10 Binomial (and other) Facts

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Recall 'Pascal's Triangle'

which we will call the

Binomial Triangle

It is convenient to display

left justified (left) rather than the typical way (right)

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

1. Who first discovered Pascal's Triangle?
2. The numbers in the binomial triangle count something. What?
3. Explain the rule 'each entry is the sum of the two above it'
4. You (probably) used the binomial triangle for computing coefficients in $(x + y)^n$. Why?
5. What are the row sums? Why?
6. What are the diagonal sums (up to a given row)?
7. What are the antidiagonal sums (look at the left justified triangle)?
8. Who first discovered Fibonacci numbers?
9. How are Fibonacci numbers related to powers of the golden ratio?
10. What is $\lim_{n\to\infty}(1 + \frac{1}{n})^n$ and why does your bank care care?

Who first discovered Pascal's Triangle?

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

- Pascal (France around 1650)

Who first discovered Pascal's Triangle?

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

- Tartaglia (Italy around 1550)
- Pascal (France around 1650)

Who first discovered Pascal's Triangle?

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

- China: Yang Hui's Triangle; Yang Hui (around 1350) based on Jia Xian (around 1050)
- Tartaglia (Italy around 1550)
- Pascal (France around 1650)

Who first discovered Pascal's Triangle?

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

- Persia: Kayyam's Triangle; Al-Karaji (around 100) and Kayyam (around 1100)
- China: Yang Hui's Triangle; Yang Hui (around 1350) based on Jia Xian (around 1050)
- Tartaglia (Italy around 1550)
- Pascal (France around 1650)

Who first discovered Pascal's Triangle?

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

- India: Hayluda Bhattotpala around 1000; commentary on Pingali 200 B.C.E. work on Sanskrit prosody
- Persia: Kayyam's Triangle; Al-Karaji (around 100) and Kayyam (around 1100)
- China: Yang Hui's Triangle; Yang Hui (around 1350) based on Jia Xian (around 1050)
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- Pascal (France around 1650)

The numbers in the binomial triangle count something.

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Row 7 column 3 entry $35 = \binom{7}{3}$ read '7 choose 3'
number of 3 element subsets of a 7 element set

Row n column k entry $\binom{n}{k}$ read 'n choose k'
number of k elements subsets of $\{1, 2, \ldots, n\}$

There is a simple numerical formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
but we do not need it today

Explain the rule ‘each entry is the sum of the two above it’

Binomial identity: $\binom{7}{3} = \binom{6}{2} + \binom{6}{3}$

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

"Proof":

The $\binom{7}{3}$=35 size 3 subsets of $\{A, B, C, D, E, F, G\}$

$=$

The $\binom{6}{2} = 15$ subsets including A $+$ The $\binom{6}{3} = 20$ subsets avoiding A

Binomial identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

You (probably) used the binomial triangle for computing coefficients in $(x+y)^n$. Why?

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx+xy+yx+yy\\ &= x^2+2xy+y^2\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy\\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y)\\ &= \ldots+xxyy+xyxy+yxxy+xyyx+yxyx+yyxx+\ldots\\ &= \ldots+6x^2y^2+\ldots\end{aligned}$$

You (probably) used the binomial triangle for computing coefficients in $(x+y)^n$. Why?

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx + xy + yx + yy\\ &= x^2 + 2xy + y^2\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y)\\ &= \ldots + xxyy + xyxy + yxxy + xyyx + yxyx + yyxx + \ldots\\ &= \ldots + 6x^2y^2 + \ldots\end{aligned}$$

- $(x+y)^n$ expands into length strings of x and y
- coefficient of $x^k y^{n-k}$ is number of choices $\binom{n}{k}$ for the x's

What are the row sums?

$$\begin{array}{rllllllll} & 1 \\ & 1 & 1 \\ & 1 & 2 & 1 \\ 8 = & 1 & 3 & 3 & 1 \\ & 1 & 4 & 6 & 4 & 1 \\ & 1 & 5 & 10 & 10 & 5 & 1 \\ & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{array}$$

What are the row sums?

