Math 242 fall 2008 notes on problem session for week of 9-15-08
This is a short overview of problems that we covered.

1. Let A be an $m \times m$ nonsingular matrix. Form $\hat{A}$ from A by adding 3 times row two to row four.

   (a) Describe an elementary matrix that encodes this.
   Let E be the $m \times m$ elementary matrix with diagonal entries 1, the $(4,2)$ entry 3 and every other entry 0. That is, $E_{i,i} = 1$ for $i = 1, 2, \ldots, m$, $E_{4,2} = 3$ and $E_{i,j} = 0$ otherwise. This is the matrix obtained from I_m by adding 3 times row two to row four. We now have $\hat{A} = EA$. This can easily be checked as follows: note that every row of E except row 4 is the corresponding identity row, so every row of $\hat{A} = EA$ except row 4 is the same as the corresponding row of A. Row 4 of E has a 3 in position 2, a 1 in position 4 and 0 otherwise. So row 4 of $\hat{A} = EA$ is 3 times row 2 of A plus 1 times row 4 of A. A more formal description of the action of elementary matrices is in the text.

   (b) Describe how to obtain $\hat{A}^{-1}$ from A^{-1}
   Since $\hat{A} = EA$ we have $\hat{A}^{-1} = (EA)^{-1} = A^{-1}E^{-1}$. It is easy to check that E^{-1} is the $m \times m$ elementary matrix with diagonal entries 1, the $(4,2)$ entry -3 and every other entry 0. One way to discover this is that we need an elementary matrix that undoes the original operation, so we add -3 times row 2 to row 4. Doing this to the identity yields E^{-1} as described. Right multiplying by E^{-1} acts on columns. Since every column of E^{-1} except column 2 is the corresponding identity column, every column of $\hat{A}^{-1} = A^{-1}E^{-1}$ is the same as the corresponding column of A^{-1}. The second column has a 1 in position 2 and a -3 in position 4 so column 2 of $\hat{A}^{-1} = A^{-1}E^{-1}$ is column 2 of A^{-1} plus -3 times column 4 of A^{-1}.

   (c) If $C = AB$ and $\hat{C} = \hat{A}B$ describe how to get $\hat{C}$ from C.
   Left multiply $C = AB$ by E to get $EC = E(AB) = (EA)B = \hat{A}B = \hat{C}$. Since $\hat{C} = EA$ and $\hat{A} = EA$, $\hat{C}$ is obtained from C in the same way that $\hat{A}$ is obtained from A; by adding 3 times row two to row four.

2. If the inverse of A^2 is B, show that A has an inverse and say what it is. Note that for A^2 to be defined A must be square. We are given $(A^2)^{-1} = B$. So $I = A^2B = A(AB)$ hence AB is the inverse of A. Note also that we could do $I = BA^2 = (BA)A$ and so also the inverse of A is BA. Since the inverse is unique we have also shown that is this case $AB = BA = A^{-1}$.

3. Let A and B be square matrices with factorizations $P_1A = L_1U_1$ and $P_2B = L_2U_2$ where P_1, P_2 are permutation matrices, L_1, L_2 are lower triangular with 1's on the diagonal and U_1, U_2 are upper triangular with nonzero entries on the diagonal. Determine the $PM = LU$ factorization of the block matrix $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Once we guess at the correct form we can show it is correct as follows, using block matrix multiplication and substituting using the given equations and using the fact that L^{-1} exists: $\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} P_1A & P_1C \\ 0 & P_2B \end{pmatrix} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}\begin{pmatrix} U_1 & L^{-1}P_1A \\ 0 & U_2 \end{pmatrix}$. From the triangular properties of L_1, L_2, U_1, U_2 it is easy to see that the block matrices on the right are lower and upper triangular as needed.

4. If $A = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 1 & -3 \\ -1 & 5 & -1 \end{pmatrix}$, determine A^{-1}. Note that the columns of A are orthogonal. Since the columns are orthogonal we know that A^TA will have 0 in all off diagonal entries. Checking we get $A^TA = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 14 \end{pmatrix}$. Recall that multiplying a row of the left matrix in a product by a constant multiplies the same row in the product by the same constant. Using this we 'adjust' the rows of A^T to get $A^{-1} = \begin{pmatrix} 1/3 & 1/3 & -1/3 \\ 4/42 & 1/42 & 5/42 \\ 2/14 & -3/14 & -1/14 \end{pmatrix}$ which is easily checked to be correct.

5. If $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$. Find a $P_1AP_2 = LU$ factorization where P_1, P_2 are permutation matrices, L is lower triangular with 1's on the diagonal and U is upper triangular. Let $P_1 = P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Left multiplying by P_1 reversed the order of the rows and right multiplying by P_2 reverses the order of the columns so we then have $L = I_4$ and $U = \begin{pmatrix} 1 & 1 & 3 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.