A MESHING TECHNIQUE FOR de BRUIJN TORI

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Abstract. An $(R, S; m, n)_k$-de Bruijn torus is a k-ary $R \times S$ toroidal array with the property that every k-ary $m \times n$ matrix appears exactly once contiguously on the torus. The torus is a generalization of de Bruijn cycles and has been extended to higher dimensions by many authors. The central question, asked by Chung, Diaconis, and Graham, is for which R, S, m, n, and k such tori exist. In this note we develop a notion of equivalence class de Bruijn cycles and we extend a technique of Iványi and Tóth. Combining these ideas we are able to construct the first examples in which R and S are not powers of k. We prove for all natural numbers s and t there is a $(4st^2, 4s^3t^2; 2, 2)_{2st}$-de Bruijn torus.

1. Introduction

All variables in this paper are assumed to be natural numbers (with $k > 1$).

An $(R, n)_k$-*de Bruijn cycle* is a cyclic k-ary sequence of length R with the property that every k-ary n-tuple appears exactly once contiguously on the cycle (of course, $R = k^n$). First invented in 1894 by Flye St. Marie [6], they were rediscovered in 1946 by de Bruijn [1] and Good [8]. An excellent survey by Fredericksen [7] introduces the reader to a vast literature on the topic.

An $(R, S; m, n)_k$-*de Bruijn torus* is a toroidal k-ary $R \times S$ array with the property that every k-ary $m \times n$ matrix appears exactly once contiguously on the torus (of course, $RS = k^{mn}$). The simplest example of such a torus is the $(4, 4; 2, 2)_2$-de Bruijn torus in Figure 1. (In addition to $RS = k^{mn}$ it is also necessary that $R > m$ and $S > n$—if $R = m$, say, then the all-0's matrix is found m times.)

0	0	1	0
0	0	0	1
0	1	1	1
1	0	1	1

Figure 1. A $(4, 4; 2, 2)_2$-de Bruijn torus.

In 1984, Ma [12] proved the binary case of the following 1988 theorem of J. C. Cock [3] (see also [4]).

Theorem 1.1. *For all m, n, and k (except $n = 2$ if k is even) there is a $(k^r, k^s; m, n)_k$-de Bruijn torus with $r = m$ and $s = m(n-1)$.* ∎

Cock's technique easily generalizes to higher dimensions, but unfortunately, each new dimension has size exponential in the previous. In particular, we can make the obvious definition of an $(\vec{R}; \vec{n})_k$-*de Bruijn d-torus* in d dimensions, with vectors $\vec{R} = \langle r_1, \ldots, r_d \rangle$ and $\vec{n} = \langle n_1, \ldots, n_d \rangle$ satisfying $\Pi r_i = k^{\Pi n_i}$ and $r_i > n_i$ for all i. One then has the following theorem, mentioned in [3] and proved in [10].

Theorem 1.1′. *For all $\vec{n}, d$ and k (except $n_2 = 2$ if k is even) there is an $(\vec{R}; \vec{n})_k$-de Bruijn d-torus with $r_1 = k^{n_1}$ and $r_j = (\prod_{i=1}^{j-1} r_i)^{n_j - 1} = k^{(n_j - 1)\prod_{i=1}^{j-1} n_i}$ for $j > 1$.* ∎

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This prompted Chung, Diaconis, and Graham [2] to ask whether it could be true that, in two dimensions, $R = S$ and $m = n$ (the so-called "square" tori). The binary case was resolved with the following theorem of Fan, Fan, Ma, and Sin [5].

Theorem 1.2. *There is a* $(2^{n^2/2}, 2^{n^2/2}; n, n)_2$*-de Bruijn torus if and only if* n *is even.* $\blacksquare$

Since an $(R, R; n, n)_k$-de Bruijn torus has $R = k^{n^2/2}$, we must have n even or k a perfect square. In [9] we settled the question for general k with the following

Theorem 1.3. *Except in the case that* k *is an even square and* $n = 3, 5, 7,$ *or* 9*, there is an* $(R, R; n, n)_k$*-de Bruijn torus if and only if* n *is even or* k *is a perfect square.* $\blacksquare$

Higher dimensional versions of Theorem 1.3 remain open. For the remainder of this paper we will concentrate on two dimensions only, seeking to make progress on the following conjecture found in [9].

