The *arithmetic mean* of a set $y_1, y_2, \ldots, y_n$ of numbers is $\frac{y_1+y_2+\cdots+y_n}{n}$. This is the usual notion of 'average'. Another notion of 'average' is the geometric mean (which works for nonnegative numbers), $(y_1 y_2 \cdots y_n)^{1/n}$. That is, we take the n^{th} root of the product of the n numbers. One typical application of the geometric mean is in averaging inflation rates or finding the 'average' rate of return in an investment.

For example, say that you invest \$100. During the first year your return is 10%, ending the year with \$110. During the second year you lose 7% and during the third year you lose 3%. What is your average rate of return. Using the arithmetic mean the average of of 10 %, -7% and -3% is 0. You should break even at the end of three years. However, doing the calculations after one year you have \$100(1.1) = \$110 after the second year you have \$110(.93) = \$102.30 and after the third year you have \$102.30(.97) = \$99.231 which is not breaking even. In this situation you multiply your principle by (1.1)(.93)(.97). Note that the arithmetic mean of 1.1 and .03 and .97 is 1 which does not give the correct average rate of return. However, the geometric mean of these three numbers is $\sqrt[3]{(1.1)(.93)(.97)} = \sqrt[3]{.99231} \approx .99743 = 1 - .00257$. The situation above would be the same as losing .257% each year. The geometric mean gives the correct interpretation for 'average' in situation where you get a total by multiplying.

In the example we just gave the arithmetic mean was greater than the geometric mean. It overapproximated the rate of return. In fact this will always be the case. For nonnegative numbers the arithmetic mean is always at least as large as the geometric mean (and equal only when the numbers are all the same). We give a proof of this arithmetic-geometric mean inequality by induction. That is we want to prove

$$\frac{y_1 + y_2 + \cdots + y_n}{n} \geq (y_1 y_2 \cdots y_n)^{1/n} \text{ for all nonnegative numbers } y_1, y_2, \ldots, y_m$$

We first transform the problem by letting $x_i = \frac{y_i}{(y_1 \cdots y_n)^{1/n}}$ for $i = 1, 2, \ldots, n$. Note that $x_1 x_2 \cdots x_n = 1$ and proving $x_1 + x_2 + \cdot + x_n \geq n$ will give the arithmetic-geometric mean inequality.

Normalized arithmetic-geometric mean inequality: If $x_1, x_2, \ldots, x_n$ are nonnegative numbers with $x_1 x_2 \cdots x_n = 1$ then $x_1 + x_2 + \cdots + x_n \geq n$. Furthermore, equality holds only if the x_i are all equal.

Proof: The proof is by induction. The inequality is trivial, if $x_1 = 1$ then $x_1 \geq 1$ when $n = 1$. Relabel the numbers so that $x_1 \geq x_2 \geq \cdots \geq x_n$. Note that since $x_1 x_2 \cdots x_n = 1$ we have $x_1 \geq 1 \geq x_n$. Consider the $(n-1)$ numbers $x_2, x_3, \ldots, x_{n-1}, x_1 x_n$. Note that $x_2 + \cdots + x_{n-1} + x_1 x_n$ satisfy $x_2 x_3 \cdots x_{n-1}(x_1 x_n) = x_1 x_2 \cdots x_n = 1$. So by induction these numbers satisfy the inequality and sum to at most $(n-1)$. Then

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= (x_2 + \cdots + x_{n-1} + x_1 x_n) + [x_1 + x_n - x_1 x_n] \\ &\geq (n-1) + [1 + (x_1 - 1) - x_n(x_1 - 1)] \\ &= n + (x_1 - 1)(1 - x_n) \\ &\geq n \end{aligned}$$

Here the last inequality follows from $(x_1 - 1)(1 - x_n) \geq 1$ since $x_1 \geq 1 \geq x_n$ and this inequality is strict unless the x_i are all equal. $\square$