Linear Programming

Linear programming refers to problems stated as maximization or minimization of a linear function subject to constraints that are linear equalities and inequalities. Although the study of algorithms for these problems is very important, the term programming in linear programming arises from the context of a particular resource allocation ‘program’ for the United States Air Force for which George Dantzig developed a linear model for and described a method, called the Simplex method, to solve. This was in the late 1940’s before the term ‘computer programming’ came into general use.

Consider the following example of a linear programming problem. In general, a linear programming problem is a maximization or minimization of a linear function subject to constraints that are linear equations or linear inequalities. Some of these may be bounds on variables. For example it is not unusual to require that variables be non-negative in applications. If there is no such constraint a variable will be called *free*. A linear programming problem is called *feasible* if there is some solution satisfying the constraints and *infeasible* otherwise. If the maximum can be made arbitrarily large the the problem is *unbounded*.

$$\begin{array}{rrcrcrcr} \max & x_1 & + & 5x_2 & + & 3x_3 & & \\ \text{s.t.} & 2x_1 & + & 3x_2 & - & 4x_3 & \leq & 5 \\ & -x_1 & - & 2x_2 & + & x_3 & \leq & -3 \\ & x_1 & + & 4x_2 & + & 2x_3 & \leq & 6 \\ & & & 3x_2 & + & 5x_3 & \leq & 2 \end{array} \tag{1}$$

We will often use matrix notation. This instance becomes

$$\max \boldsymbol{cx} \text{ s.t. } A\boldsymbol{x} \leq \boldsymbol{c}$$

where the matrix A, cost vector $\boldsymbol{c}$ and right hand side $\boldsymbol{b}$ are given and $\boldsymbol{x}$ is a vector of variables. For this example we have

$$A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & -2 & 1 \\ 1 & 4 & 2 \\ 0 & 3 & 5 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 1 & 5 & 3 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ -3 \\ 6 \\ 2 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

We will also write generic instances using sigma notation as follows:

$$\max \sum_{j=1}^{n} c_j x_j \text{ s.t. } \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, \ldots, m.$$

If we want to minimize $\boldsymbol{cx}$ we can instead maximize $-\boldsymbol{cx}$. We can replace equations $A\boldsymbol{x} = \boldsymbol{b}$ with inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$ and $-A\boldsymbol{x} \leq -\boldsymbol{b}$. To reverse the process, we can replace $A\boldsymbol{x} \leq \boldsymbol{b}$ with $A\boldsymbol{x} + I\boldsymbol{s} = \boldsymbol{b}$ and $\boldsymbol{s} \geq \boldsymbol{0}$ where I is an appropriate size identity matrix and $\boldsymbol{s}$ is a vector of *slack* *variables* which are non-negative. To replace free variables with non-negative variables we use $\boldsymbol{x} = \boldsymbol{u} - \boldsymbol{v}$ where $\boldsymbol{u0}$ and $\boldsymbol{v} \geq \boldsymbol{0}$. Alternatively we can write non-negativity constraints as simply another inequality.

Using the transformations described above we can convert any linear programming instance to one of three standard forms. It will be convenient to be able to refer to each form with the understanding that any instance can be converted to that form. The forms are

- $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}\}$
- $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- $\max\{\boldsymbol{cx} | A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$

We will later see examples of converting between the forms in the context of Farkas Lemma and duality.

Variations on Linear Programming

We have seen that we might consider various variants of linear programming problems and indicated how they are equivalent. However, the case where there are only linear equalities and all variables are free is different. We will call this a system of equations. Linear programming will refer to any problem where there is either at least one non-negative variable or at least one linear inequality. For both systems of equations and linear programming having an additional constraint that at least one variable must be integral again changes things. Thus we can distinguish four variants that are qualitatively different: Systems of Equations, Integral Systems of Equations, Linear Programming, Integer Linear Programming.

We can also distinguish between feasibility problems (find a solution to some linear system) and optimization problems (maximize or minimize a linear function over the set of solutions to some linear system — the feasible set). If we have a method for solving the optimization problem then we can easily solve feasibility using the same method. Simply maximize the zero function. Interestingly, feasibility and optimization are 'equivalent' in each of the four cases in the sense that a method for to solve the feasibility problem can be used to solve the optimization problem as will discuss below.

We consider each of the four types in turn. In each case, if there is no feasible solution then the optimization problem is infeasible. So, here we will assume feasibility when discussing optimization.

Systems of Equations. Determining $\{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b}\}$ (feasibility) and $\max\{\mathbf{c}^\mathsf{T}\mathbf{x} \mid \mathbf{Ax} = \mathbf{b}\}$ (optimization) is the material of basic linear algebra and the problems can be solved efficiently using, for example Gaussian elimination. The optimization problem is not usually discussed in linear algebra but follows easily from feasibility. The set of solutions to $\{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b}\}$ can be written as $\mathbf{x}^0 + \sum \lambda_i \mathbf{v}^i$ where the sum is over a set $\{\mathbf{v}^i\}$ of basis vectors for the nullspace of $\mathbf{A}$. If $\mathbf{c}^\mathsf{T}\mathbf{v}^i = 0$ for all $\mathbf{v}^i$, including the case that the nullspace is trivial, then the entire feasible set attains the maximum $\mathbf{c}^\mathsf{T}\mathbf{x}^0$. This includes the case that there is a unique solution and the unique solution of course solves the optimization problem. On the other hand, if $\mathbf{c}^\mathsf{T}\mathbf{v}^i \neq 0$ for some $\mathbf{v}^i$, then we can assume that we have $\mathbf{c}\mathbf{v}^i = r > 0$, as otherwise we could replace $\mathbf{v}^i$ with $-\mathbf{v}^i$ in the basis. Now we have that $\mathbf{x}^0 + t\mathbf{v}^i$ is feasible for arbitrarily large t, and hence the optimization problem is unbounded since $\mathbf{c}^\mathsf{T}(\mathbf{x}^0 + t\mathbf{v}^i) = \mathbf{c}^\mathsf{T}\mathbf{x}^0 + tr \to \infty$ as $t \to \infty$. Thus for systems of equations, the optimization problem is either infeasible, unbounded or has the entire feasible set attaining the optimal value. This material is covered in linear algebra courses and we will not discuss it here.

