Perfect Maps

Garth Isaak

Lehigh University

0 0 1 1 2 1 0 2 2 0 0 1 ...

A ternary 1-dimensional perfect map with window size 2

0 0 1 1 2 1 0 2 2 0 0 1 . . .

A ternary 1-dimensional perfect map with window size 2

$\boxed{0\ 0}\ 1\ 1\ 2\ 1\ 0\ 2\ 2\ 0\ 0\ 1\ \ldots$

00

A ternary 1-dimensional perfect map with window size 2

$$0\ \boxed{0\ 1}\ 1\ 2\ 1\ 0\ 2\ 2\ 0\ 0\ 1\ \ldots$$

$00, 01$

A ternary 1-dimensional perfect map with window size 2

$$0 \ 0 \ \boxed{1 \ 1} \ 2 \ 1 \ 0 \ 2 \ 2 \ 0 \ 0 \ 1 \ \ldots$$

$00, 01, 11$

A ternary 1-dimensional perfect map with window size 2

$$0 \; 0 \; 1 \; \boxed{1 \; 2} \; 1 \; 0 \; 2 \; 2 \; 0 \; 0 \; 1 \; \ldots$$

$00, 01, 11, 12$

A ternary 1-dimensional perfect map with window size 2

$0 \; 0 \; 1 \; 1 \; \boxed{2 \; 1} \; 0 \; 2 \; 2 \; 0 \; 0 \; 1 \; \ldots$

$00, 01, 11, 12, 21$

A ternary 1-dimensional perfect map with window size 2

$$0 \;\; 0 \;\; 1 \;\; 1 \;\; 2 \;\; \boxed{1 \;\; 0} \;\; 2 \;\; 2 \;\; 0 \;\; 0 \;\; 1 \;\; \ldots$$

$00, 01, 11, 12, 21, 10$

A ternary 1-dimensional perfect map with window size 2

$$0\ 0\ 1\ 1\ 2\ 1\ \boxed{0\ 2}\ 2\ 0\ 0\ 1\ \ldots$$

$00, 01, 11, 12, 21, 10, 02$

A ternary 1-dimensional perfect map with window size 2

$0 \; 0 \; 1 \; 1 \; 2 \; 1 \; 0 \; \boxed{2 \; 2} \; 0 \; 0 \; 1 \; \ldots$

$00, 01, 11, 12, 21, 10, 02, 22$

A ternary 1-dimensional perfect map with window size 2

$$0\ 0\ 1\ 1\ 2\ 1\ 0\ 2\ \boxed{2\ 0}\ 0\ 1\ \ldots$$

$00, 01, 11, 12, 21, 10, 02, 22, 20$

A ternary 1-dimensional perfect map with window size 2

$$0\ 0\ 1\ 1\ 2\ 1\ 0\ 2\ 2\ 0\ 0\ 1\ \ldots$$

$00, 01, 11, 12, 21, 10, 02, 22, 20$

Each size 2 ternary string appears exactly once

A ternary 1-dimensional perfect map with window size 2

0 0 1 1 2 1 0 2 2 0 0 1 . . .

00, 01, 11, 12, 21, 10, 02, 22, 20

Each size 2 ternary string appears exactly once

Also called DeBruijn cycles

History

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1946 N.G. DeBruijn - telephone engineering

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History

? Other?

1892 E. Baudot - telegraphy- binary, window size 5

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Notation

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$$0 \quad 0 \quad 1 \; \boxed{1 \quad 2} \; 1 \quad 0 \quad 2 \quad 2 \quad 0 \quad 0 \quad 1 \quad \ldots$$

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$$0 \quad 0 \quad 1 \quad \boxed{1 \quad 2} \quad 1 \quad 0 \quad 2 \quad 2 \quad 0 \quad 0 \quad 1 \quad \ldots$$

is in

$$PF_3{}^1(9;2;1)$$

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$$PF_3{}^1(9; 2; 1)$$

- PF - Perfect factor
- 3 - alphabet size 3
- 1 - dimension 1
- 9 - length 9
- 2 - window size 2
- 1 - 1 string

Notation - I apologize, will try not to rely on this notation too much

$$0 \quad 0 \quad 1 \quad \boxed{1 \quad 2} \quad 1 \quad 0 \quad 2 \quad 2 \quad 0 \quad 0 \quad 1 \quad \ldots$$

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$$A_1 = 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 2 \quad 1 \quad 2 \quad 1 \quad 0 \quad 0$$

$$A_2 = 1 \quad 1 \quad 1 \quad 2 \quad 0 \quad 0 \quad 2 \quad 0 \quad 2 \quad 1 \quad 1$$

$$A_3 = 2 \quad 2 \quad 2 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 2 \quad 2$$

is in

$$PF_3{}^1(9; 3; 3)$$

$$
\begin{array}{lccccccccccccc}
A_1 & = & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 2 & 1 & 0 & 0 \\
A_2 & = & 1 & 1 & 1 & 2 & 0 & 0 & 2 & 0 & 2 & 1 & 1 \\
A_3 & = & 2 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 2
\end{array}
$$

is in

$$PF_3{}^1(9;3;3)$$

Every ternary length 3 string appears exactly once in this collection of 3 length 9 strings

$$
\begin{aligned}
A_1 &= 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad \boxed{2 \quad 1 \quad 2} \quad 1 \quad 0 \quad 0 \\
A_2 &= 1 \quad 1 \quad 1 \quad 2 \quad 0 \quad 0 \quad 2 \quad 0 \quad 2 \quad 1 \quad 1 \\
A_3 &= 2 \quad 2 \quad 2 \quad \boxed{0 \quad 1 \quad 1} \quad 0 \quad 1 \quad 0 \quad 2 \quad 2
\end{aligned}
$$

is in

$$PF_3{}^1(9;3;3)$$

Every ternary length 3 string appears exactly once in this collection of 3 length 9 strings

For example, 212 and 011 are indicated above

$$\begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}$$

is in $PF_2{}^2((4,4);(2,2);1)$

Every binary 2 by 2 array appears exactly once in this 4 by 4, two dimensional array

$$\begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}$$

is in $PF_2{}^2((4,4);(2,2);1)$

Every binary 2 by 2 array appears exactly once in this 4 by 4, two dimensional array For example $\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$ and $\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}$ are indicated above (note the wrapping property)

Review of Basics: A construction for 1-dimensional perfect maps when k is a prime power: These are feedback shift register sequences

- Let $h(x) = x^n + h_{n-1}x^{n-1} + \cdots + h_1 x + x_0$ be a primitive polynomial of degree n over $GF(k)$
- Let $f(x_1 x_2 \ldots x_n) = -h_0 x_1 - h_1 x_2 - \cdots - h_{n-1} x_n$
- Given terms in a string $x_1 x_2 \ldots x_n$ let the next term be $f(x_1 x_2 \ldots x_n)$
- This produces a perfect map (except for omitting $000 \ldots 00$)
- This method is useful for efficient construction and also used for 2-dimensional perfect factors ...(details omitted)

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For all alphabet sizes k and window sizes n, one dimensional perfect maps exist. That is, $PF_k{}^1(k^n; n; 1)$ is non-empty. Note that the string length is determined by k and n.

