Lines and quadratics

1. Polynomials, functions in which each term is a positive integer power of the independent variable include familiar special cases, lines and quadratics. These are simple but common models. In business applications it is common to have lines and quadratics specified using constants. For example we might see $f(x) = \alpha x^2 + \beta x + \gamma$ and $p = aq - b$ instead of $f(x) = x^2 + 2x - 5$ and $y = 2x + 4$ that students may be more familiar with, although the notations $ax^2 + bx + c$ and $y = mx + b$ as general forms are probably familiar.

   Our goal is to work in the setting with constants. So, for example, we aim to be able to derive equations like the quadratic formula rather than just use it.

2. For lines, recall that slope is 'rise over run'. That is, the change in the dependent variable divided by the change in the independent variable. The symbol Δ commonly denotes 'change'.

   We use subscripts to distinguish two (x, y) points, (x_1, y_1) and (x_2, y_2). The change in the y coordinate (the 'rise') is $\Delta y = (y_2 - y_1)$ and the change in the x coordinate is $\Delta x = (x_2 - x_1)$. Then slope is written $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.
   The method that we will use makes use of this. There are several other ways to present the algebra for finding equations of lines.

   (a) To get an equation for the line through two points, for example $(3, 9)$ and $(4, 16)$ we use the slope equation twice: once with the two given points to get the slope and once with one of the points (either one will do) and a generic point (x, y).
   (b) We will write this as follows:
   Slope is $\frac{\Delta y}{\Delta x} = \frac{16-9}{4-3} = 7$. Then $7 = \frac{\Delta y}{\Delta x} = \frac{y-9}{x-3} \Rightarrow 7(x - 3) = y - 9 \Rightarrow y = 7x - 12$.
   (c) Note the use of $\Rightarrow$ as shorthand for 'implies'. Formally we should say a bit more, the first implication follows by 'cross multiplying' and the second from some basic algebra. We will typically not give these additional explanations when they are 'clear' from the manipulations shown.
   (d) We will make use of $\Rightarrow$ rather than just writing on successive lines. This is a math 81 convention so under pressure of writing quickly on quizzes and exams the order of the work is clearly indicated.

3. Treating constants as numbers we can do exactly as the previous example, manipulating the constants as we did the numbers.

   Find an equation for the line through the points $(1, k)$ and $(3, 5k)$ where k is a constant.
   Since we have not specified symbols for the variables we can pick, (x, y) is common.

Slope is $\frac{\Delta y}{\Delta x} = \frac{5k-k}{3-1} = \frac{4k}{2} = 2k$.
Then $2k = \frac{\Delta y}{\Delta x} = \frac{y-k}{x-1} \Rightarrow 2k(x-1) = y-k \Rightarrow y = 2kx - 2k + k \Rightarrow y = 2kx - k$.

This looks more complicated than the previous example, $y = 7x - 12$ but it really isn't. Instead of slope 7 and intercept -12 we have some as yet unspecified number k and we have a slope of $2k$ and intercept $-k$.

4. If we are given the slope we proceed exactly as the examples with two points but we can skip the first step of determining the slope as it is given. We do two examples, one with numbers and one with constants.

   (a) Find an equation for the line through the point $(2, 3)$ with slope -5.
   $-5 = \frac{\Delta y}{\Delta x} = \frac{y-3}{x-2} \Rightarrow -5(x-2) = y-3 \Rightarrow -5x+10 = y-3 \Rightarrow y = -5x+13$.

   (b) Find an equation for the line through the point (a, a^2) with slope $2a$.
   $2a = \frac{\Delta y}{\Delta x} = \frac{y-a^2}{x-a} \Rightarrow 2a(x-a) = y-a^2 \Rightarrow y = 2ax - 2a^2 + a^2 \Rightarrow y = 2ax - a^2$.