Integral Systems of Equations. Determining $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \in \mathbb{Z}\}$ (feasibility) and $\max\{\mathbf{c}\mathsf{T}\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \in \mathbb{Z}\}$ (optimization). The feasibility problem can be solved by a method somewhat similar to Gaussian elimination. Using what are called elementary unimodular column operations (multiply a column by -1, switch two columns and add an integral multiple of one column to another) one can reduce a given constraint matrix to what is called hermite normal form. The reduction process can be viewed as an extension of the Euclidean algorithm for greatest common divisors. This process maintains the integrality. From this all feasible solutions can be described just as for systems of equations above (except that the $\mathbf{x}^p, \mathbf{v}^i$ and λ_i are all integral) and we get a similar conclusion about the optimization problem.

Linear Programming. The various versions $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ or general linear systems with combinations of inequalities, equalities, non-negative and unrestricted variables all can easily be converted into the other forms. We will see from the duality theorem that we can solve the optimization problem by solving a related feasibility problem. Observe that these problems are 'harder' then systems of linear equations in that methods like Gaussian elimination for equations can not be directly used to solve linear optimization problems. However, a system consisting only of equations can be viewed as a special case of general linear optimization problems.

Integer Linear Optimization. If we take a linear optimization problem and in addition require that the variables take on integral values only we get integer linear optimization problems. A mixed integer programming problem requires that some, but not all of the variables be integral. Except in the case of integral systems of equations discussed above these problems are NP-complete. Fortunately, many graph and network problems formulated as integer linear programming problems are special cases where we can find nice theorems and efficient solutions.

Note also one other interesting point. If we look for integral solutions to systems of equations restricting the variables to be only 0 or 1 we have a special case of Integer Optimization which is still 'hard'. Here we have a case where there are a finite number of possible feasible solutions, yet this class of problems is much more difficult to solve than regular linear optimization problems even though in that case there are typically an infinite number of feasible solutions. In part, the difficulty lies in how large the number of finite solutions is. If there are n variables then there are 2^n potential solutions to check. If one naively attempts to check all of these problems are encountered fairly quickly. For a somewhat modest size problem with, say 150 variables, the number of potential solutions is larger than the number of atoms in the known universe. If the universe is a computer, with each atom checking billions of potential cases each second, running since the beginning of time all cases would still not have been checked. So a naive approach to solving such finite problems in general is bound to fail.

The Dual

For a given linear programming problem we will now construct a second linear programming problem whose solution bounds the original problem. Surprisingly, it will turn out that the bound is tight. That is, the optimal value to the second problem, called the dual will be equal to the optimal value for the original problem, called the primal.

Consider again the linear programming problem (1).

$$\begin{array}{lrcrcrcr}
\max & x_1 & + & 5x_2 & + & 3x_3 & & \\
\text{s.t.} & 2x_1 & + & 3x_2 & - & 4x_3 & \leq & 5 \\
 & -x_1 & - & 2x_2 & + & x_3 & \leq & -3 \\
 & x_1 & + & 4x_2 & + & 2x_3 & \leq & 6 \\
 & & & 3x_2 & + & 5x_3 & \leq & 2
\end{array}$$

If we multiply the inequalities by appropriate values and 'add' we can bound the maximum value. Note that for inequalities we must use non-negative multipliers so as not to change the direction of the inequalities. For example multiplying the first inequality by 1, the second by 3, the third by 2 and the fourth by 0 we get

$$\begin{array}{rcrcrcr}
2x_1 & + & 3x_2 & - & 4x_3 & \leq & 5 \\
-3x_1 & - & 6x_2 & + & 3x_3 & \leq & -9 \\
2x_1 & + & 8x_2 & + & 4x_3 & \leq & 12 \\
0x_1 & + & 0x_2 & + & 0x_3 & \leq & 0
\end{array}$$

Combining these results in $x_1 + 5x_2 + 3x_3 \leq 8$. Observe that we have picked the multipliers $[\ 1 \quad 3 \quad 2 \quad 0\ ]$ carefully so that we get the cost vector $\boldsymbol{c}$. Another choice would be to use the multipliers $[\ 2 \quad 4 \quad 1 \quad 1\ ]$ which yield a better bound $x_1 + 5x_2 + 3x_3 \leq 6$.

This suggests a new linear programming problem to pick multipliers so as to minimize the bound subject to picking them appropriately. In this case we must get $\boldsymbol{c}$ and use non-negative multipliers.

$$\begin{array}{lrcrcrcrcr}
\min & 5y_1 & - & 3y_2 & + & 6y_3 & + & 2y_4 & & \\
\text{s.t.} & 2y_1 & - & y_2 & + & y_3 & & & = & 1 \\
 & 3y_1 & - & 2y_2 & + & 4y_3 & + & 3y_4 & = & 5 \\
 & -4y_1 & + & y_2 & + & 2y_3 & + & 5y_4 & = & 3 \\
 & & & & & & & & & \\
 & y_1 & , & y_2 & , & y_3 & , & y_4 & \geq & 0
\end{array}$$

The constraint matrix is the transpose of the original and the roles of $\boldsymbol{c}$ and $\boldsymbol{b}$ have switched.

