1. What is the graph of \(9x^2 + 4y^2 = 1\)? It is an ellipse with points on the major and minor axes \((1/3,0), (-1/3,0),(0,1/2),(0,-1/2)\). Writing the equation in matrix form we have \((x,y)\left(\begin{array}{rr}9 & 0 \\ 0 & 4\end{array}\right)\left(\begin{array}{r}x \\ y\end{array}\right)\).

2. Graph \(8x^2 - 4xy + 5y^2 = 1\). We write this in matrix form as \((x,y)\left(\begin{array}{rr}8 & -2 \\ -2 & 5\end{array}\right)\).

We first diagonalize the matrix \(A = \left(\begin{array}{rr}8 & -2 \\ -2 & 5\end{array}\right)\). \(A\) is symmetric so we will be able to write \(A = SDS^{-1}\) where \(S\) is orthogonal and \(D\) is diagonal.

\[
det(A - \lambda I) = det\left(\begin{array}{rr}8 - \lambda & -2 \\ -2 & 5 - \lambda\end{array}\right) = (8 - \lambda)(5 - \lambda) - (-2)(-2) = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)\). So eigenvalues are 9 and 4.

To find an eigenvector associated with \(\lambda = 9\) we solve the homogeneous system \((A - 9I)x = 0\). The matrix is \:\left(\begin{array}{rr}8 & -2 \\ -2 & 5\end{array}\right) = \left(\begin{array}{rr}-1 & -2 \\ -2 & -4\end{array}\right).\) After Gaussian elimination this becomes \:\left(\begin{array}{rrr}1 & 2 & 0 \\ 0 & 0 & 0\end{array}\right)\) with solutions \((x,y) = (2, -1)y\). So multiples of \((2, -1)\) are eigenvalues associated with eigenvalue 9. Similarly we get eigenvector \((1, 2)\) associated with eigenvalue 4. We normalize these and put them as columns of \(S\) and make \(D\) the diagonal matrix with the eigenvalues on the diagonal. Then we have \(A = \left(\begin{array}{rr}8 & -2 \\ -2 & 5\end{array}\right) = \left(\begin{array}{rr}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{array}\right)\left(\begin{array}{rr}9 & 0 \\ 0 & 4\end{array}\right)\left(\begin{array}{rr}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{array}\right)^{-1} = SDS^{-1}\). The graph is the same ellipse as before but now with axes given by the eigenvectors. To see this let \(u_1\) be the eigenvector with eigenvalue 9 and \(u_2\) the eigenvector with eigenvalue 4. These are orthogonal and hence form a basis for \(\mathbb{R}^2\). So for any \(x\) we can find \(r,s\) so that \(x = ru_1 + su_2\). Using \(Au_1 = 9u_1\) and \(Au_2 = 4u_2\) as these are eigenvectors and using \(u_1^T u_2 = u_2^T u_1 = 0\) since they are orthogonal and \(u_1^T u_1 = u_2^T u_2 = 1\) since they have been normalized we get that our graph is

\[
1 = x^T Ax = (ru_1 + su_2)^T(Aru_1 + su_2) = (ru_1^T + su_2^T)(9ru_1 + 4su_2) = 9r^2u_1^T u_1 + 4rsu_1^T u_2 + 4rsu_2^T u_1 + 4s^2u_2^T u_2 = 9r^2 + 4s^2.
\]

So the graph is an ellipse with points on the major and minor axes \(\pm 1/3\) of a unit in the direction \(u_1 = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})\) and \(\pm 1/2\) of a unit in the direction \(u_2 = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})\).

3. Let \(y_0\) be the population of California at time \(t = 0\) and \(z_0\) the population of the rest of the world. If during any year \(1/10\) of California’s population moves out of
the state and 2/10 of the population of the rest of the world moves into the state 
write equations for the populations $y_1, z_1$ after 1 year. We get $y_1 = .9y_0 + .2z_0$ and 
z$1 = .1y_0 + .8z_0$. In matrix form these are \[
\begin{pmatrix}
y_1 \\
z_1
\end{pmatrix} = \begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix} \begin{pmatrix} y_0 \\
z_0
\end{pmatrix}.
\]
Calling the matrix $A$ and the vectors $x_1$ and $x_0$ we can extend this to time $t$. The populations 
are given by $x_t = A^t x_0$. The matrix $A$ has eigenvalues 1 and 0.7 with associated 
eigenvectors $(2/3, 1/3)$ and $(1/3, -1/3)$. So we diagonalize $A = \begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix} = 
\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ 0 & .7 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} = SDS^{-1}$. In this case $S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$.

Observe that $A^t = (SDS^{-1})^T = (SDS^{-1})(SDS^{-1}) \cdots (SDS^{-1}) = SD(S^{-1}S)D(S^{-1}S)D \cdots S^{-1} = 
SD^tS^{-1}$. So we can compute $A^t = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1^t & 0 \\ 0 & (.7)^t \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} = 
\begin{pmatrix} 2/3 + 1/3(.7)^t & 2/3 - 2/3(.7)^t \\ 1/3 - 1/3(.7)^t & 1/3 + 2/3(.7)^t \end{pmatrix}$. As $t \to \infty$ this becomes $\begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$, the 
matrix with the eigenvector associated with 1 in each column. The long term state is 
y$= 2/3$ and $z = 1/3$. This is an example of what is called a Markov chain.

4. Let $B$ have eigenvalue $\lambda$ and associated eigenvector $v$. If $B^{-1}$ exists show that $v$ 
is an eigenvector of $B^{-1}$ associated with eigenvalue $1/\lambda$. We have $Bv = \lambda v$. Then 
$\frac{1}{\lambda} v = \frac{1}{\lambda} Iv = \frac{1}{\lambda} B^{-1} Bv = \frac{1}{\lambda} B^{-1} \lambda v = B^{-1} v$.

Let $A$ have eigenvalue $\lambda$ and associated eigenvector $v$. Show that $(I - A)$ and $(I - A)^{-1}$ 
(assuming it exists) also have $v$ as an eigenvector and determine the eigenvalues.

$(I - A)v = Iv - Av = v - \lambda v = (1 - \lambda)v$. So the eigenvalue is $(1 - \lambda)$. Then, from 
the previous paragraph $v$ is an eigenvector with eigenvalue $\frac{1}{1-\lambda}$ for $(I - A)^{-1}$.

Finally we noted the matrix equation (similar to the Taylor expansion of $\frac{1}{1-x} = (1 - x)^{-1}$) $(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$ and noted (comparing to the previous example) 
that we get convergence when the powers of $A$ go to the zero matrix, which occurs 
if all eigenvalues have magnitude less than 1. This equation appears in the Leontief model in economics.