ and the adjacencies for z. The adjacencies between intervals unchanged relative to each other are the same. In a manner similar to case 2, note that the left endpoint of I_x is

$$\begin{aligned} c_x - h_x &= 2c_x - c_{4m} \\ &= 2m\zeta + 2\binom{m}{2}\eta - \varepsilon - (2m - 1)\zeta - \binom{2m}{2}\eta + \varepsilon \\ &= \zeta - m^2\eta. \end{aligned}$$

By an appropriate choice of ζ, η and ε this left endpoint will be greater than $c_1 = 0$ and less than $c_2 = \zeta - \varepsilon$. Then I_x contains every center except c_1 and c_2, as needed.

Case 9: Delete $2i$ for $i = m+1, m+2, \ldots, 2m-1$. This case is nearly identical to case 8. Let $I_j = I_j'$ for $j \in V \setminus \{2i, 2i+1, \ldots, 4m+1, x\}$. Shift the remaining intervals to the left as follows. For $j = 2i+2, 2i+3, \ldots, 4m+1$ let $h_j = h_j'$. Let $h_{2i+1} = 0$. For $j = 2i+1, 2i+2, \ldots, 4m+1$ let

$$c_j = c_j' - \zeta.$$

Let $c_x = c_x'$ and $h_x = c_{4m} - c_x$. (Here we use the new c_{4m} not c_{4m}'.) The rest is identical to case 8.

Case 10: Delete $4m$. Let $I_j = I_j'$ for $j \in V \setminus \{4m, x\}$. Let $c_x = c_x'$ and $h_x = c_{4m-1} - c_x$. Note that $c_{4m+1} = c_{4m-2} + \varepsilon$ so h_x is as in case 2. As in that case, ζ and η can be chosen so that x has the proper adjacencies. □

6. Conclusion

We have answered a question of Golumbic et al. [3] by constructing proper tolerance graphs that are not unit tolerance graphs. We showed also that the class of tolerance graphs with tolerance equal to the half length (50% tolerance graphs) is equal to the class of unit tolerance graphs. This equivalence was useful in answering the question Golumbic et al.

We note that each of these families has an order theoretic analog. For example, bounded (proper, unit) tolerance orders are partial orders for which $x \succ y$ if and only if $c_x > c_y$ and $|I_x \cap I_y| > \min\{t_x, t_y\}$ in a bounded (proper, unit) tolerance representation. The 50% tolerance orders can also be viewed in the following manner: $x \succ y$ if and only if $c_x - c_y > \max\{t_x, t_y\}$. Although we have discussed only graphs in this paper, their order theoretic analogs have aided our thinking about graphs. Our results for graphs can easily be translated to order versions.

Finally, we note that our initial examples were motivated by the fact that certain inequalities (or one of the two inequalities) using endpoints and tolerances as variables must be satisfied in a tolerance representation with a fixed order of the centers. Our initial example arose from working with Farkas' lemma (the theorem of the alternative) and these systems.

We have constructed an infinite family of minimal graphs that are proper tolerance but not unit tolerance graphs. The smallest member of this family has 11 vertices. This raises some open problems. Are there smaller examples? What other graphs or families of graphs are minimal proper and not unit tolerance graphs? Can one characterize all minimal proper tolerance graphs which are forbidden unit tolerance graphs? We have also shown that paths have at most eight possible orderings in a proper tolerance representation. Are there graphs which have only two orderings (unique up to duality)? Finally, we ask whether or not there are efficient recognition algorithms for any of the classes that we have discussed.

Appendix

In this appendix we include details on checking that the representation given in Lemma 9 is indeed that of G^m.

To check that the intervals represent G^m, note that an edge is present if the length of the intersection of the intervals is greater or equal to than at least one of the tolerances. When considering vertices a and b we will say that a is *adjacent from* b if $t_b \leqslant |I_a \cap I_b|$. Thus a and b are adjacent if and only if at least one is adjacent from the other.