Conjecture 1.4. *If* $R, S, m, n,$ *and* k *satisfy* $R > m$*,* $S > n$*, and* $RS = k^{mn}$ *then there is an* $(R, S; m, n)_k$*-de Bruijn torus.*

Until now, every result has had R and S, both powers of the base k. In this paper we prove the following theorem. Because of the nature of the techniques we use in proving Theorems 1.3 and 1.6 below, we expect this theorem will help extend those results.

Theorem 1.5. *For all* s *and* t *there is a* $(4st^2, 4s^3t^2; 2, 2)_{2st}$*-de Bruijn torus.*

Outside of Theorems 1.3 and 1.5, the most progress toward conjecture 1.4 is found in [10], namely

Theorem 1.6. *Let* k *have prime factorization* $\Pi p_i^{\alpha_i}$ *and let* $q = k\Pi p_i^{\lfloor \log_{p_i} m \rfloor}$*. Then for all* m, n *there is a* $(q, k^{mn}/q; m, n)_k$*-de Bruijn torus.* $\blacksquare$

In section 2 we describe the "*meshing*" method of Iványi and Tóth [11] on which Theorem 1.5 rests. Since their paper uses significantly different notation, and since we generalize their method, we sketch their proof of the following result.

Theorem 1.7. *For all* k *there is a* $(k^2, k^2; 2, 2)_k$*-de Bruijn torus.*

In section 3 we include a discussion of what we call *equivalence-class de Bruijn cycles* and present results about them which we use in section 4 to prove Theorem 1.5. For convenience, we let $dB_k(k^n; n)$ be the set of all k-ary de Bruijn cycles of order n, and $dB_k(R, S; m, n)$ be the set of all $(R, S; m, n)_k$-de Bruijn tori.

2. Meshing Method

Given two sequences $\vec{a} = a_0a_1 \ldots a_{t-1}$ and $\vec{b} = b_0b_1 \ldots b_{t-1}$ of equal length we define the $t \times t$ matrix $\mathrm{Mesh}\,(\vec{a}, \vec{b}) = [c_{ij}]$ by letting $c_{ij} = b_j$ when $i + j$ is even, a_i when $i + j$ is odd. For example, if $\vec{a} = 0011021220313233$, then $\mathrm{Mesh}\,(\vec{a}, \vec{a})$ is as in Figure 2. Note that $\vec{a} \in dB_4(16; 2)$, but that $\mathrm{Mesh}\,(\vec{a}, \vec{a}) \notin dB_4(16, 16; 2, 2)$. This is because $\vec{a}$ is not what we call an *even* de Bruijn sequence. For k even, we call $\vec{a} \in dB_k(k^n; k)$ *even* if for all $\alpha, \beta \in \{0, 1, \ldots, k-1\}$ we have that $(i-j) \bmod k^n$ is even whenever the sequence $\alpha\beta = a_ia_{i+1}$ and the sequence $\beta\alpha = a_ja_{j+1}$. Now choose an even $\vec{a} = 0011021331203223 \in dB_4(16; 2)$ and notice that $\mathrm{Mesh}\,(\vec{a}, \vec{a}) \in dB_4(16, 16; 2, 2)$, as in Figure 3. The sequence $\vec{a}$ in Figure 2 fails because for

$\alpha\beta = 13$ we have odd $i - j = 1$.