	1							
	1	1						
	1	2	1					
$8 =$	1	3	3	1				
$16 =$	1	4	6	4	1			
	1	5	10	10	5	1		
	1	6	15	20	15	6	1	
	1	7	21	35	35	21	7	1

What are the row sums?

		1								
		1	1							
		1	2	1						
8	=	1	3	3	1					
16	=	1	4	6	4	1				
32	=	1	5	10	10	5	1			
		1	6	15	20	15	6	1		
		1	7	21	35	35	21	7	1	

What are the row sums?

$$\begin{array}{rcl}
1 &=& 1\\
2 &=& 1\ 1\\
4 &=& 1\ 2\ 1\\
8 &=& 1\ 3\ 3\ 1\\
16 &=& 1\ 4\ 6\ 4\ 1\\
32 &=& 1\ 5\ 10\ 10\ 5\ 1\\
64 &=& 1\ 6\ 15\ 20\ 15\ 6\ 1\\
128 &=& 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1
\end{array}$$

What are the row sums?

$$\begin{aligned}
1 &= 1 \\
2 &= 1\ 1 \\
4 &= 1\ 2\ 1 \\
8 &= 1\ 3\ 3\ 1 \\
16 &= 1\ 4\ 6\ 4\ 1 \\
32 &= 1\ 5\ 10\ 10\ 5\ 1 \\
64 &= 1\ 6\ 15\ 20\ 15\ 6\ 1 \\
128 &= 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1
\end{aligned}$$

Row sums are powers of 2

What are the row sums?

$$\begin{array}{rcl} 1 &=& 1 \\ 2 &=& 1\ 1 \\ 4 &=& 1\ 2\ 1 \\ 8 &=& 1\ 3\ 3\ 1 \\ 16 &=& 1\ 4\ 6\ 4\ 1 \\ 32 &=& 1\ 5\ 10\ 10\ 5\ 1 \\ 64 &=& 1\ 6\ 15\ 20\ 15\ 6\ 1 \\ 128 &=& 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1 \end{array}$$

Row sums are powers of 2

"Proof": $128 = 2^7$, number of subsets of $\{1, 2, \ldots, 7\}$
row sums over choices of subset size

What are the diagonal sums (up to a given row)

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

What are the diagonal sums (up to a given row)

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

What are the diagonal sums (up to a given row)

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Diagonal sums are binomial coefficients:

$1 + 3 + 6 + 10 + 15 = 35$

Diagonal sums are binomial coefficients:

"Proof":

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Diagonal sums are binomial coefficients:

"Proof":

1									1							
1	1								1	1						
1	2	1							1	2	1					
1	3	3	1					=	1	3	3	1				
1	4	6	4	1					1	4	6	4	1			
1	5	10	10	5	1				1	5	10	10	5	1		
1	6	15	20	15	6	1			1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1		1	7	21	35	35	21	7	1

Diagonal sums are binomial coefficients:

"Proof":

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

=

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

=

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Diagonal sums are binomial coefficients:

"Proof":

1							
1	1						
1	2	1					
1	**3**	3	1				
1	4	**6**	4	1			
1	5	10	**10**	5	1		
1	6	15	20	**15**	6	1	
1	7	21	35	35	21	7	1

=

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	**20**	**15**	6	1	
1	7	21	35	35	21	7	1

=

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	**35**	21	7	1

By Mathematical Induction

What are the antidiagonal sums?

$$
\begin{array}{rl}
 & 1 \\
 & 1\ 1 \\
 & 1\ \color{magenta}{2}\ 1 \\
\color{magenta}{3 =} & \color{magenta}{1}\ 3\ 3\ 1 \\
 & 1\ 4\ 6\ 4\ 1 \\
 & 1\ 5\ 10\ 10\ 5\ 1 \\
 & 1\ 6\ 15\ 20\ 15\ 6\ 1 \\
 & 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1
\end{array}
$$

What are the antidiagonal sums?

	1							
	1	1						
	1	2	1					
3 =	1	3	3	1				
5 =	1	4	6	4	1			
	1	5	10	10	5	1		
	1	6	15	20	15	6	1	
	1	7	21	35	35	21	7	1

What are the antidiagonal sums?