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Construct a digraph $D(k, n)$:
Vertices ' $=$ ' k-ary strings of length n
Arcs: $(s_1 s_2 \ldots s_n) \longrightarrow (s_2 s_3 \ldots s_n s_{n+1})$ between strings that can appear as consecutive windows

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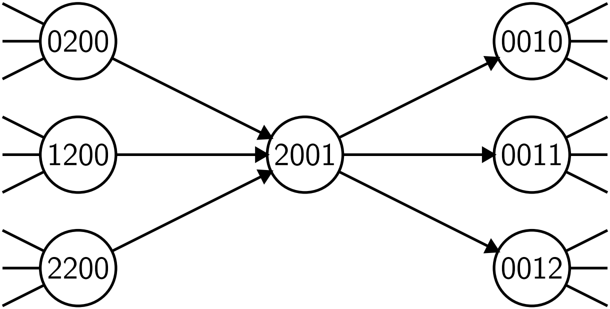
Part of digraph $D(3,4)$:



Construct a digraph $D(k,n)$:

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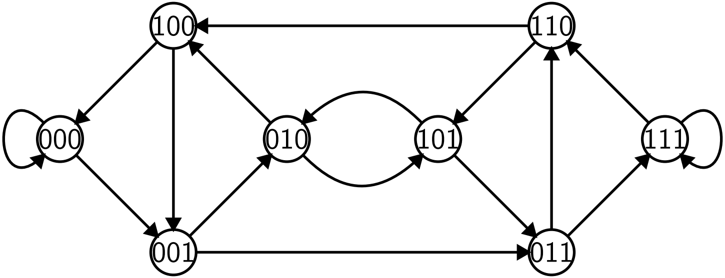
A Hamiltonian cycle in $D(k, n)$ corresponds to a perfect map



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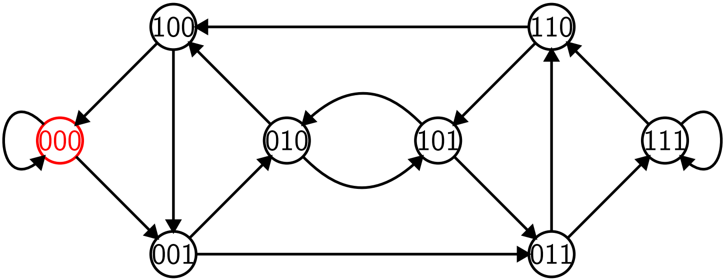

000

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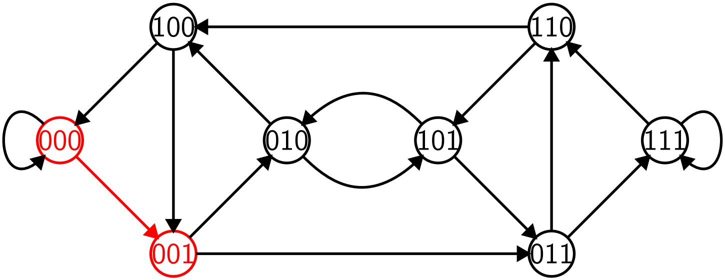

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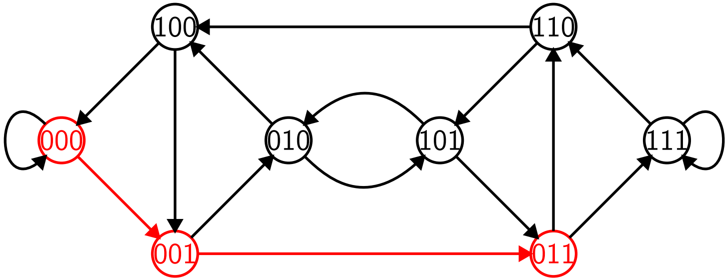

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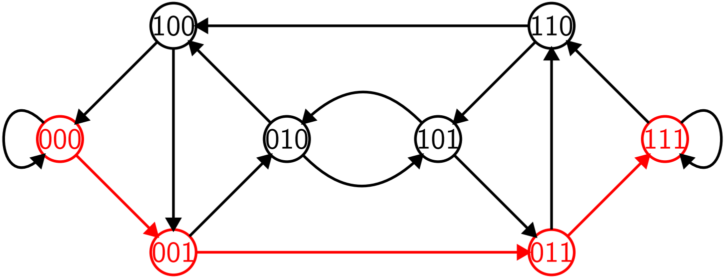

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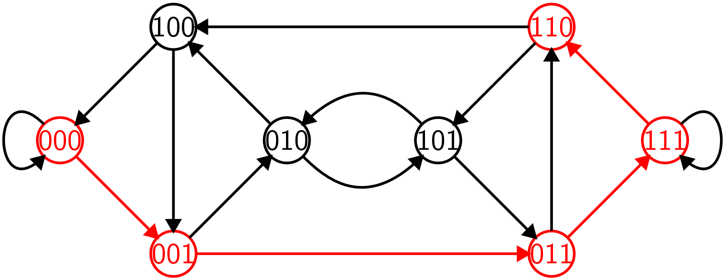

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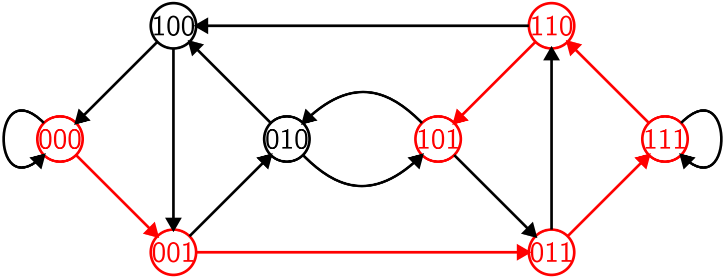

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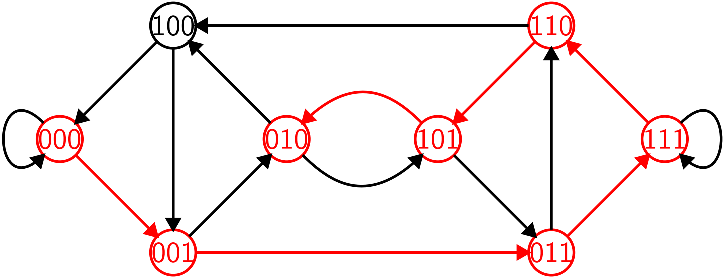

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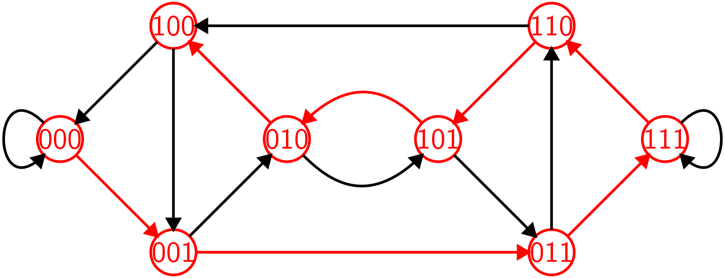

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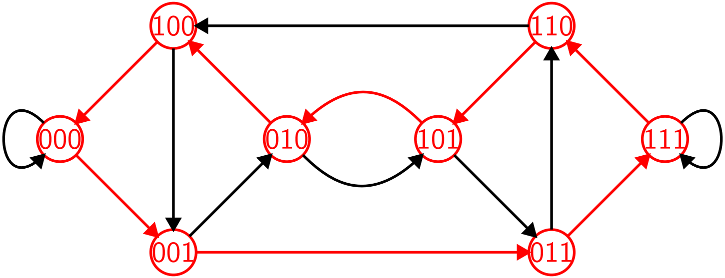

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Finding Hamiltonian cycles is 'hard'

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BUT ...