$\vec{a} =$	0	0	1	1	0	2	1	2	2	0	3	1	3	2	3	3
0	0	0	1	0	0	0	1	0	2	0	3	0	3	0	3	0
0	0	0	0	1	0	2	0	2	0	0	0	1	0	2	0	3
1	0	1	1	1	0	1	1	1	2	1	3	1	3	1	3	1
1	1	0	1	1	1	2	1	2	1	0	1	1	1	2	1	3
0	0	0	1	0	0	0	1	0	2	0	3	0	3	0	3	0
2	2	0	2	1	2	2	2	2	2	0	2	1	2	2	2	3
1	0	1	1	1	0	1	1	1	2	1	3	1	3	1	3	1
2	2	0	2	1	2	2	2	2	2	0	2	1	2	2	2	3
2	0	2	1	2	0	2	1	2	2	2	3	2	3	2	3	2
0	0	0	0	1	0	2	0	2	0	0	0	1	0	2	0	3
3	0	3	1	3	0	3	1	3	2	3	3	3	3	3	3	3
1	1	0	1	1	1	2	1	2	1	0	1	1	1	2	1	3
3	0	3	1	3	0	3	1	3	2	3	3	3	3	3	3	3
2	2	0	2	1	2	2	2	2	2	0	2	1	2	2	2	3
3	0	3	1	3	0	3	1	3	2	3	3	3	3	3	3	3
3	3	0	3	1	3	2	3	2	3	0	3	1	3	2	3	3

Figure 2. $\vec{a}$ not even, Mesh $(\vec{a}, \vec{a}) \notin dB_4(16, 16; 2, 2)$

$\vec{a} =$	0	0	1	1	0	2	1	3	3	1	2	0	3	2	2	3
0	0	0	1	0	0	0	1	0	3	0	2	0	3	0	2	0
0	0	0	0	1	0	2	0	3	0	1	0	0	0	2	0	3
1	0	1	1	1	0	1	1	1	3	1	2	1	3	1	2	1
1	1	0	1	1	1	2	1	3	1	1	1	0	1	2	1	3
0	0	0	1	0	0	0	1	0	3	0	2	0	3	0	2	0
2	2	0	2	1	2	2	2	3	2	1	2	0	2	2	2	3
1	0	1	1	1	0	1	1	1	3	1	2	1	3	1	2	1
3	3	0	3	1	3	2	3	3	3	1	3	0	3	2	3	3
3	0	3	1	3	0	3	1	3	3	3	2	3	3	3	2	3
1	1	0	1	1	1	2	1	3	1	1	1	0	1	2	1	3
2	0	2	1	2	0	2	1	2	3	2	2	2	3	2	2	2
0	0	0	0	1	0	2	0	3	0	1	0	0	0	2	0	3
3	0	3	1	3	0	3	1	3	3	3	2	3	3	3	2	3
2	2	0	2	1	2	2	2	3	2	1	2	0	2	2	2	3
2	0	2	1	2	0	2	1	2	3	2	2	2	3	2	2	2
3	3	0	3	1	3	2	3	3	3	1	3	0	3	2	3	3

Figure 3. $\vec{a}$ even, Mesh $(\vec{a}, \vec{a}) \in dB_4(16, 16; 2, 2)$

Proof of Theorem 1.7. The case when k is odd involves the same method as found in Cock's construction in Theorem 1.1, so we will not concern ourselves with this case. It is when k is even that the mesh is used, and this case is handled with the following two claims.

Claim 2.1. *For each even k there is an even $\vec{a} \in dB_k(k^2;2)$.*

Proof. We use induction, increasing k by 2 at each step. The sequence $\vec{a} = 0011$ works for $k = 2$ (in fact Mesh $(\vec{a}, \vec{a})$ is figure 1 in this case). Now let $\vec{a}$ satisfy the case for k, and for the sequence $\vec{b} = b_0 b_1 \cdots b_t$ denote its reverse sequence by $\vec{b}^{-1} = b_t \cdots b_2 b_1$. We now adjoin the two new letters α and β.

Now let $\vec{b} = 0\alpha 1\beta 2\alpha 3\beta \cdots \beta(k-2)\alpha(k-1)\beta$ and assume that $\vec{a}$ begins with a 0. Then the sequence $\vec{a}\vec{b}\vec{b}^{-1}\beta\alpha\alpha\beta$ satisfies the case for $k+2$. $\blacksquare$

Claim 2.2. *If $\vec{a} \in dB_k(k^2;2)$ is even then* Mesh $(\vec{a}, \vec{a}) \in dB_k(k^2, k^2; 2, 2)$.