The new problem is called the *dual* and the original problem is called the *primal*.

The multiplier for an equation need not be constrained to be non-negative. For variables that are non-negative we can use $\geq$ in the new problem rather than $=$. For example, if the above problem had non-negative variables then if the combination resulted in $2x_1 + 5x_2 + 7x_3 \leq 10$ we would also have $x_1 + 5x_2 + 4x_4 \leq 10$ by adding the inequalities $-x_1 \leq 0$ and $-3x_3 \leq 0$ that result from non-negativity.

To summarize we have the following versions of dual problems written in matrix form:

Primal	Dual
$\max\{\boldsymbol{cx} \mid A\boldsymbol{x} \leq \boldsymbol{b}\}$	$\min\{\boldsymbol{yb} \mid \boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$
$\max\{\boldsymbol{cx} \mid A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$	$\min\{\boldsymbol{yb} \mid \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$
$\max\{\boldsymbol{cx} \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$	$\min\{\boldsymbol{yb} \mid \boldsymbol{y}A \geq \boldsymbol{c}\}$

We motivated the formulation of the dual as a bound on the primal. We state this now as the following:

Weak Duality Theorem of Linear Programming: If both the primal and dual are feasible then $\max\{\boldsymbol{cx} \mid A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} \leq \min\{\boldsymbol{yb} \mid \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$.

Proof: For any feasible $\boldsymbol{x}^*$ and $\boldsymbol{y}*$ we have

$$\boldsymbol{cx}^* \leq (\boldsymbol{y}^*A)\boldsymbol{x}^* = \boldsymbol{y}^*(A\boldsymbol{x}^*) \leq \boldsymbol{y}^*\boldsymbol{b}$$

where the first inequality follows since $\boldsymbol{x}^* \geq \boldsymbol{0}$ and $\boldsymbol{y}^*A \geq \boldsymbol{c}$ and the second inequality follows since $\boldsymbol{y}^* \geq \boldsymbol{0}$ and $A\boldsymbol{x}^* \leq \boldsymbol{b}$. $\square$

Similar versions hold for each of the primal-dual pairs. We give an alternate proof using summation notation:

$$\sum_{j=1}^{n} c_j x_j^* \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{i,j} y_i^*\right) x_j^* = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i,j} x_j^*\right) y_i^* \leq \sum_{i=1}^{m} b_i y_i^*.$$

We will next work toward showing strong duality: we in fact get equality in the weak duality theorem.

Linear systems

We will first examine linear systems and will use results about feasibility to show strong duality.

Consider first the following systems of linear equations.

$$\begin{array}{rcrcrcr} x & + & 4y & - & z & = & 2 \\ -2x & - & 3y & + & z & = & -1 \\ -3x & - & 2y & + & z & = & 0 \\ 4x & + & y & - & z & = & -1 \end{array} \qquad \begin{array}{rcrcrcr} x & + & 4y & - & z & = & 2 \\ -2x & - & 3y & + & z & = & 1 \\ -3x & - & 2y & + & z & = & 0 \\ 4x & + & y & - & z & = & -1 \end{array}$$

A solution to the system on the left is $x = 0, y = 1, z = 2$. The system on the right has no solution. A typical approach to verify that the system on the right has no solution by noting something about the row echelon form of the reduced augmented matrix having a row of zeros with a non-zero right side or some other variant on this. This can be directly stated by producing a certificate of inconsistency such as $(-1, -3, 3, 1)$. Multiplying the first row by -1, the second by -3, the third by 3 and the fourth by 1 and adding we get the inconsistency $0 = -6$. So the system must not have a solution. The fact that a system of linear equations either has a solution or a certificate of inconsistency is presented in in various guises in typical undergraduate linear algebra texts and often proved as a result of the correctness of Gaussian elimination.

Consider the following systems of linear inequalities

$$\begin{array}{rcrcrcr} x & + & 4y & - & z & \leq & 2 \\ -2x & - & 3y & + & z & \leq & 1 \\ -3x & - & 2y & + & z & \leq & 0 \\ 4x & + & y & - & z & \leq & -1 \end{array} \tag{2}$$

$$\begin{array}{rcrcrcr} x & + & 4y & - & z & \leq & 1 \\ -2x & - & 3y & + & z & \leq & -2 \\ -3x & - & 2y & + & z & \leq & 1 \\ 4x & + & y & - & z & \leq & 1 \end{array} \tag{3}$$

A solution to (2) is $x = 0, y = 1, z = 2$. The system (3) has no solution but how do we show this?

In order to get a certificate of inconsistency consisting of multipliers for the rows as we did for systems of equations we need to be a bit more careful with the multipliers. Try using the same multipliers $(-1, -3, 3, 1)$ from the equations for the inequalities. Multiplying the first row by -1, the second by -3, the third by 3 and the fourth by 1 and combining we get $0 \leq 11$. This is not an inconsistency. As before we need a left side of 0 but because of the $\leq$ we need the right side to be negative in order to get an inconsistency. So we try the multipliers $(1, 3, -3, -1)$ and would seem to get the inconsistency $0 \leq -11$. However, this is not a certificate of inconsistency. Recall that multiplying an inequality by a negative number also changes the direction of the inequality. In order for our computations to be valid for a system of inequalities the multipliers must be non-negative.