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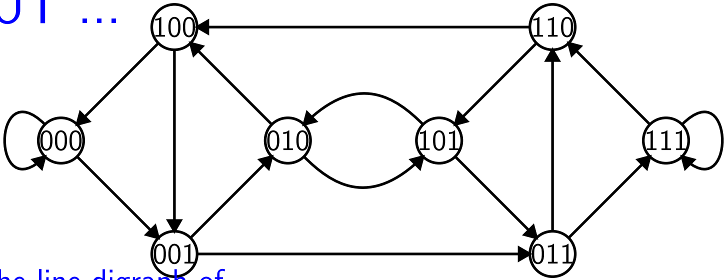
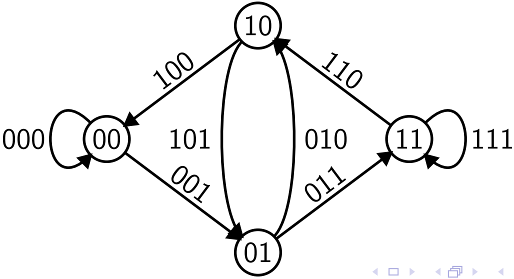
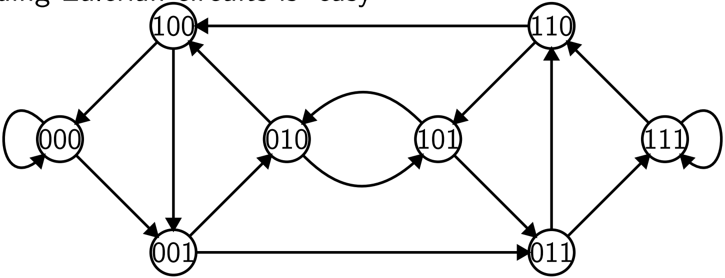
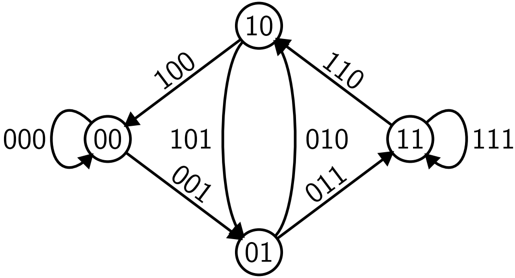


is the line digraph of



Finding Eulerian circuits is 'easy'





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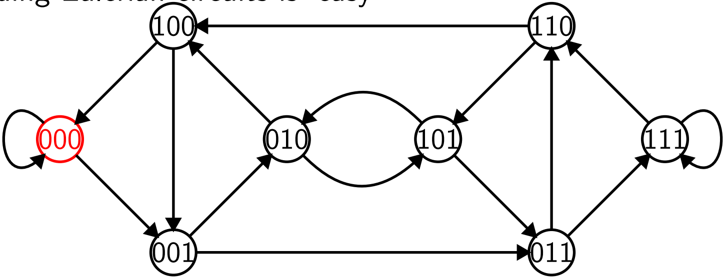

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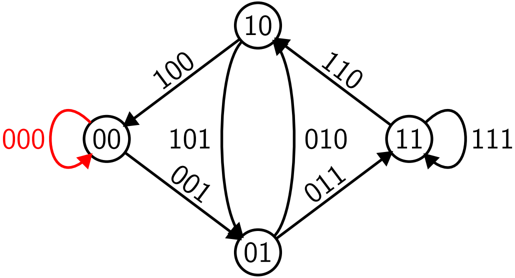

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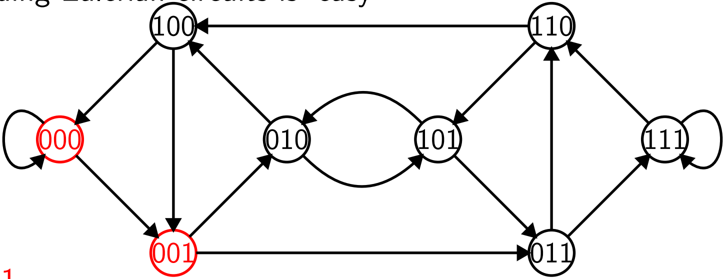

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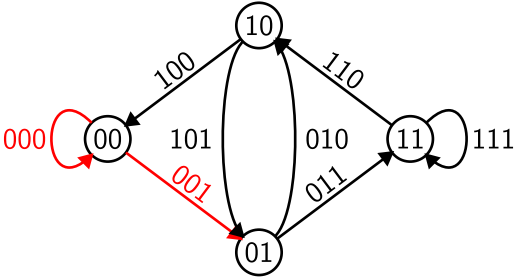

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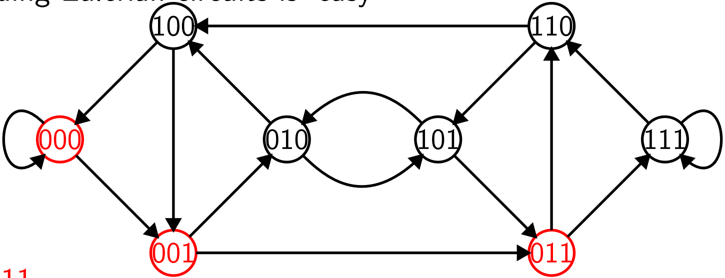

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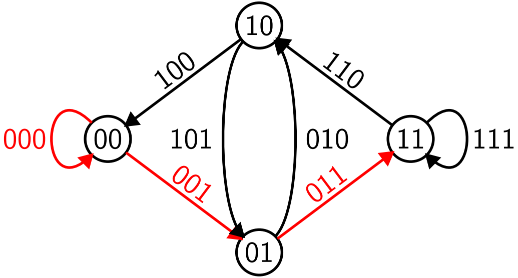

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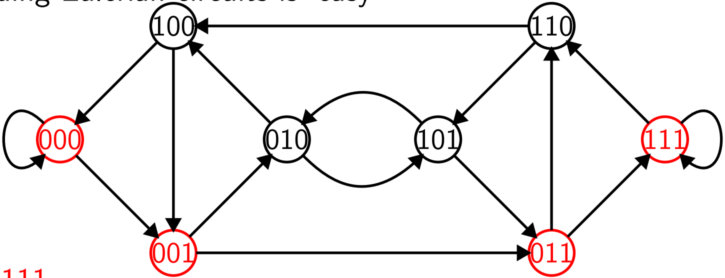