   (c) In these examples we should really say that we are using the slope formula but we will agree to shorthand that $\frac{\Delta y}{\Delta x}$ indicates we are using this. You should either use this form or say something like 'using slope = rise/run'.

5. Although it's something we do not need for developing calculus, finding the intersection of two lines does come up in economics examples making use of equations found using calculus. One example is finding Nash equilibria. (Nash is a mathematician who won the Nobel prize in economics and also had a movie made about his life dealing with mental illness.) It also good algebra practice working with constants.

   There are various ways to present the algebra. We will use one fairly compact version.

   In the second example do not get confused by the constants with subscripts, Q_B and Q_S. They are still just symbols representing some number. In this case quantity produced by two firms B and S.

   (a) Solve the linear system $2x+3y = 4$ and $5x+7y = 9$. That is, find the intersection of the corresponding lines.
   There are various ways to present the algebra. Here is one.

$$\begin{array}{rcrcr} 2x & + & 3y & = & 4 \\ 5x & + & 7y & = & 9 \end{array} \Rightarrow \begin{array}{rcrcr} 5(2x & + & 3y & = & 4) \\ -2(5x & + & 7y & = & 9) \end{array} \Rightarrow \begin{array}{rcrcr} 10x & + & 15y & = & 20 \\ -10x & + & -14y & = & -18 \\ \hline & & y & = & 2 \end{array}$$

   So $y = 2$ then $2x + 3 \cdot 2 = 4 \Rightarrow 2x = 4 - 6 = -2 \Rightarrow x = -1$.

   (b) Solve the linear system $Q_B = 5000 - \frac{Q_S}{2}$ and $Q_S = 5000 - \frac{Q_B}{2}$. This gives the Nash equilibrium for a particular problem.

$$\begin{array}{rcrcl} Q_B & + & \frac{Q_S}{2} & = & 5000 \\ \frac{Q_B}{2} & + & Q_S & = & 5000 \end{array} \Rightarrow \begin{array}{rcrcl} Q_B & + & \frac{Q_S}{2} & = & 5000 \\ -2(\frac{Q_B}{2} & + & Q_S & = & 5000) \end{array} \Rightarrow \begin{array}{rcrcr} Q_B & + & \frac{Q_S}{2} & = & 5000 \\ -Q_B & - & 2Q_S & = & -10000 \\ \hline & & \frac{-3}{2}Q_S & = & -5000 \end{array}$$

So $Q_S = \frac{-2}{3} \cdot (-5000) = \frac{10000}{3}$ then $Q_B + \frac{10000/3}{2} = 5000 \Rightarrow Q_B = 5000 - \frac{10000}{6} = \frac{10000}{3} \Rightarrow Q_B = \frac{10000}{3}$.

6. Quadratic polynomials can be written in several forms. Recall function transformations. The graph of $f(x) = (x+5)^2 - 3$ is the basic parabola $y = x^2$ shifted left 5 units and down 3 units.

   Expanding the squared term we have $f(x) = (x+5)^2 - 3 = (x^2 + 10x + 25) - 3 = x^2 + 10x + 22$. The original form is useful, for example to find the vertex of the parabola and its roots. So we will want to be able to reverse the process of expanding.

   First we will practice a bit with expanding quadratics. Some students will have learned ‘foil’ to help recall the process. The second example really just records this generic formula.

   Once we introduce constants we need to be careful with the term corresponding to $5^2 = 25$ in the example above. It helps to write a little more to be sure we don’t make mistakes by trying to do too much in our head.

   (a) $(2x+3)^2 = (2x)^2 + 2 \cdot 2x \cdot 3 + 3^2 = 4x^2 + 12x + 9$.
   (b) $(x+h)^2 = x^2 + 2xh + h^2$
   (c) $(3\alpha x - \beta y)^2 = (3\alpha x)^2 + 2 \cdot 3\alpha x \cdot (-\beta y) + (-\beta y)^2 = 9\alpha^2 x^2 - 6\alpha\beta xy + \beta^2 y^2$.
   (d) $(x + \frac{b}{2a})^2 = x^2 + 2 \cdot x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}$.