It is not difficult to check that $(3, 4, 1, 2)$ is a certificate of inconsistency for the system on the right above. Multiplying the first row by 3, the second by 4, the third by 1 and the fourth by 2 and combining we get the inconsistency $0 \leq -2$. So the system of inequalities has no solution. In general, for a system of inequalities, a certificate of inconsistency consists of non-negative multipliers and results in $0 \leq b$ with b negative. For a mixed system with equalities and inequalities we can drop the non-negativity constraint on multipliers for the equations.

The fact that a system of linear inequalities either has a solution or a certificate of inconsistency is often called Farkas's lemma. It can be proved by an easy induction using Fourier-Motzkin elimination. Fourier-Motzkin elimination in some respects parallels Gaussian elimination, using (non-negative) linear combinations of inequalities to create a new system in which a variable is eliminated. From a solution to the new system a solution to the original can be determined and a certificate of inconsistency to the new system can be used to determine a certificate of inconsistency to the original.

Fourier-Motzkin Elimination

We will start with a small part of an example of Fourier-Motzkin elimination for illustration.

Consider the system of inequalities (3). Rewrite each inequality so that it is of the form $x \geq$ or $x \leq$ depending of the sign of the coefficient of x.

$$\begin{array}{rcrcrcrcrcrcr}
 & & & & & & x & \leq & 1 & - & 4y & + & z \\
1 & - & 3y/2 & + & z/2 & \leq & x & & & & & & \\
-1/3 & - & 2y/3 & + & z/3 & \leq & x & & & & & & \\
 & & & & & & x & \leq & 1/4 & - & y/4 & + & z/4
\end{array}$$

Then pair each upper bound on x with each lower bound on x.

$$\begin{array}{rcrcrcrcrcr}
1 & - & 3y/2 & + & z/2 & \leq & 1 & - & 4y & + & z \\
1 & - & 3y/2 & + & z/2 & \leq & 1/4 & - & y/4 & + & z/4 \\
-1/3 & - & 2y/3 & + & z/3 & \leq & 1 & - & 4y & + & z \\
-1/3 & - & 2y/3 & + & z/3 & \leq & 1/4 & - & y/4 & + & z/4
\end{array} \tag{4}$$

Simplify to obtain a new system in which x is eliminated.

$$\begin{array}{rcrcr}
5y/2 & - & z/2 & \leq & 0 \\
-5y/4 & + & z/4 & \leq & -3/4 \\
10y/3 & - & 2z/3 & \leq & 4/3 \\
-5y/12 & + & z/12 & \leq & 7/12
\end{array} \tag{5}$$

The new system (5) has a solution if and only if the original (3) does. The new system is inconsistent. A certificate of inconsistency is $(2, 6, 1, 2)$. Observe that the first row of (5) is obtained from 1/2 the second row and the first row of (3). Similarly, the second row of (5) comes from 1/2 the second row and 1/4 the fourth row of (3). The other inequalities in (5) do not involve the second row of (3). Using the multipliers 2,6 for the first two rows of (5) we translate to a multiplier of $2 \cdot 1/2 + 6 \cdot 1/2 = 3$ for the second row of (3). Looking at the other rows in a similar manner we translate the certificate of inconsistency $(2, 6, 1, 2)$ for (5) to the certificate of inconsistency $(3, 4, 1, 2)$ for (3).

In a similar manner any certificate of inconsistency for the new system determines a certificate of inconsistency for the original.

Now, consider the system of inequalities (2). Eliminating x as in the previous example we get the system

$$\begin{array}{rcrcr} 5y/2 & - & z/2 & \leq & 5/2 \\ -5y/4 & + & z/4 & \leq & 1/4 \\ 10y/3 & - & 2z/3 & \leq & 2 \\ -5y/12 & + & z/12 & \leq & -1/4 \end{array} \tag{6}$$

A solution to (6) is $y = 1$, $z = 2$. Substituting these values into (2) we get

$$\begin{array}{rcl} x & \leq & 0 \\ -2x & \leq & 2 \\ -3x & \leq & 0 \\ 4x & \leq & 0 \end{array}$$

So $y = 1$, $z = 2$ along with $x = 0$ gives a solution to (2). In general, each solution to the new system substituted into the original yields an interval of possible values for x. Since we paired upper and lower bounds to the get the new system, this interval will be well defined for each solution to the new system.

Below we will give an inductive proof of Farkas' lemma following the patterns of the examples above. The idea is to eliminate one variable to obtain a new system. A solution to the new system can be used to determine a solution to the original and a certificate of inconsistency to the new system can be used to determine a certificate of inconsistency to the original.

Note that in these example we get the same number of new inequalities. In general, with n inequalities we might get as many as $n^2/4$ new inequalities if upper and lower bounds are evenly split. Iterating to eliminate all variables might then yield an exponential number of inequalities in the end. This is not a practical method for solving systems of inequalities, either by hand or with a computer. It is interesting as it does yield a simple inductive proof of Farkas' Lemma.

Farkas Lemma

Consider the system $A\boldsymbol{x} \leq \boldsymbol{b}$ which can also be written as $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ for $i = 1, 2 \ldots, m$. Let $U = \{i | a_{in} > 0\}$, $L = \{i | a_{in} < 0\}$ and $N = \{i | a_{in} = 0\}$. We prove Farkas' Lemma with the following steps:

(i) We give a system with variables $x_1, x_2, \ldots, x_{n-1}$ that has a solution if and only if the original system does. We will then use induction since there is one less varaible.

(ii) We show that Farkas' lemma holds for systems with 1 variable. This is the basis of the induction. (This is 'obvious', however there is some work to come up with appropriate notation to make it a real proof. One could also use the no variable case as a basis for induction but the notation is more complicated for this.)