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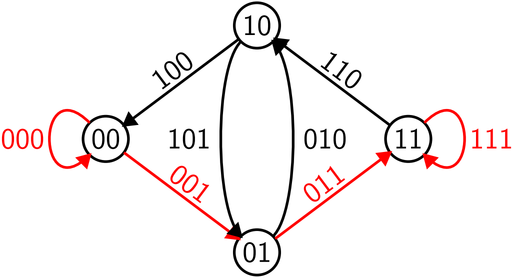

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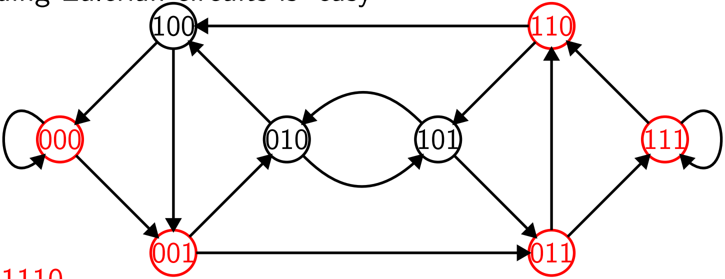

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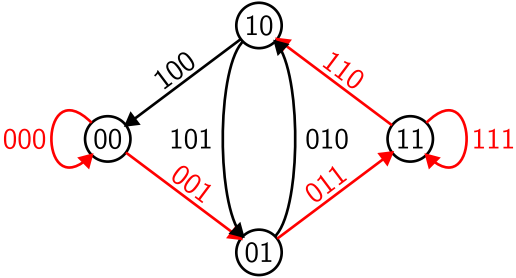

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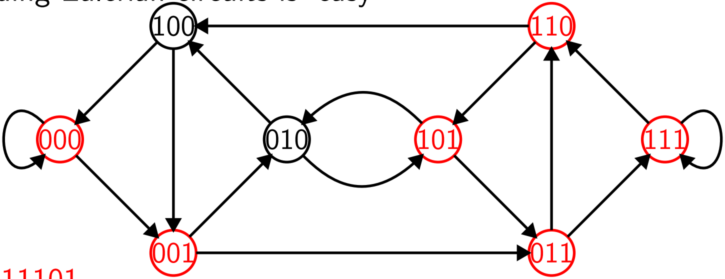

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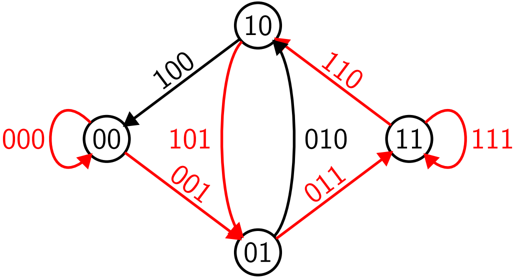

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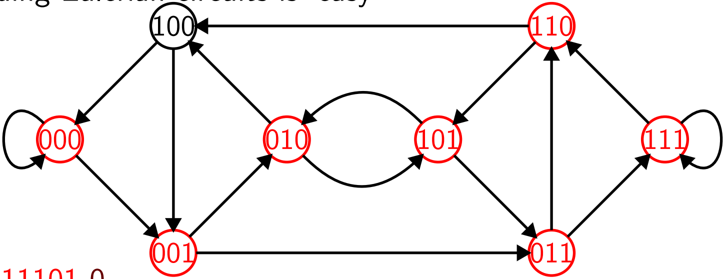
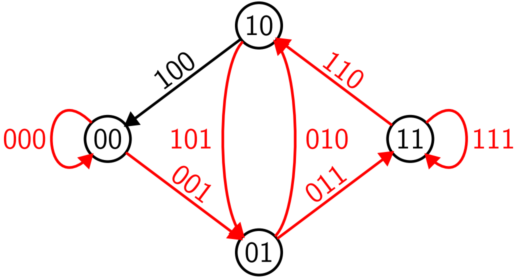


00011101 0



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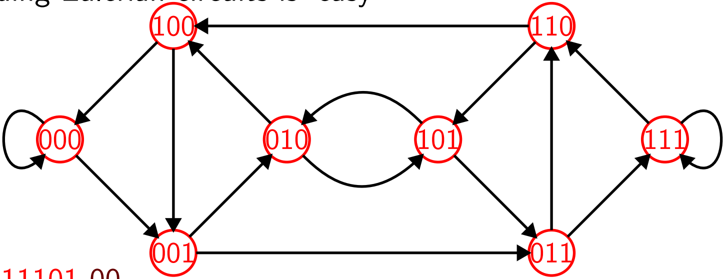
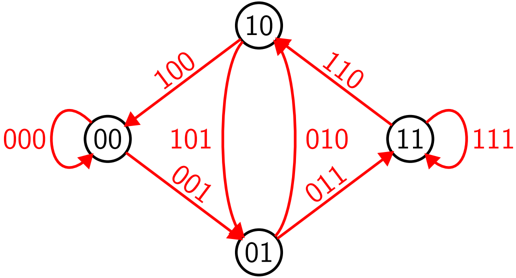


00011101 00



It is easy to check that the digraphs $D(k, n-1)$ are Eulerian: they are connected and each vertex has indegree and outdegree k. The Eulerian circuits correspond to Hamiltonian cycles in $D(k, n)$.

Enumeration

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Label the arcs leaving a given vertex in order that they are traversed in the Eulerian circuit starting from $000\ldots00$. The arcs traversed last form a spanning in-tree rooted at $000\ldots00$.

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The number of perfect maps for k-ary windows of size n is

$$[(k-1)!]^{k^{n-1}}\, k^{k^{n-1}-n}$$

Universal Cycles

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We illustrate with permutations ('easier' than subsets ...) : Look at length 3-permutations of $\{1, 2, 3, 4, 5\}$:

123421423215241321354153253452314531542543245134124351431251234

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Every length 3-permutation of $\{1, 2, 3, 4, 5\}$ appears exactly once

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$$\boxed{1\ 2\ 3}\ 4\ 2\ 1\ 4\ 2\ 3\ 5\ 2\ 1\ \ldots$$

123

Every length 3-permutation of $\{1, 2, 3, 4, 5\}$ appears exactly once

$$1 \; \boxed{2 \; 3 \; 4} \; 2 \; 1 \; 4 \; 2 \; 3 \; 5 \; 2 \; 1 \; \ldots$$

$123, 234$

Every length 3-permutation of $\{1, 2, 3, 4, 5\}$ appears exactly once

$$1 \ 2 \ \boxed{3 \ 4 \ 2} \ 1 \ 4 \ 2 \ 3 \ 5 \ 2 \ 1 \ \ldots$$

$123, 234, 342$

Every length 3-permutation of $\{1, 2, 3, 4, 5\}$ appears exactly once

$$1 \ 2 \ 3 \ \boxed{4 \ 2 \ 1} \ 4 \ 2 \ 3 \ 5 \ 2 \ 1 \ \ldots$$

$123, 234, 342, 421$

Every length 3-permutation of $\{1, 2, 3, 4, 5\}$ appears exactly once

1 2 3 4 $\boxed{2\ 1\ 4}$ 2 3 5 2 1 ...

