Math 242 fall 2008 notes on problem session for week of 9-8-08

This is a short overview of problems that we covered.

1. Prove that $A^T A$ is symmetric. Using $(BC)^T = C^T B^T$ and $(A^T)^T = A$ we get $(A^T A)^T = A^T (A^T)^T = A^T A$. Thus $A^T A$ is symmetric (since it is equal to its transpose).

2. Prove $(AB)^T = B^T A^T$. Assume that $A$ is $m \times n$ and $B$ is $n \times p$.

   Use subscripts like $A_{ij}$ to indicate entries in matrix $A$. We need to show that $(AB)^T_{ij} = (B^T A^T)_{ij}$. From the definition of transpose, the $ij$ entry of the transpose is the $ji$ entry of the original. Use this and the definition of matrix multiplication.

   $$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} A_{kj}^T B_{ik}^T = \sum_{k=1}^{n} B_{ik}^T A_{kj}^T = (B^T A^T)_{ij}.$$ 

3. The trace of a square matrix is the sum of its diagonal elements. Writing $\text{Tr}(A)$ for the trace of an $n \times n$ matrix this is $\text{Tr}(A) = \sum_{i=1}^{n} A_{ii}$. Prove that $\text{Tr}(AB) = \text{Tr}(BA)$ when both are defined. Assume that $A$ is $m \times n$ then $B$ is $n \times m$ in order for both to be defined. Using the definition of trace and matrix multiplication we get

   $$\text{Tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{m} A_{ik} B_{ki} = \sum_{i=1}^{n} \sum_{k=1}^{m} B_{ki} A_{ik} = \sum_{k=1}^{m} (BA)_{kk} = \text{Tr}(BA).$$

4. Find an equation for the plane $ax + by + cz = d$ through the points $(0, 2, -1)$, $(-2, 4, 3)$, $(2, -1, -3)$.

   Substituting for $x, y, z$ from each of the three points we get three equations

   $$2b - c - d = 0$$
   $$-2a + 4b + 3c - d = 0$$
   $$2a - b - 3c - d = 0$$

   We get solutions $a = 4d/3, b = 2d/3, c = d/3, d = d$ for any choice of $d$. Setting $d = 3$ yields the plane $4x + 2y + z = 3$.

5. Let $Ax = \lambda x$ for some scalar $\lambda$. Prove that $A^k x = \lambda^k x$. Use induction. The case $k = 1$ is given. For $k > 1$, using the fact that scalars commute with matrices and the induction hypothesis $A^{k-1} x = \lambda^{k-1} x$ we get $A^k x = A A^{k-1} x = A(\lambda^{k-1} x) = \lambda^{k-1} A x = \lambda^{k-1} (\lambda x) = \lambda^k x$. Hence by induction the result holds for all positive integers $k$. 