(iii) We show that if the system in (i) is empty show that the original system has a solution. (It is empty if $L \cup N$ or $U \cup N$ is empty.)

(iv) If the system in (i) is inconsistent and multipliers u_{rs} for $r \in L, s \in U$, v_t for $t \in N$ provide a

certificate of inconsistency, we describe in terms of these multipliers a certificate of inconsistency for the original system.
(v) If the system in (i) has a solution $x_1^*, x_2^*, \ldots, x_{n-1}^*$ we describe a non-empty set of solutions to the original problem that agrees with the x_j^* for $j = 1, 2, \ldots n-1$.

(i) We start with

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \text{ for } i = 1, 2 \ldots, m. \tag{7}$$

Let $U = \{i | a_{in} > 0\}$, $L = \{i | a_{in} < 0\}$ and $N = \{i | a_{in} = 0\}$. Then

$$\begin{aligned} &\frac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right) \leq x_n \text{ for } r \in L \\ &x_n \leq \frac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right) \text{ for } s \in U \\ &\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t \text{ for } t \in N \end{aligned} \tag{8}$$

is just a rearrangement of (7). Note that the direction of the inequality changes when $r \in L$ since $a_{rn} < 0$ and we multiply by this. We pair each upper bound with each lower bound and carry along the inequalities not involving x_n to get

$$\begin{aligned} &\frac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right) \leq \frac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right) \text{ for } r \in L, s \in U \\ &\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t \text{ for } t \in N \end{aligned} \tag{9}$$

Due to the construction (7) has a solution if and only if (9) does. (This will also follow from parts (iv) and (v) below.)

(ii) With one variable we have three types of inequalities as above using $n = 1$ for L, U, N: $a_{s1}x_1 \leq b_s$ for $s \in U$; $a_{r1}x_1 \leq b_r$ for $r \in L$ and inequalities with no variable ($t \in N$) of the form $0 \leq b_t$. If we have $0 \leq b_t$ for some $b_t < 0$ then the system is inconsistent, there is clearly no solution to the system. Set all multiplies $u_{i1} = 0$ except $u_{t1} = 1$ to get a certificate of inconsistency. We can drop inequalities $0 \leq b_t$ for $b_t \geq 0$ so we can assume now that all a_{i1} are nonzero. Rewriting the system we have $x_1 \leq b_s/a_{s1}$ for $s \in U$ and $b_r/a_{r1} \leq x_1$ for $r \in L$. There is a solution if and only if $\max_{r \in L} b_r/a_{r1} \leq \min_{s \in U} b_s/a_{s1}$. If this holds then any $x_1 \in [\max_{r \in L} b_r/a_{r1}, \min_{s \in U} b_s/a_{s1}]$ satisfies all inequalities. If not then for some r^*, s^* we have $b_{s^*}/a_{s^*1} < b_{r^*}/a_{r^*1}$. For a certificate of inconsistency take all $u_{i1} = 0$ except $u_{r^*1} = a_{s^*1} > 0$ and $u_{s^*1} = -a_{r^*1} > 0$. Multiplying we have $a_{s^*1}(a_{r^*1}x_1 \leq b_{r^*})$ and $-a_{r^*1}(a_{s^*1}x_1 \leq b_{s^*})$. Combining these we get $0 \leq (b_{r^*}a_{s^*1} - b_{s^*}a_{r^*1}) < 0$. The last < 0 follows from $b_{s^*}/a_{s^*1} < b_{r^*}/a_{r^*1}$ (again noting that the direction of the inequality changes as $s_{s^*1} < 0$).

(iii) If $L \cup N$ is empty the original system is $x_n \leq \frac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right)$ for $s \in U$. For a solution take $x_1 = x_2 = \cdots x_{n-1} = 0$ and $x_n = \min_{s\in U}(b_s/a_{sn})$. It is straightforward to check that this is a solution.

If $U \cup N$ is empty the original system is $\frac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right) \leq x_n$ for $r \in L$. For a solution take $x_1 = x_2 = \cdots x_{n-1} = 0$ and $x_n = \max_{r\in L}(b_r/a_{rn})$. It is straightforward to check that this is a solution.

(iv) If (9) is inconsistent with multipliers u_{rs} for $r \in L, s \in U$ and u_t for $t \in N$ construct a certificate $\boldsymbol{y}$ of inconsistency for (7): For $t \in N$ let $y_t = u_t$. For $r \in L$ let $y_r = -\frac{1}{a_{rn}}\sum_{s\in U} u_{rs}$ and for $s \in U$ let $y_s = \frac{1}{a_{sn}}\sum_{r\in L} u_{rs}$. Since we started with a certificate for (9) we have that combining the inequalities

$$\begin{aligned}
&u_{rs}\left(\frac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right) \leq \frac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right)\right) \quad \text{for } r \in L, s \in U\\
&u_t\left(\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t\right) \quad \text{for } t \in N
\end{aligned}$$

yields $0 < b$ for some $b < 0$. For (8) which is a rearrangement of (7), combining

$$\begin{aligned}
&\left(-\frac{1}{a_{rn}}\sum_{s\in U} u_{rs}\right)\left(\frac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right) \leq x_n\right) \quad \text{for } r \in L\\
&\left(\frac{1}{a_{sn}}\sum_{r\in L} u_{rs}\right)\left(x_n \leq \frac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right)\right) \quad \text{for } s \in U\\
&(u_t)\sum_{j=1}^{n-1} a_{tj}x_j \leq b_t \text{ for } t \in N
\end{aligned}$$

(v) Given a solution $(x_1^*, x_2^*, \ldots, x_{n-1}^*)$ to (9) take x_n^* to be any value in the interval

$$\left[\max r \in L\frac{1}{a_{rn}}\left(b_r - \sum_{j=1}^{n-1} a_{rj}x_j\right), \min s \in U\frac{1}{a_{sn}}\left(b_s - \sum_{j=1}^{n-1} a_{sj}x_j\right)\right]$$

This complete the proof of Farkas' lemma.