Proof. Suppose not. Let

$$A_{i,j} = \begin{pmatrix} c_{i,j} & c_{i,j+1} \\ c_{i+1,j} & c_{i+1,j+1} \end{pmatrix}$$

(subscripts mod k^2) and suppose that $A_{i,j} = A_{x,y}$ with $(i,j) \neq (x,y)$. If $i+j$ is even then $(c_{i,j}, c_{i+1,j+1}) = (a_j, a_{j+1})$, and if also $x+y$ is even then $(a_j, a_{j+1}) = (a_y, a_{y+1})$ and so $y = j$. But then $(a_i, a_{i+1}) = (a_x, a_{x+1})$ and so $i = x$, a contradiction. Hence, $x+y$ must be odd. By a similar argument, $i+j$ odd implies $x+y$ even, and so we will assume that $i+j$ is even and $x+y$ is odd. This implies that $(a_x, a_{x+1}) = (c_{x,y}, c_{x+1,y+1}) = (c_{i,j}, c_{i+1,j+1}) = (a_j, a_{j+1})$, and so $x = j$. Then also, $(a_y, a_{y+1}) = (c_{x+1,y}, c_{x,y+1}) = (c_{i+1,j}, c_{i,j+1}) = (a_{i+1}, a_i)$, and so $y = i+1$. But this contradicts the fact that $\vec{a}$ is even since $y - i = 1$. $\blacksquare$

With these two claims, the proof of Theorem 1.7 is complete. $\blacksquare$

3. Equivalence-class de Bruijn cycles

If $\vec{J}_x$ is the all 1's sequence of length x, we say that the length x k-ary sequences $\vec{a}$ and $\vec{b}$ are *equivalent* if $\vec{b} - \vec{a}$ is a multiple of $\vec{J}_x$. We then say that the cyclic sequence $\vec{a}$ is a *k-ary equivalence-class de Bruijn cycle of order n and length x* if its equivalence class $[\vec{\alpha}] = \{\vec{a}, \vec{a} + \vec{J}_x, \ldots, \vec{a} + (m-1)\vec{J}_x\}$ $(m|k,\ x = k^n/m)$ has the property that each k-ary m-tuple appears exactly once as a contiguous n-tuple in precisely one of the $\vec{a} + i\vec{J}_x$, and we denote the set of all such cycles $EC_k(x,n)$. These cycles were used very successfully in [9] to help prove Theorem 1.3, and they are easily constructed as follows.

0	0	1	5	2	4	3	3	4	2	5	1
1	1	2	0	3	5	4	4	5	1	0	2
2	2	3	1	4	0	5	5	0	2	1	3

Figure 4. An equivalence class of rows from $EC_6(12,2)$ generated by $014325 \in dB_6(6;1)$.

Let $\vec{c} = c_1 \ldots c_{k^{m-1}} \in dB_k(k^{m-1}; m-1)$ and define, for any $a_1 \in \{0,1,\ldots,k-1\}$, the sequence $\vec{a} = a_1 \ldots a_{k^{m-1}}$ by the relations $a_{j+1} = a_j + c_j$ $(1 \leq j < k^{m-1})$, addition being performed modulo k. Of course, $a_1 = a_{k^m} + c_{k^m}$ since $\sum_{i=1}^{k^{m-1}} c_i \equiv 0 \bmod k$, unless k is even and $m = 2$. In this case, $\sum_{i=1}^{k^{m-1}} c_i \equiv \left(\frac{k}{2}\right) \bmod k$ and so the sequence $\vec{a}$ generated by $\vec{c}$ has length $2k$ rather than k. Figure 4 shows the three cycles generated by $\vec{c} = 014325$ with $k = 6$. Notice that each row $\vec{a}$ of figure 4 satisfies $\vec{a} = \vec{a} + 3\vec{J}_{12}$, with equality allowing for cyclic shifts (of 6 digits, in particular). This is why we have $m|k$ and $x = k^n/m$ in the definition. See [10] for further generalizations and uses of this idea. We will use the following proposition in the next section.

