1: [8] Let \( \mathbf{v} \) and \( \mathbf{w} \) be elements of an inner product space. Prove that \( \|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \) if and only if \( \mathbf{v} \) and \( \mathbf{w} \) are orthogonal.

2: [7] Prove that the transpose of an orthogonal matrix is orthogonal.

3: [17] Let \( P = A(A^T A)^{-1} A^T \) be the projection matrix onto the column space of \( A \).
   (a) Prove that \( P^2 = P \).
   (b) Prove that \( \text{Range}(P) = \text{Range}(A) \).

4: [11] Prove that if \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are nonzero mutually orthogonal vectors in a vector space then they are linearly independent.

5: [22] Let \( K \) be a symmetric matrix and \( \mathbf{x}^* \) a solution to \( K \mathbf{x} = \mathbf{f} \).
   Show/explain each of the following:
   (a) The quadratic form \( p(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2 \mathbf{x}^T \mathbf{f} + c \) is equal to \( (\mathbf{x} - \mathbf{x}^*)^T K (\mathbf{x} - \mathbf{x}^*) + [c - (\mathbf{x}^*)^T K \mathbf{x}^*] \).
   (b) If \( K \) is positive definite then \( \mathbf{x}^* \) exists and is the unique minimizer to \( p(\mathbf{x}) \).
   (c) If \( K \) is positive semidefinite and \( \mathbf{x}^* \) exists (i.e., \( \mathbf{f} \) is in the range of \( K \)) then \( \mathbf{x} \) such that \( K \mathbf{x} = \mathbf{f} \) minimize \( p(\mathbf{x}) \) and the minimizer is not unique.
   (d) If \( K \) is positive semidefinite and \( K \mathbf{x} = \mathbf{f} \) has no solution (i.e., \( \mathbf{f} \) is not in the range of \( K \)) then \( p(\mathbf{x}) \) has no global minimum.
   (e) If \( K \) is not positive semidefinite then \( p(\mathbf{x}) \) has no global minimum.

6: [11] Derive the normal equations for the least squares solution to the system \( A \mathbf{x} = \mathbf{b} \).
   A least squares solution minimizes \( \|A \mathbf{x} - \mathbf{b}\| \) and the orthogonal projection is the vector \( \mathbf{w} \in W \) such that \( \mathbf{z} = \mathbf{v} - \mathbf{w} \) is orthogonal to every vector in \( W \). You may assume the geometry that the closest point \( \mathbf{b} \) in a subspace \( W \) to \( \mathbf{v} \) is the orthogonal projection of \( \mathbf{b} \) onto \( W \).

7: [24] (a) Find the projection of \((15, -5, 0)^T \) onto \((3/5, 4/5, 0)^T \).
   (b) Find the projection of \((5, -10, 2)^T \) onto the plane spanned by \((3/5, 4/5, 0)^T \) and \((4/5, -3/5, 0)^T \).
   (c) Find a \( QR \) factorization of \[
   \begin{pmatrix}
   6 & 15 & 5 \\
   8 & -5 & -10 \\
   0 & 0 & 2
   \end{pmatrix}
   \].