Versions of Farkas' Lemma

We will state three versions of Farkas' Lemma. This is also called the Theorem of the alternative for linear inequalities.

A: Exactly one of the following holds:

$$\begin{array}{l} \text{(I) } A\boldsymbol{x} \le \boldsymbol{b}, \text{ has a solution } \boldsymbol{x} \\ \text{(II) } \boldsymbol{y}A = \boldsymbol{0}, \boldsymbol{y} \ge \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0 \text{ has a solution } \boldsymbol{y} \end{array}$$

B: Exactly one of the following holds:

$$\begin{array}{l} \text{(I) } A\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0} \text{ has a solution } \boldsymbol{x} \\ \text{(II) } \boldsymbol{y}A \ge \boldsymbol{0}, \boldsymbol{y} \ge \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0 \text{ has a solution } \boldsymbol{y} \end{array}$$

C: Exactly one of the following holds:

$$\begin{array}{l} \text{(I) } A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0} \text{ has a solution } \boldsymbol{x} \\ \text{(II) } \boldsymbol{y}A \ge \boldsymbol{0}, \boldsymbol{y}\boldsymbol{b} < 0 \text{ has a solution } \boldsymbol{y} \end{array}$$

Of course the most general version would allow mixing of these three, with equalities and inequalities and free and non-negative variables. However, it is easier to consider these. In the general case, the variables in II corresponding to inequalities in I are constrained to be non-negative and variables corresponding to equalities in I are free. In II there are inequalities corresponding to non-negative variables in I and equalities corresponding to free variables in I.

We will show the equivalence of the versions A and B. Showing other equivalences is similar. The ideas here are the same as those discussed in the conversions between various forms of linear programming problems.

Note - there are at least two ways to take care of the 'at most one of the systems has a solution' part of the statements. While it is a bit redundant we will show both ways below. First we show it directly and we also show it by the equivalent systems. If the 'at most one system holds' is shown first then only the $\Leftarrow$'s are needed for the equivalent systems.

First we note that for each it is easy to show that at most one of the systems holds for A and B.

If both IA and IIA hold then

$$0 = \boldsymbol{0}\boldsymbol{0} = (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) \le \boldsymbol{y}\boldsymbol{b} < 0$$

a contradiction. We have used $\boldsymbol{y} \ge \boldsymbol{0}$ in the $\le$.

If both IB and IIB hold then

$$0 = \boldsymbol{0}\boldsymbol{0} \le (\boldsymbol{y}A)\boldsymbol{x} = \boldsymbol{y}(A\boldsymbol{x}) \le \boldsymbol{y}\boldsymbol{b} < 0$$

a contradiction. We have used $\boldsymbol{y} \geq \mathbf{0}$ in the second $\leq$.

So for the remainder we will seek to show *at least one of the following holds.*

$(A \Rightarrow B)$: Note the following equivalences.

$$\text{(IB)} \quad \begin{array}{c} A\boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{x} \geq \mathbf{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} A\boldsymbol{x} \leq \boldsymbol{b} \\ -I\boldsymbol{x} \leq \mathbf{0} \end{array} \quad \Leftrightarrow \quad \left[\begin{array}{c} A \\ -I \end{array}\right] \boldsymbol{x} \leq \left[\begin{array}{c} \boldsymbol{b} \\ \mathbf{0} \end{array}\right] \quad \text{(IB')}$$

and

$$\text{(IIB)} \quad \begin{array}{c} \boldsymbol{y}A \geq \mathbf{0} \\ \boldsymbol{y} \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y}A - \boldsymbol{s}I = \mathbf{0} \\ \boldsymbol{y} \geq \mathbf{0}, \boldsymbol{s} \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} - \boldsymbol{s}\mathbf{0} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \left[\begin{array}{cc} \boldsymbol{y} & \boldsymbol{s} \end{array}\right] \left[\begin{array}{c} A \\ -I \end{array}\right] = \mathbf{0} \\ \left[\begin{array}{cc} \boldsymbol{y} & \boldsymbol{s} \end{array}\right] \geq \mathbf{0} \\ \left[\begin{array}{cc} \boldsymbol{y} & \boldsymbol{s} \end{array}\right] \left[\begin{array}{c} \boldsymbol{b} \\ \mathbf{0} \end{array}\right] < 0 \end{array} \quad \text{(IIB')} \ .$$

Applying A, we get that exactly one of (IB') and (IIB') has a solution since they are a special case of A. The equivalences then show that exactly one of (IB) and (IIB) has a solution.