7. Rewrite $t^2 + 8t - 5$ as $(t-q)^2 + r$. Use this to sketch the graph $f(t) = t^2 + 8t - 5$ and to solve $t^2 + 8t - 5 = 0$. Do *not* use the quadratic formula (you can check your answer with it). Instead, complete the square. This aids both in the graphing and in understanding how the quadratic formula arises.

   Completing the square gives $t^2 + 8t - 5 = t^2 + 8t + (16 - 16) - 5 = (t^2 + 8t + 16) - 21 = (t+4)^2 - 21$.

   This is the parabola t^2 shifted left 4 units and down 21 units. The vertex of the parabola is $(-4, -21)$.

   For solving the equation to find the roots we get
   $t^2 + 8t - 5 = 0 \Rightarrow (t+4)^2 - 21 = 0 \Rightarrow (t+4)^2 = 21 \Rightarrow t + 4 = \pm\sqrt{21} \Rightarrow t = -4 \pm \sqrt{21}$.

8. Solve quadratics by completing the square. On the left we do it with numbers and the right with symbols deriving the quadratic formula.

$$\begin{array}{ccc} 3x^2+5x+7=0 & \qquad & ax^2+bx+c=0 \\ \Downarrow & & \Downarrow \\ x^2+\frac{5}{3}x=-\frac{7}{3} & & x^2+\frac{b}{a}x=-\frac{c}{a} \\ \Downarrow & & \Downarrow \\ x^2+\frac{5}{3}x+\frac{5^2}{4\cdot 3^2}=-\frac{7}{3}+\frac{5^2}{4\cdot 3^2} & & x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2} \\ \Downarrow & & \Downarrow \\ (x+\frac{5}{2\cdot 3})^2=\frac{5^2-4\cdot 3\cdot 7}{4\cdot 3^2} & & (x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2} \\ \Downarrow & & \Downarrow \\ x+\frac{5}{2\cdot 3}=\pm\sqrt{\frac{5^2-4\cdot 3\cdot 7}{4\cdot 3^2}} & & x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}} \\ \Downarrow & & \Downarrow \\ x=\frac{-5}{2\cdot 3}\pm\frac{\sqrt{5^2-4\cdot 3\cdot 7}}{2\cdot 3} & & x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} \end{array}$$

9. We practiced expanding quadratic polynomials above. Students may recall expanding higher powers using Pascal's triangle (incidently discovered in China, India and the middle east well before Pascal). The binomial theorem is a more formal statement of this expansion which is encountered in probability (any situation with two outcomes, say polling, flipping a coin etc).

   We will quickly introduce the binomial theorem to get practice with $\sum$ notation needed later in the course and as we will refer to it for understanding one of our basic derivative shortcuts. We will not do exercises with these higher powers.

   (a) Recall that $(x+h)^2 = x^2+2xh+h^2$. Then $(x+h)^3 = (x+h)(x^2+2xh+h^2) = x^3+2x^2h+xh^2+hx^2+2xh^2+h^3 = x^3+3x^2h+3xh^2+h^3$.

   (b) In general how do we expand $(x+h)^n$ for positive integers n?

   This will be a sum of all terms where the exponents on x and h add up to n. The coefficient of the term $x^k h^{n-k}$ is denoted $\binom{n}{k}$ and called the binomial coefficient.

   (c) These are the numbers in what is called in the west, Pascal's triangle (discovered in China, Persia, India much earlier). They count the number of size k subsets of an n element set. The exact numerical value is $\binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{k\cdot(k-1)\cdots 2}$.

   (d) This binomial theorem is also used in statistics when understanding probabilities in coin flips, polls etc.

