Homework solutions for Math 242, Linear Algebra, Lehigh University fall 2008
Here are solutions to a few of the more abstract homework problems. Please remember that there is often more than one way to do a proof and more than one way to present a particular proof. So these are examples of answers. Not the only correct way to do them.

1.2.30: Prove that matrix multiplication is associative: $A(BC) = (AB)C$ when it is defined.

Assume that A is $m \times n$, B is $n \times p$ and C is $p \times q$. We will show that both have the same (i, j) entry for every i, j. Using the definition of matrix multiplication we get

$$[A(BC)]_{ij} = \sum_{r=1}^{n} A_{ir}(BC)_{rj} = \sum_{r=1}^{n} A_{ir}(\sum_{s=1}^{p} B_{rs}C_{sj}) = \sum_{r=1}^{n}\sum_{s=1}^{p} A_{ir}B_{rs}C_{sj} =$$
$$\sum_{s=1}^{p}(\sum_{r=1}^{n} A_{ir}B_{rs})C_{sj} = \sum_{s=1}^{p}(AB)_{is}C_{sj} = [(AB)C]_{ij}$$

1.3.13b: A matrix is nilpotent if $A^k = 0$ for some k. A matrix A is strictly upper triangular if $A_{ij} = 0$ for $i \geq j$. Prove that strictly upper triangular matrices are nilpotent.

We will prove, by induction, that if A is strictly upper triangular then $A^k_{ij} = 0$ for $i > j - k$. This implies that $A^k = 0$ for $k \geq m$ if A is $m \times m$. The basis for the induction is $A^1 = 0$ for $i > j - 1$ follows from the assumption that A is strictly upper triangular (since $i \geq j$ if and only if $i > j - 1$). We assume, by induction that $A^{k-1}_{ij} = 0$ for $i > j - (k - 1)$ and show that $A^k_{ij} = 0$ for $i > j - k$. The result then follows by induction.

From the definition of matrix multiplication we get for $i > j - k$:

$$A^k_{ij} = (AA^{k-1})_{ij} = \sum_{r=1}^{m} A_{ir}A^{k-1}_{rj} = \sum_{r=1}^{i} A_{ir}A^{k-1}_{rj} + \sum_{r=i+1}^{m} A_{ir}A^{k-1}_{rj} == \sum_{r=1}^{i} 0 \cdot A^{k-1}_{rj} + \sum_{r=i+1}^{m} A_{ir} \cdot 0 = 0.$$

Here we have used that in the first sum $r \leq i$ and hence $A_{ir} = 0$ since A is strictly upper triangular. In the second sum $r \geq i + 1 > (j - k) + 1 = j - (k - 1)$ and hence $A^{k-1}_{rj} = 0$ by the induction hypothesis.

1.3.21c: Prove that the product of two special lower triangular matrices is special lower triangular. If L and M are $m \times m$ special lower triangular matrices then $L_{ij} = M_{ij} = 0$ for $m \geq j > i \geq 1$ and $L_{ii} = M_{ii} = 0$ for $m \geq i \geq 1$. We need to show that $(LM)_{ij} = 0$ for $m \geq j > i \geq 1$ and $(LM)_{ii} = 0$ for $m \geq i \geq 1$.

For $j > i$

$$(LM)_{ij} = \sum_{k=1}^{m} L_{ik}M_{kj} = \sum_{k=1}^{i} L_{ik}M_{kj} + \sum_{k=i+1}^{m} L_{ik}M_{kj} = \sum_{k=1}^{i} L_{ik} \cdot 0 + \sum_{k=i+1}^{m} 0 \cdot M_{kj} = 0.$$

Here we have used that in the first sum $k \leq i < j$ so $M_{kj} = 0$ and in the second sum $k \geq i + 1 > i$ so $L_{ik} = 0$.

1.5.18c: Write $A \sim B$ if there exists an invertible matrix S such that $B = S^{-1}AS$. Prove that if $A \sim B$ and $B \sim C$ then $A \sim C$. Since $A \sim B$ and $B \sim C$ there are invertible matrices S, T such that $B = S^{-1}AS$ and $C = T^{-1}BT$. Then $(ST)^{-1}A(ST) = T^{-1}(S^{-1}AS)T = T^{-1}BT = C$. So using ST we see that $A \sim C$.