$(B \Rightarrow A)$: Note following equivalences.

$$\text{(IA)} \quad A\boldsymbol{x} \leq \boldsymbol{b} \quad \Leftrightarrow \quad \begin{array}{c} A(\boldsymbol{r} - \boldsymbol{s}) \leq \boldsymbol{b} \\ \boldsymbol{r} \geq \mathbf{0}, \boldsymbol{s} \geq \mathbf{0} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \left[\begin{array}{cc} A & -A \end{array}\right] \left[\begin{array}{c} \boldsymbol{r} \\ \boldsymbol{s} \end{array}\right] \leq \boldsymbol{b} \\ \left[\begin{array}{c} \boldsymbol{r} \\ \boldsymbol{s} \end{array}\right] \geq \mathbf{0} \end{array} \quad \text{(IA')}$$

and

$$\text{(IIA)} \quad \begin{array}{c} \boldsymbol{y}A = \mathbf{0} \\ \boldsymbol{y} \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y}A \geq \mathbf{0} \\ -\boldsymbol{y}A \geq \mathbf{0} \\ \boldsymbol{y} \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \boldsymbol{y} \left[\begin{array}{cc} A & -A \end{array}\right] \geq \mathbf{0} \\ \boldsymbol{y} \geq \mathbf{0} \\ \boldsymbol{y}\boldsymbol{b} < 0 \end{array} \quad \text{(IIA')} \ .$$

In the first line, given $\boldsymbol{x}$ one can easily pick non-negative $\boldsymbol{r}, \boldsymbol{s}$ such that $\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{s}$ so the first $\Leftrightarrow$ in the first line does hold.

Applying B, we get that exactly one of (IA') and (IIA') has a solution since they are a special case of B. The equivalences then show that exactly one of (IA) and (IIA) has a solution.

Linear Programming Duality Theorem from the Theorem of the Alternative for Inequalities:

We will assume the Theorem of the Alternative for Inequalities in the following form:

Exactly one of the following holds:

(I) $A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}$ has a solution $\boldsymbol{x}$
(II) $\boldsymbol{y}A \geq \mathbf{0}, \boldsymbol{y} \geq \mathbf{0}, \boldsymbol{y}\boldsymbol{b} < 0$ has a solution $\boldsymbol{y}$

and use this to prove the following duality theorem for linear programming.

In what follows we will assume that A is an $m \times n$ matrix, $\boldsymbol{c}$ is a length n row vector, $\boldsymbol{x}$ is a length n column vector of variables, $\boldsymbol{b}$ is a length m column vector and $\boldsymbol{y}$ is a length n row vector of variables. We will use $\mathbf{0}$ for zero vectors, 0 for zero matrices, and I for identity matrices where appropriate sizes will be assumed and clear from context.

We will consider the following *primal* linear programming problem $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\}$ and its *dual* $\min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \mathbf{0}\}$. (It can be shown that we can use maximum instead of supremum and minimum instead of infimum as these values are attained if they are finite.)

We repeat here the statement of the weak duality theorem in one of its forms.

Weak Duality Theorem of Linear Programming: If both the primal and dual are feasible then $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\} \leq \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \mathbf{0}\}$.

Strong Duality Theorem of Linear Programming: If both the primal and dual are feasible then $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \mathbf{0}\}$.

Proof: By weak duality we have max $\leq$ min. Thus it is enough to show that there are primal feasible $\boldsymbol{x}^*$ and dual feasible $\boldsymbol{y}^*$ with $\boldsymbol{cx}^* \geq \boldsymbol{y}^*\boldsymbol{b}$. We get this if and only $\boldsymbol{x}^*, \boldsymbol{y}^*$ is a feasible solution to

$$A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}, \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \mathbf{0}, \boldsymbol{cx} \geq \boldsymbol{yb}. \tag{10}$$

We can write (10) as $A'\boldsymbol{x}' \leq \boldsymbol{b}', \boldsymbol{x}' \geq \mathbf{0}$ where

$$A' = \left[\begin{array}{c|c} A & 0 \\ \hline -\boldsymbol{c} & \boldsymbol{b}^T \\ \hline 0 & -A^T \end{array}\right] \text{ and } \boldsymbol{x}' = \left[\begin{array}{c} \boldsymbol{x} \\ \hline \boldsymbol{y}^T \end{array}\right] \text{ and } \boldsymbol{b}' = \left[\begin{array}{c} \boldsymbol{b} \\ \hline 0 \\ \hline -\boldsymbol{c}^T \end{array}\right] \tag{11}$$

By the Theorem of the Alternative for Inequalities if (11) has no solution then

$$\boldsymbol{y}'A' \geq \mathbf{0}, \boldsymbol{y}' \geq \mathbf{0}, \boldsymbol{y}'\boldsymbol{b}' < 0 \tag{12}$$

has a solution. Writing

$$\boldsymbol{y}' = \left[\begin{array}{c|c|c} \boldsymbol{r} & s & \boldsymbol{t}^T \end{array}\right]$$

(12) becomes

$$\boldsymbol{r}A \geq s\boldsymbol{c}, A\boldsymbol{t} \leq s\boldsymbol{b}, \boldsymbol{r} \geq \mathbf{0}, s \geq 0, \boldsymbol{t} \geq \mathbf{0}, \boldsymbol{rb} - \boldsymbol{ct} < 0. \tag{13}$$

If we show that (13) has no solution then (10) must have a solution and we will be done. We will assume that (13) has a solution $\boldsymbol{r}^*, s^*, \boldsymbol{t}^*$ and reach a contradiction.

Observe that s is a scalar (a number). $\boldsymbol{tc}^T$. (We have already used that $\boldsymbol{t}^T\boldsymbol{c}^T = \boldsymbol{ct}$ is a scaler in writing (13)).

Case 1: $s^* = 0$. From (13) with $s^* = 0$ we get $\boldsymbol{r}^*A \geq \mathbf{0}, \boldsymbol{r}^* \geq \mathbf{0}$ and $A\boldsymbol{t}^* \leq \mathbf{0}, \boldsymbol{t}^* \geq \mathbf{0}$. Applying the Theorem of the Alternative to primal feasibility $A\boldsymbol{x} \leq \mathbf{0}, \boldsymbol{x} \geq \mathbf{0}$ yields $\boldsymbol{r}^*\boldsymbol{b} \geq 0$. Applying the Theorem of the Alternative to dual feasibility $\boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \mathbf{0}$ yields $\boldsymbol{ct}^* \leq 0$. Then $\boldsymbol{r}^*\boldsymbol{b} \geq 0$ and $\boldsymbol{ct}^* \leq 0$ contradicts $\boldsymbol{r}^*\boldsymbol{b} - \boldsymbol{ct}^* < 0$.