(e) We will use shorthand notation $\sum_{k=0}^{n} f(k)$ to represent what we get by successively setting $k = 0, 1, 2, \ldots, n$ and adding all of these terms. For example we get

$$\begin{aligned}(x+h)^4 &= \tbinom{4}{4}x^4 + \tbinom{4}{3}x^3h + \tbinom{4}{2}x^2h^2 + \tbinom{4}{1}xh^4 + \tbinom{4}{0}h^4 \\ &= x^4 + 4x^3h + 6x^2h^2 + 4xh^4 + h^4 \\ &= \sum_{k=0}^{4} \tbinom{4}{k} x^{n-k}h^k\end{aligned}$$

and

$$\begin{aligned}(x+h)^7 &= \tbinom{7}{7}x^7 + \tbinom{7}{6}x^6h + \tbinom{7}{5}x^5h^2 + \tbinom{7}{4}x^4h^3 + \tbinom{7}{3}x^3h^4 + \tbinom{7}{2}x^2h^5 + \tbinom{7}{1}x^1h^6 + \tbinom{7}{0}h^7 \\ &= x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7 \\ &= \sum_{k=0}^{7} \tbinom{7}{k} x^{n-k}h^k\end{aligned}$$

(f) The binomial theorem in general gives for positive integers n that

$$(x+h)^n = \sum_{k=0}^{n} \tbinom{n}{k} x^{n-k}h^k.$$

(g) What matters to us is introducing the $\sum$ notation and the fact the the first two terms are $x^n + nx^{n-1}h$ with power of h at least 2 in all other terms. We will use these facts later.

Supplementary problems

By Math 81 conventions we assume constants are such that we do not divide by 0 in our algebra. For lines, unless otherwise stated, we use x and y for the variables.

Equations of lines

P3.1 (a) Find an equation for the line through $(11, -13)$ and parallel to the x-axis.
(b) Find an equation for the line through $(11, -13)$ and parallel to the y-axis.

P3.2 Find an equation for the line through the points $(-1, Q_0 - Q)$ and $(1, Q + Q_0)$.

P3.3 Find an equation for the line through the points (a, a^2) and $(a + h, (a + h)^2)$.

P3.4 Find an equation for the line through the points $(0, \alpha)$ and $(2, \beta)$

P3.5 Find an equation of the line through the points $(\alpha, 7)$ and $(4\alpha, 6)$

Solving linear systems

P3.6 Solve the linear system $2x + y = 3a$ and $x - 4y = a$ for x and y in terms of a.

P3.7 Solve the linear system $ax + y = 2b$ and $x + cy = 0$ for x and y in terms of a, b, c. Assume $ac \neq 1$.

Quadratics

P3.8 Solve $4x^2 + 6x + 3 = 0$ for x by completing the square. Do not use the quadratic formula.

P3.9 Solve $x^2 + 6x + K = 0$ for x by completing the square. Do not use the quadratic formula.

P3.10 Solve $x^2 + 2\delta x + 1 = 0$ by completing the square. Do not use the quadratic formula.

Solutions to supplementary problems

Equations of lines

S3.1 (a) $y = -13$.
(b) $x = 11$.

S3.2 Slope is $m = \frac{\Delta y}{\Delta x} = \frac{(Q+Q_0)-(Q_0-Q)}{1-(-1)} = \frac{2Q}{2} = Q$. Then
$Q = \frac{\Delta y}{\Delta x} = \frac{y-(Q+Q_0)}{x-1} \Rightarrow Q(x-1) = y-(Q+Q_0) \Rightarrow Qx - Q = y - Q - Q_0 \Rightarrow y = Qx + Q_0$.

S3.3 Slope is $\frac{\Delta y}{\Delta x} = \frac{(a+h)^2-a^2}{(a+h)-a} = \frac{(a^2+2ah+h^2)-a^2}{h} = \frac{2ah+h^2}{h} = 2a+h$.
Then $2a+h = \frac{\Delta y}{\Delta x} = \frac{y-a^2}{x-a} \Rightarrow (2a+h)(x-a) = y-a^2 \Rightarrow y = (2a+h)x-2a^2-ah+a^2 \Rightarrow$
$y = (2a+h)x - (a^2+ah)$.