1.6.13a: Suppose that $\boldsymbol{v}^T A\boldsymbol{w} = \boldsymbol{v}^T B\boldsymbol{w}$ for all vectors $\boldsymbol{w}, \boldsymbol{w}$. Prove that $A = B$. Let f_i^T denote the i^{th} row of I_m and e_j denote the j^{th} column of I_n. Now $\boldsymbol{f_i}^T A\boldsymbol{e_j} = A_{ij}$, the (i, j) entry of A. This follows since $\boldsymbol{f_i}^T A$ is the i^{th} row of A and $Row_i(A)e_j$ is the j^{th} entry of $Row_i(A)$. Similarly, $\boldsymbol{f_i}^T B\boldsymbol{e_j} = B_{ij}$. So for any $1 \leq i \leq m, 1 \leq j \leq n$ we have $A_{ij} = \boldsymbol{f_i}^T A\boldsymbol{e_j} = \boldsymbol{f_i}^T A\boldsymbol{e_j} = B_{ij}$. Hence $A = B$.

1.8.15a: Let $A = \boldsymbol{v}\boldsymbol{w}^T$ be the product of an $m\times 1$ column vector $\boldsymbol{w}$ith $\boldsymbol{v}^T = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix}$ and a $1 \times n$ row vector $\boldsymbol{w}^T = \begin{pmatrix} w_1 & w_2 & \cdots & w_m \end{pmatrix}$. Prove that the rank of A is 1. You may assume that $w_1 \neq 0$ and $v_1 \neq 0$ to simplify notation.

We show that the rank of A is 1 by showing that U has 1 nonzero row in a factorization $PA = LU$. We will give a factorization with $P = I$. Write $\hat{\boldsymbol{v}}^T = \begin{pmatrix} v_2 & v_3 & \cdots & v_m \end{pmatrix}$ and $\hat{\boldsymbol{w}}^T = \begin{pmatrix} w_2 & w_3 & \cdots & w_n \end{pmatrix}$. These are $\boldsymbol{v}$ and $\boldsymbol{w}$ with the first entry deleted. Then consider the following block matrix multiplication where the 0 matrices and vectors and identity matrix are of the appropriate sizes.

$$A = \boldsymbol{v}\boldsymbol{w}^T = \begin{pmatrix} v_1 \\ \hat{\boldsymbol{v}} \end{pmatrix} \begin{pmatrix} w_1 & \hat{\boldsymbol{w}}^T \end{pmatrix} = \begin{pmatrix} v_1 w_1 & v_1 \hat{\boldsymbol{w}}^T \\ \hat{\boldsymbol{v}} w_1 & \hat{\boldsymbol{v}}\hat{\boldsymbol{w}}^T \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \frac{1}{v_1}\hat{\boldsymbol{v}} & I \end{pmatrix} \begin{pmatrix} v_1 w_1 & v_1 \hat{\boldsymbol{w}}^T \\ 0 & 0 \end{pmatrix}$$

This is a $A = LU$ factorization with U having one nonzero row. So the rank of A is 1.

Alternate proof: A is $\begin{pmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_n \\ \vdots & & \ddots & \vdots \\ v_m w_1 & v_m w_2 & \cdots & v_m w_n \end{pmatrix}$. We see that each row is a multiple of $\boldsymbol{w}^T$ with the multipliers specified by $\boldsymbol{v}$. Pivoting on the $(1, 1)$ entry which we have assumed to be nonzero we add $\frac{-v_i}{v_1}$ times row 1 to row i resulting in a zero row. So pivoting produces a matrix with the same first row as A and every other row a zero row. Hence A has rank 1.

1.9.8 Prove that if A is $n \times n$ and c is a scalar then $det(cA) = c^n det(A)$. Note that $cA = cIA = \hat{I}A$ where $\hat{I}$ is a diagonal matrix with every diagonal entry c. Since $\hat{I}$ is diagonal its determinant is the product of these diagonal entries. That is $det(\hat{I}) = c^n$. Then $det(cA) = det(\hat{I}A) = det(\hat{I})det(A) = c^n det(A)$.