Case 2: $s^* \neq 0$. Let $\boldsymbol{r'} = \boldsymbol{r}^*/s^*$ and $\boldsymbol{t'} = (\boldsymbol{t}^*/s^*)^T$. Then, from (13) we have

$$\boldsymbol{r}'A \geq \boldsymbol{c}, A\boldsymbol{t}' \leq \boldsymbol{b}, \boldsymbol{r}' \geq \boldsymbol{0}, \boldsymbol{t}' \geq \boldsymbol{0}, \boldsymbol{r}'\boldsymbol{b} - \boldsymbol{c}\boldsymbol{t}' < 0.$$

But $\boldsymbol{r}'\boldsymbol{b} - \boldsymbol{c}\boldsymbol{t}' < 0$ implies $\boldsymbol{c}\boldsymbol{t}' > \boldsymbol{r}'\boldsymbol{b}$ contradicting weak duality. Thus, (13) has no solution and hence (10) has solution. $\square$

(Equivalently, since $\boldsymbol{y}^* + M\boldsymbol{r}^*$ is dual feasible for any dual feasible $\boldsymbol{y}*$ and number $M \geq 0$ we must have $\boldsymbol{r}^*\boldsymbol{b} \geq 0$ or the dual is unbounded and the primal infeasible.) (Equivalently, since $\boldsymbol{x}^* + N\boldsymbol{t}^*$ is primal feasible for any primal feasible $\boldsymbol{x}*$ and number $N \geq 0$ we must have $\boldsymbol{c}^*\boldsymbol{t} \leq 0$ or the primal is unbounded and the dual infeasible.)

We can in fact easily show that if either the primal or the dual has a finite optimum then so does the other. Weak duality shows that if the primal or dual is unbounded then the other must be infeasible. Thus there are four possibilities for a primal-dual pair: both infeasible; primal unbounded and dual infeasible; dual unbounded and primal infeasible; both primal and dual with equal finite optima.

As with Farkas' Lemma we can show that the various versions of duality are equivalent. Here is an example where we prove the equivalence of the strong duality theorems for the following primal-dual pairs:
$\max\{\boldsymbol{cx}|A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A \geq \boldsymbol{c}\}$ and
$\max\{\boldsymbol{cx}|A\boldsymbol{x} \leq \boldsymbol{b}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$

Let (I) be the statement $\max\{\boldsymbol{cx}|A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A \geq \boldsymbol{c}\}$ (when both are feasible) and (II) the statement $\max\{\boldsymbol{cx}|A\boldsymbol{x} \leq \boldsymbol{b}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$ (when both are feasible).

To show (II) implies (I): Assuming the first and last LPs below are feasible we have

$$\begin{aligned}
\max\{\boldsymbol{cx}|A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} &= \max\left\{\left[\begin{array}{c} A \\ \hline -A \\ \hline -I \end{array}\right]\boldsymbol{x} \leq \left[\begin{array}{c} \boldsymbol{b} \\ \hline -\boldsymbol{b} \\ \hline \boldsymbol{0} \end{array}\right]\right\} \\
&= \min\left\{\left[\begin{array}{c|c|c} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{array}\right]\left[\begin{array}{c} \boldsymbol{b} \\ \hline -\boldsymbol{b} \\ \hline \boldsymbol{0} \end{array}\right] | \left[\begin{array}{c|c|c} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{array}\right]\left[\begin{array}{c} A \\ \hline -A \\ \hline -I \end{array}\right] = \boldsymbol{c}, \left[\begin{array}{c|c|c} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{array}\right] \geq \boldsymbol{0}\right\} \\
&= \min\{\boldsymbol{yb}|\boldsymbol{y}A \geq \boldsymbol{c}\}
\end{aligned}$$

The first and third equalities follow from basic manipulations. The second follows from (II).

To show (I) implies (II): Assuming the first and last LPs below are feasible we have

$$\begin{aligned}
\max\{\boldsymbol{cx}|A\boldsymbol{x} \leq \boldsymbol{b}\} &= \max\left\{\left[\begin{array}{c|c|c} \boldsymbol{c} & -\boldsymbol{c} & \boldsymbol{0} \end{array}\right]\left[\begin{array}{c} \boldsymbol{u} \\ \hline -\boldsymbol{v} \\ \hline \boldsymbol{w} \end{array}\right] | \left[\begin{array}{c|c|c} A & -A & I \end{array}\right]\left[\begin{array}{c} \boldsymbol{u} \\ \hline -\boldsymbol{v} \\ \hline \boldsymbol{w} \end{array}\right] = \boldsymbol{b}, \left[\begin{array}{c} \boldsymbol{u} \\ \hline -\boldsymbol{v} \\ \hline \boldsymbol{w} \end{array}\right] \geq \boldsymbol{0}\right\} \\
&= \min\left\{\boldsymbol{yb}|\boldsymbol{y}\left[\begin{array}{c|c|c} A & -A & I \end{array}\right] \geq \left[\begin{array}{c|c|c} \boldsymbol{c} & -\boldsymbol{c} & \boldsymbol{0} \end{array}\right]\right\} \\
&= \min\{\boldsymbol{yb}|\boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}
\end{aligned}$$

The first and third equalities follow from basic manipulations. The second follows from (I).