S3.4 Slope is $\frac{\Delta y}{\Delta x} = \frac{\beta-\alpha}{2-0} = \frac{\beta-\alpha}{2}$. Then $\frac{\beta-\alpha}{2} = \frac{\Delta y}{\Delta x} = \frac{y-\alpha}{x-0} \Rightarrow x\,\frac{\beta-\alpha}{2} = y-\alpha \Rightarrow y = \frac{\beta-\alpha}{2}\,x+\alpha$.

S3.5 Slope is $\frac{\Delta y}{\Delta x} = \frac{6-7}{4\alpha-\alpha} = \frac{-1}{3\alpha}$.
Then $\frac{-1}{3\alpha} = \frac{\Delta y}{\Delta x} = \frac{y-7}{x-\alpha} \Rightarrow \frac{-1}{3\alpha}(x-\alpha) = y-7 \Rightarrow y = \frac{-1}{3\alpha}\,x + \frac{1}{3} + 7 \Rightarrow y = \frac{-1}{3\alpha}\,x + \frac{22}{3}$

Solving linear systems

S3.6
$$\begin{array}{rcrcr} 2x & + & y & = & 3a \\ x & - & 4y & = & a \end{array} \Rightarrow \begin{array}{rcrcr} 2x & + & y & = & 3a \\ -2(x & - & 4y & = & a) \end{array} \Rightarrow \begin{array}{rcrcr} 2x & + & y & = & 3a \\ -2x & + & 8y & = & -2a \\ \hline & & 9y & = & a \end{array}$$

So $y = \frac{a}{9}$ then $2x + \frac{a}{9} = 3a \Rightarrow 2x = 3a - \frac{a}{9} = \frac{26a}{9} \Rightarrow x = \frac{13a}{9}$.

S3.7
$$\begin{array}{rcrcr} ax & + & y & = & 2b \\ x & + & cy & = & 0 \end{array} \Rightarrow \begin{array}{rcrcr} ax & + & y & = & 2b \\ -a(x & + & cy & = & 0) \end{array} \Rightarrow \begin{array}{rcrcr} ax & + & y & = & 2b \\ -ax & + & -acy & = & 0 \\ \hline & & (1-ac)y & = & 2b \end{array}$$

So $y = \frac{2b}{1-ac}$ then $x + c \cdot \frac{2b}{1-ac} = 0 \Rightarrow x = \frac{-2bc}{1-ac}$.

Quadratics

S3.8 $4x^2+6x+1 = 0 \Rightarrow x^2 + \frac{3}{2}x + \frac{1}{4} = 0 \Rightarrow x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \left(\frac{3}{4}\right)^2 - \frac{1}{4}$
$\Rightarrow (x+\frac{3}{4})^2 = \frac{9}{16} - \frac{4}{16} = \frac{5}{16} \Rightarrow x + \frac{3}{4} = \pm\sqrt{\frac{5}{16}} \Rightarrow x = -\frac{3}{4} \pm \frac{\sqrt{5}}{4} = \frac{-3\pm\sqrt{5}}{4}$

S3.9 $x^2+6x+K = 0 \Rightarrow x^2+6x+9 = 9-K \Rightarrow (x+3)^2 = 9-K \Rightarrow x+3 = \pm\sqrt{9-K} \Rightarrow$
$x = -3 \pm \sqrt{9-K}$

S3.10 $x^2+2\delta x+1 = 0 \Rightarrow x^2+2\delta x+\delta^2 = \delta^2-1 \Rightarrow (x+\delta)^2 = \delta^2-1 \Rightarrow x+\delta = \pm\sqrt{\delta^2-1} \Rightarrow$
$x = -\delta \pm \sqrt{\delta^2-1}$