Proposition 3.1. *If k is even and $\vec{a} \in EC_k(2k;2)$ is generated by $\vec{c} = (1, -2, 3, -4, \ldots, k-1, 0) \in dB_k(k;1)$, where the digits are written* mod$k$*, then $\vec{a}$ is even.*

Proof. Follows from the fact that $\vec{a} = \vec{b}\vec{b}^{-1} + m\vec{J}_{2k}$ for $\vec{b} = (0, 1, -1, 2, -2, \ldots, \frac{k}{2})$ and some constant m. $\blacksquare$

We will use various cyclic permutations of $\vec{c} = (0.1. -2, 3, -4, \ldots, k-1)$ in generating two families of equivalence-class cycles which will be meshed in Section 4, so we define, for a sequence $\vec{u} = u_1 u_2 \ldots u_x$, its cyclic shift by i digits, $\vec{u}(i) = u_{i+1} u_{i+2} \cdots u_x u_1 \cdots u_i$. For $0 \leq i \leq (k-2)/2$, k even, let $\vec{\alpha}_i$ begin with $(2i)$

and be generated by $\vec{c}(2i)$, and for $0 \le j \le (k-2)/2$ let $\vec{\beta}_j$ begin with 0 and be generated by $\vec{c}(-2j)$. Notice that $\vec{\alpha}_i(2) = \vec{\beta}_i^{-1}$. Now we are ready to prove our main result.

4. $dB_{2st}(4st^2, 4s^3t^2; 2, 2)$

Proof of Theorem 1.5. Let $k = 2st$ and for each $0 \le i < st$ and $0 \le j < st$ define the matrix $M_{i,j} = \text{Mesh}\,(\vec{\alpha}_i, \vec{\beta}_j)$, where $\vec{\alpha}_i$ and $\vec{\beta}_j$ are as above. We will use these s^2t^2 $(4st) \times (4st)$ matrices as building blocks to construct the necessary de Bruijn torus.

Claim 4.1. *When viewed as a torus, each $M_{i,j}$ covers a set of distinct 2×2 k-ary matrices.*

Proof. Since $\vec{\alpha}_i$ and $\vec{\beta}_j$ are both even we can easily mimic the proof of Claim 2.2 to show no two 2×2 matrices in $M_{i,j}$ are identical. ∎

Claim 4.2. *The matrices $M_{i,j}$ cover mutually disjoint sets of 2×2 k-ary matrices.*

Proof. Let $M = M_{i,j}$ have entries $m_{x,y}$, $M' = M_{i',j'}$ have entries $m'_{x',y'}$, and suppose

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

is found in both M and M'. Then there are x, y, x', and y' so that

$$M(x,y) = \begin{pmatrix} m_{x,y} & m_{x,y+1} \\ m_{x+1,y} & m_{x+1,y+1} \end{pmatrix} = D = \begin{pmatrix} m'_{x',y'} & m'_{x',y'+1} \\ m'_{x'+1,y'} & m'_{x'+1,y'+1} \end{pmatrix} = M'(x',y').$$

If $x + y$ and $x' + y'$ are both even then $(d_{11}, d_{22}) = (b_y, b_{y+1})$ and $(d_{11}, d_{22}) = (b'_{y'}, b'_{y'+1})$, where $\vec{\beta}_j = b_0 b_1 \cdots b_{4st}$ and $\vec{\beta}_{j'} = b'_0 b'_1 \cdots b'_{4st}$. But because the sequences $\vec{\beta}_j$ cover mutually disjoint sets of pairs, this implies that $j' = j$ and $y' = y$. Likewise, we surmise $i' = i$ and $x' = x$ from $(a_x, a_{x+1}) = (d_{12}, d_{21}) = (a'_{x'}, a'_{x'+1})$, where $\vec{\alpha}_i = a_0 a_1 \cdots a_{4st}$ and $\vec{\alpha}_{i'} = a'_0 a'_1 \cdots a'_{4st}$. A similar argument works when $x + y$ and $x' + y'$ are both odd.