$123, 234, 342, 421, 214$

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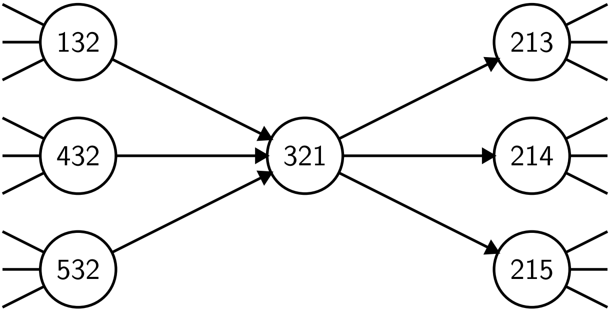
$$1\ 2\ 3\ 4\ 2\ 1\ 4\ 2\ 3\ 5\ 2\ 1\ \ldots$$

$123, 234, 342, 421, 214, \ldots$

Find a Hamiltonian cycle in a particular graph $Q(n, k)$



Graph for 3 permutations of $\{1, 2, 3, 4, 5\}$

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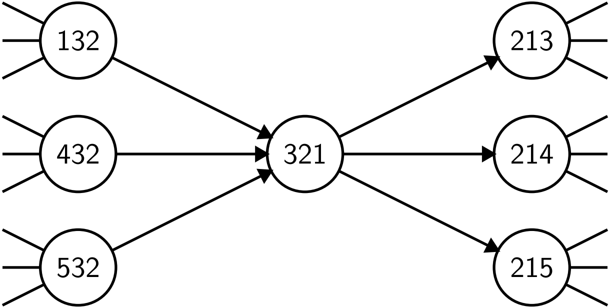
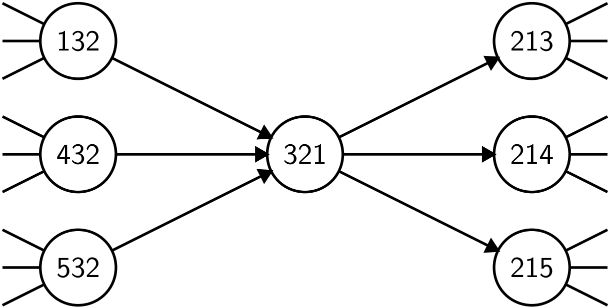


Graph for 3 permutations of $\{1, 2, 3, 4, 5\}$

Like the perfect map case these are line digraphs and similar methods work to show existence of universal cycles for k permutations of $\{1, 2, \ldots, n\}$

Find a Hamiltonian cycle in a particular graph $Q(n, k)$

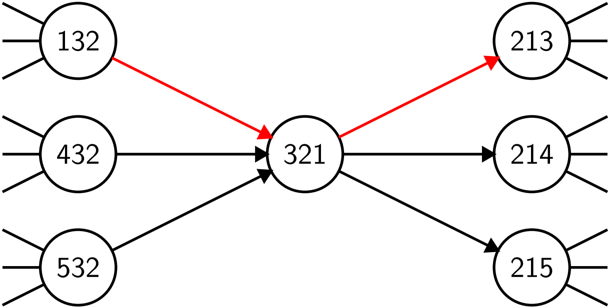


Graph for 3 permutations of $\{1, 2, 3, 4, 5\}$

Like the perfect map case these are line digraphs and similar methods work to show existence of universal cycles for k permutations of $\{1, 2, \ldots, n\}$

But $Q(n, k)$ is the line digraph of some other digraph $P(n, k)$ and not of $Q(n, k - 1)$

These other digraphs $P(n, k)$ omit edges that do not correspond to permutations



Omit the red edges

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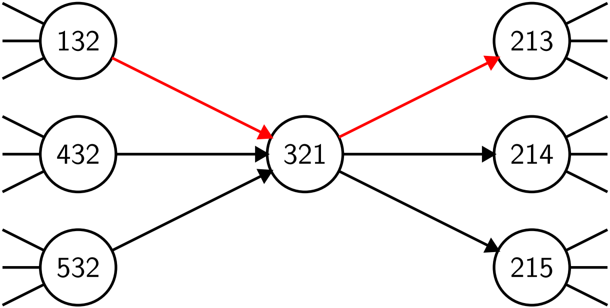

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- If yes then we get universal cycles for k-permutations for which the $k+1$ strings are also permutations
- These digraphs were introduced by Fiol et al. in a different context and the question of Hamiltonicity asked by Klerlein, Carr and Starling (at a Southeast conference)

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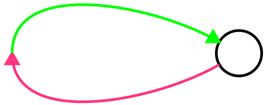
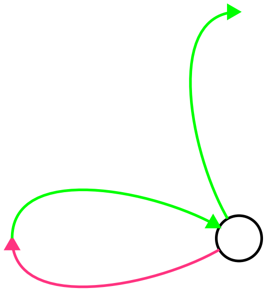
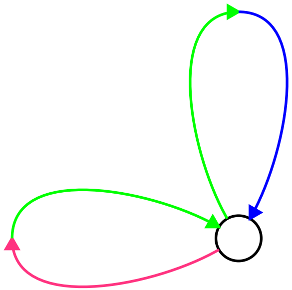
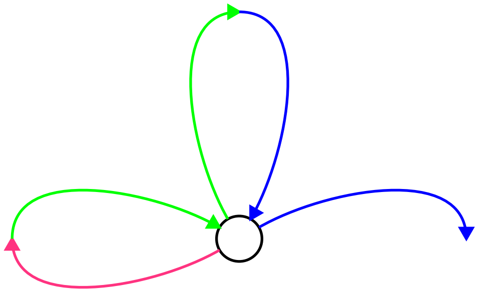
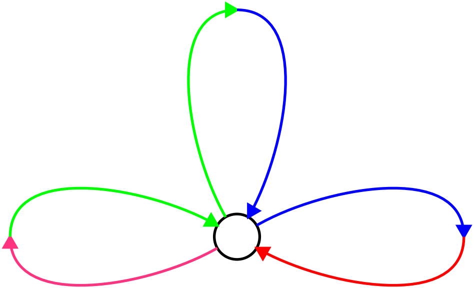
- Obtain $P(n, k)$ from the line digraph of $P(n, k-1)$ by deleting a few arcs
- An Eulerian circuit in $P(n, k)$ that avoids certain turns produces a Hamiltonian cycle in $P(n, k)$