Case 1: $\{2i-1, 2i\} \notin E$ for $i = 1, 2, \ldots, m$,

$$|I_{2i-1} \cap I_{2i}| = 2 - \varepsilon - i\delta < 2 - i\delta = \min\{2, 2 - i\delta\}.$$

Note also that a is not adjacent from $2i-1$ for $2i \leqslant a$ since $|I_a \cap I_{2i-1}| \leqslant |I_{2i} \cap I_{2i+1}|$. Finally, note that x is not adjacent from 1 since $|I_1 \cap I_2| > |I_1 \cap I_x|$.

Case 2: $\{2m+2i-1, 2m+2i\} \notin E$ for $i = 1, 2, \ldots, m$,

$$|I_{2m+2i-1} \cap I_{2m+2i}| = 1 - \varepsilon - i\gamma < 1 - i\gamma = \min\{1, 1 - i\gamma\}.$$

Note also that a is not adjacent from $2i-1$ for $2m+2i \leqslant a$ since $|I_a \cap I_{2m+2i-1}| \leqslant |I_{2m+2i} \cap I_{2m+2i+1}|$.

Case 3: $\{2i-1, 2i+1\} \in E$ for $i = 1, 2, \ldots, m-1$,

$$|I_{2i-1} \cap I_{2i+1}| = 2 - (i+1)\delta = t_{2i+1} = \min\{t_{2i-1}, t_{2i+1}\}.$$

Note that this also gives $\{2i, 2i+1\} \in E$ for $i = 1, \ldots, m-1$ since $|I_{2i-1} \cap I_{2i+1}| < |I_{2i} \cap I_{2i+1}|$. Also, a is not adjacent from $2i+1$ for $a < 2i-1$ since $|I_a \cap I_{2i+1}| < |I_{2i-1} \cap I_{2i+1}| = t_{2i+1}$.

Case 4: $\{2m+2i-1, 2m+2i+1\} \in E$ for $i = 1, 2, \ldots, m-1$,

$$|I_{2m+2i-1} \cap I_{2m+2i+1}| = 1 - (i+1)\gamma = t_{2m+2i+1}.$$

Note that this also gives $\{2m+2i, 2m+2i+1\} \in E$ for $i = 1, \ldots, m-1$ since $|I_{2m+2i-1} \cap I_{2m+2i+1}| < |I_{2m+2i} \cap I_{2m+2i+1}|$. Also, a is not adjacent from $2m+2i+1$ for $a < 2m+2i-1$ since $|I_a \cap I_{2m+2i+1}| < |I_{2m+2i-1} \cap I_{2m+2i+1}| = t_{2m+2i+1}$.

Case 5: $\{2m+1, 2m-1\} \in E$,

$$|I_{2m-1} \cap I_{2m+1}| = 1 \geqslant 1 - \gamma = t_{2m+1}.$$

Note that this also gives $\{2m, 2m+1\} \in E$ since $|I_{2m} \cap I_{2m+1}| > |I_{2m-1} \cap I_{2m+1}|$.

Case 6: Nothing is adjacent from $2i$ for $i = 1, 2, \ldots, 2m$ since $t_{2i} = |I_{2i}|$.

Case 7: $\{2m+1, 2m-2\} \notin E$,

$$|I_{2m+1} \cap I_{2m-2}| = 1 + \varepsilon - \delta < 1 - \gamma$$

by the choice of ε, δ and γ. Note also that a is not adjacent from $2m+1$ for $a \leqslant 2m-2$ since $|I_a \cap I_{2m+1}| < |I_{2m-2} \cap I_{2m+1}|$.

Case 8: $\{x, 1\} \notin E$. Since $|I_1 \cap I_x| < |I_2 \cap I_x| = t_x$ and since x is not adjacent from 1 by case 1.

Case 9: $\{x, i\} \in E$ for $i = 2, \ldots, 2m$ since $t_x = |I_x \cap I_2| \leqslant |I_x \cap I_i|$.

Case 10: $\{x, i\} \in E$ for $i = 2m + 1, \ldots, 4m$,

$$|I_x \cap I_i| \geqslant |I_x \cap I_{4m}| > t_x$$

by the choice of t_x (see the end of the proof of Lemma 9).

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