Now suppose, without loss of generality, that $x+y$ is even and $x'+y'$ is odd. Then we have the equality

$$\begin{pmatrix} b_y & a_x \\ a_{x+1} & b_{y+1} \end{pmatrix} = \begin{pmatrix} a'_{x'} & b'_{y'+1} \\ b'_{y'} & a'_{x'+1} \end{pmatrix}.$$

Thus $(b_y, b_{y+1}) = (a'_{x'}, a'_{x'+1})$, and since $\vec{\beta}_j$ covers the same pairs as $\vec{\alpha}_j$ only, we must have $i' = j$. Likewise $(b'_{y'}, b'_{y'+1}) = (a_{x+1}, a_x)$ implies $j' = i$, and so $M = M_{i,j}$ and $M' = M_{j,i}$. Thus we may write

$$M'(x',y') = \text{Mesh}\,(\vec{\alpha}_{i'}, \vec{\beta}_{j'})(x',y') = \begin{pmatrix} a'_{x'} & b'_{y'+1} \\ b'_{y'} & a'_{x'+1} \end{pmatrix}$$

as

$$\text{Mesh}\,(\vec{\alpha}_j, \vec{\beta}_i)(x',y') = \begin{pmatrix} a_{x'} & b_{y'+1} \\ b_{y'} & a_{x'+1} \end{pmatrix}.$$

The next thing to notice about $\vec{\alpha}_j$ and $\vec{\beta}_j$ is that if $(a_{x'}, a_{x'+1}) = (b_y, b_{y+1})$ then x' and y have the same parity. This follows from the relation $\vec{\alpha}_j(2) = \vec{\beta}_j^{-1}$ and the fact that both $\vec{\alpha}_j$ and $\vec{\beta}_j$ are even. Similarly, in $\vec{\alpha}_i$ and $\vec{\beta}_i$, $(a_{x+1}, a_x) = (b_{y'}, b_{y'+1})$ implies that x and y' have the same parity. But then $x + y$ and $x' + y'$ have the same parity, contadicting the fact that the former is even and the latter is odd. Hence the claim is proven. ∎

Claim 4.3. *If $M_{i,j}$ covers a set S of 2×2 matrices and if $M_{i',j'}$ covers a set S' of 2×2 matrices, and if their first columns are identical, then their horizontal juxtaposition $M_{i,j}M_{i',j'}$ covers $S \cup S'$. Likewise, if their last rows are identical, then their vertical juxtaposition $(M_{i,j}^T M_{i',j'}^T)^T$ covers $S \cup S'$.*

Proof. Only those matrices on the boundary of the fundamental region can be affected, but since we are covering only 2×2 matrices, these are unaffected if the first columns (last rows) match. $\blacksquare$

We are now in the position of knowing that if S is the set of all k-ary 2×2 matrices and S_{ij} are those covered by $M_{i,j}$, then S is the disjoint union of the $S_{i,j}$ (since $(s^2t^2)(4st)^2 = k^4$), and if the columns and/or rows of the $M_{i,j}$ match up properly, we can juxtapose them carefully to form the necessary torus.

Claim 4.4. *The first column of every $M_{i,j}$ is identical.*

Proof. It is a matter of verification that each $\vec{\alpha}_i = \_0\_(k-1)\_\cdots\_3\_2\_1$, where only the blank spaces vary. Since the first digit of $\vec{\beta}_j$ is always 0, this yields a first column of $(000(k-1)\cdots 030201)^T$. $\blacksquare$

Thus we never need to worry about horizontal juxtaposition. Our only care is vertically, as follows.

Claim 4.5. *For fixed j, the last row of each $M_{i,j}$ is identical.*

Proof. The last digit of each $\vec{\alpha}_i$ is a 1. $\blacksquare$

The proof of Theorem 1.5 is complete with only a little notation. For $0 \le r < s$ we let $\Gamma_{r,j}$ be the vertical juxtaposition $[(M_{tr,j}^T)(M_{tr+1,j}^T)\cdots(M_{tr+t-1,j}^T)]^T$, Q_j the horizontal juxtaposition $\Gamma_{0,j}\Gamma_{1,j}\cdots\Gamma_{s-1,j}$, and finally $Q = Q_0Q_1\cdots Q_{st-1} \in dB_{2st}(4st^2, 4s^3t^2; 2, 2)$. $\blacksquare$

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