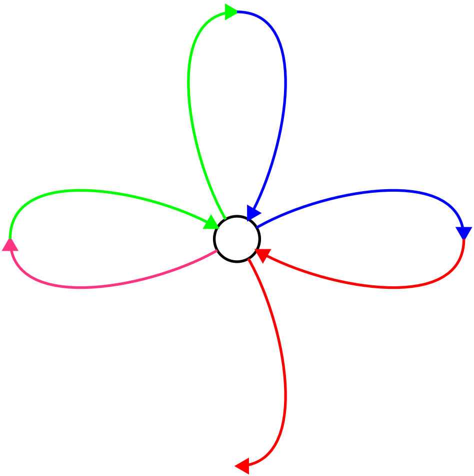


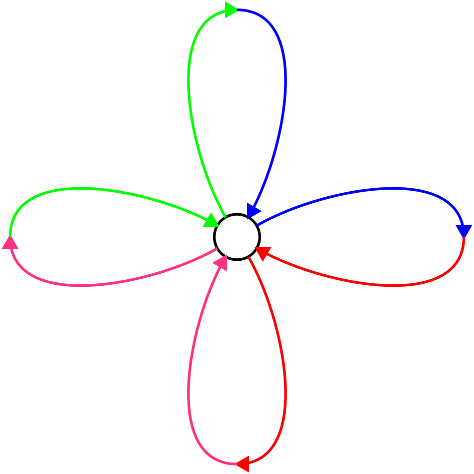


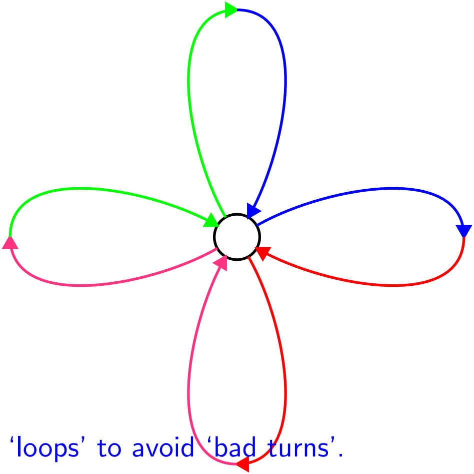




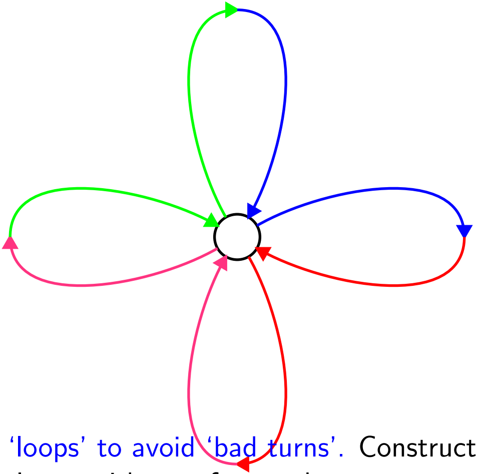




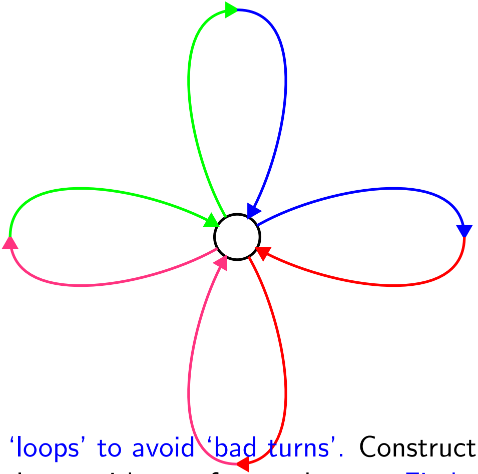




Rearrange the ‘loops’ to avoid ‘bad turns’.



Rearrange the ‘loops’ to avoid ‘bad turns’. Construct a new digraph on the loops with arcs for good turns.



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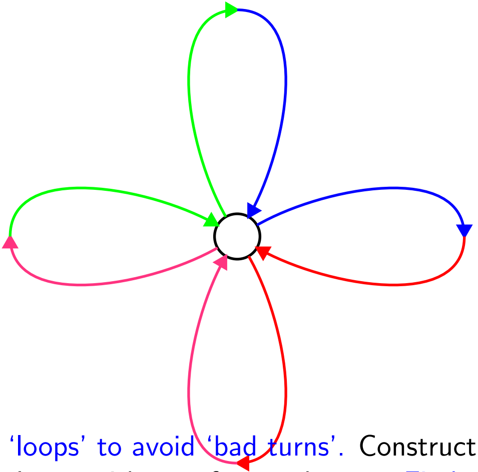

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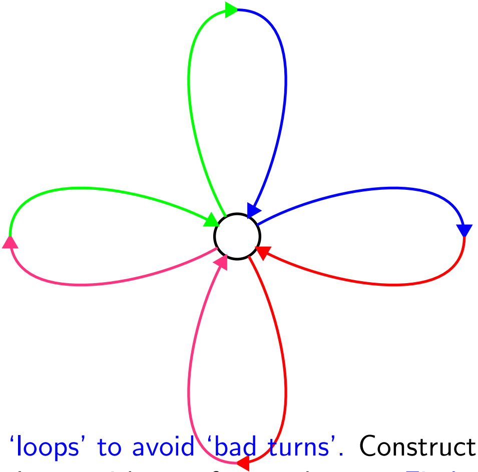

Rearrange the ‘loops’ to avoid ‘bad turns’. Construct a new digraph on the loops with arcs for good turns. Find a Hamiltonian cycle to give the rearrangement. The Hamiltonian cycle exists in the new digraph as degrees are high enough. Repeat for all vertices of $P(n, k)$ to get a Hamiltonian cycle

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- Rankin's Theorem 1948 application to camponology (bell ringing)

Back to perfect maps

Perfect Maps and Factors in higher dimensions - a partial history:

1961 Reed and Stewart

1985 Fan,Fan,Ma,Sui and Etzion

1988 Cock

1988 Ivanyi and Toth

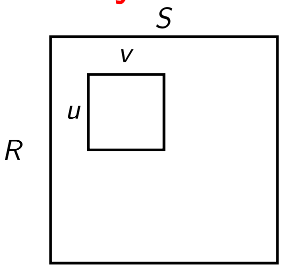
1993 Hurlbert and Isaak

1994 Mitchell and Paterson

1996 Paterson

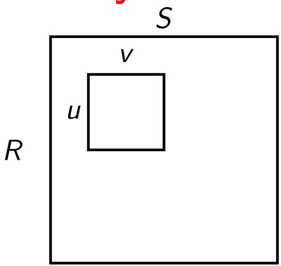
- and many others

Necessary Conditions



- For 2-dimensional $k-$ary perfect maps
-
-
-
-

Necessary Conditions



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- There are RS entries/windows and k^{uv} possible windows
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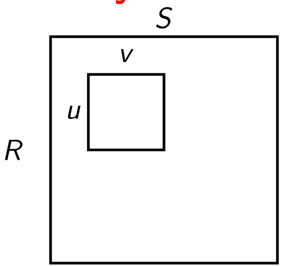

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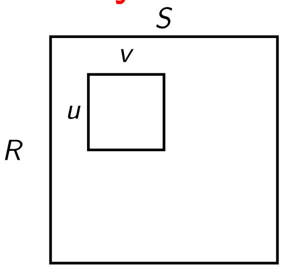

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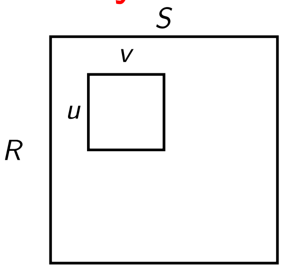

- For 2-dimensional $k-$ary perfect maps
- There are RS entries/windows
  and k^{uv} possible windows
- So $RS = k^{uv}$
- The all 0 window is repeated if $u = R$ or $v = S$
- So $R > u$ and $S > v$

Necessary Conditions

$$RS = k^{uv} \text{ and } R > u, S > v$$

Similar conditions for perfect factors and for higher dimensions

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Similar conditions for perfect factors and for higher dimensions

