Notes on bases for the four fundamental spaces of a matrix
Notes for Math 242, Linear Algebra, Lehigh University fall 2008

Except for the method for the cokernel what we do here is also covered in more detail in the text on pages 116-119. Parts here are the same as the text. We will assume that A is an $m \times n$ matrix and that we have a factorization $PA = LU$. The rank of A is the number of pivots (which is the number of nonzero rows of U) is r. Concrete examples using these rules to compute bases are in the notes to the problem session on 9-29-08.

Kernel: To find a basis for $ker(A) = \{\boldsymbol{x} | A\boldsymbol{x} = \boldsymbol{0}\}$ we write down the general solution to $A\boldsymbol{x} = \boldsymbol{0}$ in terms of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{n-r}$ (which we obtain by back substitution) times the free variables. By construction the vectors span the kernel. The vectors are linearly independent since since if x_i is a free variable then the i^{th} coordinate for its vector is 1 and the i^{th} coordinate of the other vectors is 0. Then looking at the i^{th} coordinate of $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_{n-r}\boldsymbol{v}_{n-r} = \boldsymbol{0}$ we get $c_i = 0$. So these are linearly independent. The dimension is the number of free variables $n - r$.

Corange: The corange is $\{\boldsymbol{c} | \boldsymbol{y}^T A = \boldsymbol{c}^T$ for some $\boldsymbol{y}\}$. We have shown (for example in the problem session notes) that left multiplication by elementary matrices does not change the corange (the row space). Thus the rows of U have the same span as the rows of A. The span of the nonzero rows of U is clearly the same as the span of the rows of U and hence these vectors span the corange. Let $\hat{U}$ be the submatrix consisting of the pivot columns of U. Since these are the pivot columns this is an upper triangular matrix with nonzero diagonal entries. Thus $\hat{U}^{-1}$ exists. To show linear independence we must show that $\boldsymbol{y}^T U = \boldsymbol{0}^T$ implies $\boldsymbol{y} = \boldsymbol{0}$. If $\boldsymbol{y}^T U = \boldsymbol{0}$ then $\boldsymbol{y}^T \hat{U} = \boldsymbol{0}^T$ since $\hat{U}$ is a matrix formed by a subset of columns of U. Since $\hat{U}^{-1}$ exists we have $\boldsymbol{y}^T = \boldsymbol{0}^T \hat{U}^{-1} = \boldsymbol{0}^T$. So the vectors are linearly independent. The dimension is the number of nonzero rows, which is the number of pivots r.

Cokernel: The cokernel is $\{\boldsymbol{y} | \boldsymbol{y}^T A = \boldsymbol{0}\}$. We have $PA = LU$ and since L^{-1} exists $L^{-1}PA = U$. The last $m - r$ rows of U are zero rows. Thus the last $m - r$ rows of $L^{-1}P$, $\boldsymbol{y}_1^T, \boldsymbol{y}_2^T, \ldots, \boldsymbol{y}_{m-r}^T$ satisfy $\boldsymbol{y}_i^T A = \boldsymbol{0}$ as each product is the corresponding zero row of U. To show linear independence of these vectors we must show that $\boldsymbol{u}^T W = \boldsymbol{0}^T$ implies $\boldsymbol{u} = \boldsymbol{0}$ where W consists of the last $m - r$ rows of $L^{-1}P$. Since right multiplication by a permutation matrix permutes columns this implies that $\boldsymbol{u}^T \hat{W} = \boldsymbol{0}$ where $\hat{W}$ is the submatrix consisting of the last $m - r$ row and columns of L^{-1}. Since L^{-1} is lower triangular with 1's on the diagonal $\hat{W}^{-1}$ exists. Then $\boldsymbol{u}^T = \boldsymbol{0}\hat{W}^{-1} = \boldsymbol{0}$. So the vectors are independent.

We will show below that the dimension of the range is r. Hence using the results above that the dimension of kernel plus the dimension of the corange is the number of columns we have $dim(Cokernel A) = dim(kernal A^T) = m - dim(corange A^T) = m - dim(range A) = m - r$. So the last $m - r$ rows of $L^{-1}P$ are $m - r$ independent vectors in the cokernel which has dimension $m - r$ hence they are a basis. The dimension is $m - r$.

Range: The range is $\{\boldsymbol{b} | A\boldsymbol{x} = \boldsymbol{b}$ for some $\boldsymbol{x}\}$. Consider the pivot columns of U and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ be the corresponding columns of A. Writing the general solution to $A\boldsymbol{x} = \boldsymbol{b}$ we have a solution that is of the form $\boldsymbol{z} + s_1\boldsymbol{v}_1 + s_2\boldsymbol{v}_2 + \cdots + s_{n-r}\boldsymbol{v}_{n-r}$ where $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-r}$ are a basis for the kernel and the s_i are arbitrarily selected values of the free variables. Setting the s_i to 0 gives a solution for which only the pivot variables are nonzero. That is, we have a particular solution so that $A\boldsymbol{x}^* = \boldsymbol{b}$ and only entries corresponding to pivot variables are nonzero. So an arbitrary element of the range can be written in terms of the columns of A corresponding to pivot variables. To show these columns are linearly independent consider the submatrix $\hat{A}$ of these columns. We must show that $\hat{A}\boldsymbol{c} = \boldsymbol{0}$ implies $\boldsymbol{c} = \boldsymbol{0}$. We have $P\hat{A} = L\hat{U}$ where $\hat{U}$ consists of pivot columns of U. Note that $\hat{U}$ is upper triangular with nonzero diagonal entries. So $\hat{U}^{-1}$ exists. So if $\hat{A}\boldsymbol{c} = \boldsymbol{0}$ then $L\hat{U}\boldsymbol{c} = P\hat{A}\boldsymbol{c} = \boldsymbol{0}$. Left multiplying by $L^{-1}\hat{U}^{-1}$ we get $\boldsymbol{c} = \boldsymbol{0}$. So the vectors are linearly independent. The dimension is r.