- Are these sufficient?
- 1-dimensional perfect maps - YES
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- Otherwise? - partial results
- Difficulty with sizes like $2^{12} \times 3^{12}$ with window size 3×4 and 6-ary

Non-prime powers alphabets from prime power alphabets:

$$\begin{array}{ccc} 0011 & \in & PF_2^1(4;2;1) \\ \{001, 112, 220\} & \in & PF_3^1(3;2;3) \end{array}$$

'Combine'

$$\begin{array}{ccccccccccccc} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \oplus & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ \hline & 1 & 1 & 5 & 4 & 1 & 2 & 4 & 4 & 2 & 1 & 4 & 5 \end{array}$$

Using also 001 and 220 this gives

$$\left\{ \begin{array}{c} 115412442145 \\ 004301331034 \\ 223520550253 \end{array} \right\} \in PF_6^1(12;2;3)$$

Higher Dimensional Perfect Maps

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- Two basic techniques have been implicit in many results
- 'Integration' 'grows' the window size
- 'Concatenation' increases the dimension
- Both use as tools perfect factors, perfect multifactors, equivalence class perfect multifactors ...

Concatenation

Start with 1-dimensional perfect map 001121022 for columns and shift sequence 01023456789 a perfect map with window size 1

	0	1	2	3	4	5	6	7	8
0									
0									
1									
1									
2									
1									
0									
2									
2									

Concatenation

Start with 1-dimensional perfect map 001121022 for columns and shift sequence 01023456789 a perfect map with window size 1

0	1	2	3	4	5	6	7	8
0	0							
0	0							
1	1							
1	1							
2	2							
1	1							
0	0							
2	2							
2	2							

Concatenation

Start with 1-dimensional perfect map 001121022 for columns and shift sequence 01023456789 a perfect map with window size 1

0	1	2	3	4	5	6	7	8
0	0	0						
0	0	1						
1	1	1						
1	1	2						
2	2	1						
1	1	0						
0	0	2						
2	2	2						
2	2	0						

Concatenation

Start with 1-dimensional perfect map 001121022 for columns and shift sequence 01023456789 a perfect map with window size 1

0	1	2	3	4	5	6	7	8
0	0	0	1					
0	0	1	2					
1	1	1	1					
1	1	2	0					
2	2	1	2					
1	1	0	2					
0	0	2	0					
2	2	2	0					
2	2	0	1					

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Start with 1-dimensional perfect map 001121022 for columns and shift sequence 01023456789 a perfect map with window size 1

0 1 2 3 4 5 6 7 8

0	0	0	1	0
0	0	1	2	2
1	1	1	1	2
1	1	2	0	0
2	2	1	2	0
1	1	0	2	1
0	0	2	0	1
2	2	2	0	2
2	2	0	1	1

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Start with 1-dimensional perfect map 001121022 for columns and shift sequence 01023456789 a perfect map with window size 1

0 1 2 3 4 5 6 7 8

0	0	0	1	0	0
0	0	1	2	2	1
1	1	1	1	2	1
1	1	2	0	0	2
2	2	1	2	0	1
1	1	0	2	1	0
0	0	2	0	1	2
2	2	2	0	2	2
2	2	0	1	1	0

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	0	1	2	3	4	5	6	7	8
0	0	0	1	0	0	0			
0	0	1	2	2	1	2			
1	1	1	1	2	1	2			
1	1	2	0	0	2	0			
2	2	1	2	0	1	0			
1	1	0	2	1	0	1			
0	0	2	0	1	2	1			
2	2	2	0	2	2	2			
2	2	0	1	1	0	1			

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0	1	2	3	4	5	6	7	8
0	0	0	1	0	0	0	1	
0	0	1	2	2	1	2	2	
1	1	1	1	2	1	2	1	
1	1	2	0	0	2	0	0	
2	2	1	2	0	1	0	2	
1	1	0	2	1	0	1	2	
0	0	2	0	1	2	1	0	
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0	1	2	3	4	5	6	7	8
0	0	0	1	0	0	0	1	0
0	0	1	2	2	1	2	2	1
1	1	1	1	2	1	2	1	1
1	1	2	0	0	2	0	0	2
2	2	1	2	0	1	0	2	1
1	1	0	2	1	0	1	2	0
0	0	2	0	1	2	1	0	2
2	2	2	0	2	2	2	0	2
2	2	0	1	1	0	1	1	0

$PF_3^2((9,9);(2,2);1)$

Find $\begin{matrix} 1 & 2 \\ 0 & 2 \end{matrix}$ where there is a shift of 2

The shift in location from 10 to 22 in 001121022

0	1	2	3	4	5	6	7	8
0	0	0	1	0	0	0	1	0
0	0	1	2	2	1	2	2	1
1	1	1	1	2	1	2	1	1
1	1	2	0	0	2	0	0	2
2	2	1	2	0	1	0	2	1
1	1	0	2	1	0	1	2	0
0	0	2	0	1	2	1	0	2
2	2	2	0	2	2	2	0	2
2	2	0	1	1	0	1	1	0

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- In general needs 1-dimensional perfect factors for the shifts

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- Concatenation of perfect factors requires two 1-dimensional factors; one for shifts and one to pick which factor

'Integrating' to produce perfect factors (the inverse of Lempel's homomorphism, finite difference operator):

For 1-dimensional perfect factors:

	0	0	1	1	2	1	0	2	2
0									

The first row 001121022 is a $PF_3^1(9;2;1)$ (window size 2) and gives the differences for (part of) a perfect factor with window size 3. Start with 0

‘Integrating’ to produce perfect factors (the inverse of Lempel’s homomorphism, finite difference operator):

For 1-dimensional perfect factors:

	0	0	1	1	2	1	0	2	2
0	0								

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	0		0		1	1	2	1	0	2	2
0		0		0							

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0		0		0		1						

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$$\begin{array}{cccccccccccccccc} & 0 & & 0 & & 1 & & 1 & & 2 & & 1 & 0 & 2 & 2 \\ 0 & & 0 & & 0 & & 1 & & 2 & & 1 & & & & \end{array}$$

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0		0		0		1		2		1		2				

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	0		0		1		1		2		1		0		2		2
0		0		0		1		2		1		2		2		1	

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	0		0		1		1		2		1		0		2		2
0		0		0		1		2		1		2		2		1	
1		1		1		2		0		2		0		0		2	
2		2		2		0		1		0		1		1		0	

The top row 001121022 is a $PF_3^1(9;2;1)$ and gives the differences for each of the other rows. The other rows differ by the constant ‘starter’ in the first column.

The other rows form a $PF_3^1(9;3;3)$

	0	0	1	1	2	1	0	2	2
0	0	0	1	2	1	2	2	1	
1	1	1	$\boxed{2}$	$\boxed{0}$	$\boxed{2}$	0	0	2	
2	2	2	0	1	0	1	1	0	

Find 202: its differences are 12 so it will appear (in one of the rows) in a location ‘below’ 12

	0		0		1		1		2		1		0		2		2
0		0		0		1		2		1		2		2		1	
1		1		1		2		0		2		0		0		2	
2		2		2		0		1		0		1		1		0	

Find 202: its differences are 12 so it will appear (in one of the rows) in a location 'below' 12

Requires sum of the entries to be 0 mod k. If not then number of factors will be decreased and length of each factor increased

	0	0	1	1	2	1	0	2	2
0	0	0	1	2	1	2	2	1	
1	1	1	$\boxed{2}$	$\boxed{0}$	$\boxed{2}$	0	0	2	
2	2	2	0	1	0	1	1	0	

Find 202: its differences are 12 so it will appear (in one of the rows) in a location 'below' 12

Requires sum of the entries to be 0 mod k. If not then number of factors will be decreased and length of each factor increased

We will use integration along 'directions' for higher dimensional perfect factors

$$\begin{array}{cccccccccccccccccc}
 & 0 & & 0 & & 1 & & 1 & & 2 & & 1 & & 0 & & 2 & & 2 \\
0 & & 0 & & 0 & & 1 & & 2 & & 1 & & 2 & & 2 & & 1 & \\
1 & & 1 & & 1 & & \multicolumn{5}{c}{\boxed{2 \quad\quad 0 \quad\quad 2}} & & 0 & & 0 & & 2 & \\
2 & & 2 & & 2 & & 0 & & 1 & & 0 & & 1 & & 1 & & 0 &
\end{array}$$

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Requires sum of the entries to be 0 mod k. If not then number of factors will be decreased and length of each factor increased

We will use integration along ‘directions’ for higher dimensional perfect factors

In general for higher dimensions we need a perfect multifactor for a ‘starter’

Perfect multifactors

000011210220112102201121022

is obtained by writing three 0's followed by three copies of the string 01121022. In this string every 3-ary window of length 2 appears exactly 3 times, once in each position modulo 3. We call this a perfect multifactor. Shifting by 3 and by 6 we get two additional strings

022000011210220112102201121 121022000011210220112102201

for a set of 3, length 27 strings in which each length 2 window appears appears exactly once in each position modulo 9

Integration

Use the second string of the previous example and the 9×9 array from the example preceding that:

$$
\begin{array}{ccccccccc|ccccccccc|ccccccccc}
0&2&2&0&0&0&0&1&1&2&1&0&2&2&0&1&1&2&1&0&2&2&0&1&1&2&1\\
\hline
0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&1&0\\
0&0&1&2&2&1&2&2&1&0&0&1&2&2&1&2&2&1&0&0&1&2&2&1&2&2&1\\
1&1&1&1&2&1&2&1&1&1&1&1&1&2&1&2&1&1&1&1&1&1&2&1&2&1&1\\
1&1&2&0&0&2&0&0&2&1&1&2&0&0&2&0&0&2&1&1&2&0&0&2&0&0&2\\
2&2&1&2&0&1&0&2&1&2&2&1&2&0&1&0&2&1&2&2&1&2&0&1&0&2&1\\
1&1&0&2&1&0&1&2&0&1&1&0&2&1&0&1&2&0&1&1&0&2&1&0&1&2&0\\
0&0&2&0&1&2&1&0&2&0&0&2&0&1&2&1&0&2&0&0&2&0&1&2&1&0&2\\
2&2&2&0&2&2&2&0&2&2&2&2&0&2&2&2&0&2&2&2&2&0&2&2&2&0&2\\
2&2&0&1&1&0&1&1&0&2&2&0&1&1&0&1&1&0&2&2&0&1&1&0&1&1&0
\end{array}
$$

Integrate down the columns:

integration down the columns yields:

$$
\begin{array}{ccccccccccccccccccccccccccc}
0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 2 & 1 \\
0 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 & 2 & 0 & 1 & 2 & 2 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 2 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 2 & 0 & 2 & 2 \\
1 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\
2 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 2 \\
1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 0 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 2 & 2 & 0 \\
2 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\
2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 2 & 1 & 1 & 2 \\
1 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 1
\end{array}
$$

Doing the same thing with the other two possible starters produces three 3-ary 9×27 arrays in which we claim that every 3-ary 3×2 subarray appears exactly once.

The examples we have given hint at several general methods which give hope that that necessary conditions can be shown sufficient in higher dimensions at least for prime power alphabets.

The examples we have given hint at several general methods which give hope that that necessary conditions can be shown sufficient in higher dimensions at least for prime power alphabets.

For non prime power alphabet sizes new tools will probably be needed

Let A be a $(\vec{R};\vec{V};\tau)^d_G[\vec{N}]$ PMF (perfect multifactor). Let $H = Z_{r_1/n_1} \times Z_{r_2/n_2} \times \cdots Z_{r_d/n_d}$ and let $H' = \{1, 2, \ldots, \tau\}$. Let $(B : C)$ be a $(Q;(U-1,U);\rho)_{H,H'}[M]$ PMFP (perfect multifactor pair) with the following property. There exists $c \in H$ such that each string $B(j)$ in B satisfies $\sum_{h=1}^{Q}[B(j)]_h = c$. That is, the entries in each fundamental block sum to c.

Then,

- If $c = 0 \in H$, concatenation using $(B : C)$ as indexer yields a $(\vec{R}^+;\vec{V}^+;\rho)^{d+1}_G[\vec{N}^+]$ PMF (perfect multifactor) where the first d coordinates of $\vec{N}^+$, $\vec{R}^+$ and $\vec{V}^+$ are the same as $\vec{N}$, $\vec{R}$ and $\vec{V}$ and $n^+_{d+1} = M$, $r^+_{d+1} = Q$ and $v^+_{d+1} = U$.
- If $c \neq 0 \in H$ and additionally we have the following: If c is viewed as a vector $\vec{C} = (c_1, c_2, \ldots, c_d)$ with entries from Z and for $i = 1, 2, \ldots, d$ we have $\eta_i = \dfrac{r_i/n_i}{\gcd(r_i/n_i, c_i)}$ (i.e., the

Let A be a $(\vec{Q};\vec{U};\rho)^d_G[\vec{N}]$ PMF (perfect multifactor) with the sum of entries in each (one dimensional) projection along direction d equal to a constant $c \in G$. Let $\vec{Q}^-$ and $\vec{U}^-$ be obtained from $\vec{Q}$ and $\vec{U}$ by deleting the d^{th} dimension.
Then,

- If $c = 0$, let B be a $(\vec{R};\vec{U}^-;\tau)^{d-1}_{G|H}[\vec{Q}^-]$ EPMF (equivalence class perfect multifactor modulo H). Integrating A with starter B yields a $(\vec{R}^+;\vec{U}^*;\rho\tau)^d_{G|H}[\vec{N}]$ EPMF (equivalence class perfect multifactor modulo H) where $\vec{U}^* = \vec{U} + \vec{e}(d)$ and $r^+_d = q_d$.
- If $c \neq 0$, let H' be the subgroup generated by c. Let B be a set of representatives modulo H' of a $(\vec{R};\vec{U}^-;\tau)^{d-1}_{G|H'}[\vec{Q}^-]$ EPMF (equivalence class perfect multifactor modulo H'). Integrating A with starter B yields a $(\vec{R}^+;\vec{U}^*;\rho\tau/|H'|)^d_G[\vec{N}]$ PMF (perfect multifactor) where $\vec{U}^* = \vec{U} + \vec{e}(d)$ and $r^+_d = |H'|q_d$.