		1							
		1	1						
		1	2	1					
3	=	1	3	3	1				
5	=	1	4	6	4	1			
8	=	1	5	10	10	5	1		
		1	6	15	20	15	6	1	
		1	7	21	35	35	21	7	1

What are the antidiagonal sums?

		1							
		1	1						
		1	2	1					
3	=	1	3	3	1				
5	=	1	4	6	4	1			
8	=	1	5	10	10	5	1		
13	=	1	6	15	20	15	6	1	
		1	7	21	35	35	21	7	1

What are the antidiagonal sums?

		1							
		1	1						
		1	2	1					
3	=	1	3	3	1				
5	=	1	4	6	4	1			
8	=	1	5	10	10	5	1		
13	=	1	6	15	20	15	6	1	
21	=	1	7	21	35	35	21	7	1

What are the antidiagonal sums?

$$
\begin{array}{rcl}
1 &=& 1 \\
1 &=& 1\ 1 \\
2 &=& 1\ 2\ 1 \\
3 &=& 1\ 3\ 3\ 1 \\
5 &=& 1\ 4\ 6\ 4\ 1 \\
8 &=& 1\ 5\ 10\ 10\ 5\ 1 \\
13 &=& 1\ 6\ 15\ 20\ 15\ 6\ 1 \\
21 &=& 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1 \\
34 &=& 1\ 8\ 28\ 56\ 70\ 56\ 28\ 8\ 1
\end{array}
$$

What are the antidiagonal sums?

$$
\begin{array}{rcl}
1 &=& 1 \\
1 &=& 1\ 1 \\
2 &=& 1\ 2\ 1 \\
3 &=& 1\ 3\ 3\ 1 \\
5 &=& 1\ 4\ 6\ 4\ 1 \\
8 &=& 1\ 5\ 10\ 10\ 5\ 1 \\
13 &=& 1\ 6\ 15\ 20\ 15\ 6\ 1 \\
21 &=& 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1 \\
34 &=& 1\ 8\ 28\ 56\ 70\ 56\ 28\ 8\ 1
\end{array}
$$

Anti-diagonal sums are Fibonacci numbers

What are the antidiagonal sums?

1	=	1								
1	=	1	1							
2	=	1	2	1						
3	=	1	3	3	1					
5	=	1	4	6	4	1				
8	=	1	5	10	10	5	1			
13	=	1	6	15	20	15	6	1		
21	=	1	7	21	35	35	21	7	1	
34	=	1	8	28	56	70	56	28	8	1

Anti-diagonal sums are Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

What are the antidiagonal sums?

$$\begin{array}{rcl}
1 &=& 1 \\
1 &=& 1\ 1 \\
2 &=& 1\ 2\ 1 \\
3 &=& 1\ 3\ 3\ 1 \\
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\end{array}$$

Anti-diagonal sums are Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0, F_1 = 1$.

What are the antidiagonal sums?

$$
\begin{array}{rcl}
1 &=& 1 \\
1 &=& 1\ \ 1 \\
2 &=& 1\ \ 2\ \ 1 \\
3 &=& 1\ \ 3\ \ 3\ \ 1 \\
5 &=& 1\ \ 4\ \ 6\ \ 4\ \ 1 \\
8 &=& 1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 \\
13 &=& 1\ \ 6\ \ 15\ \ 20\ \ 15\ \ 6\ \ 1 \\
21 &=& 1\ \ 7\ \ 21\ \ 35\ \ 35\ \ 21\ \ 7\ \ 1 \\
34 &=& 1\ \ 8\ \ 28\ \ 56\ \ 70\ \ 56\ \ 28\ \ 8\ \ 1
\end{array}
$$

Anti-diagonal sums are Fibonacci numbers

"Proof": Use binomial identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
Each anti-diagonal is sum of previous two, satisfies same recurrence

Who first discovered Fibonacci numbers?

Pingali 200 BCE in Sanskrit prosody

Fibonacci numbers count the number of 1,2 strings with sum n (long and short beats)

- (1) sum 1: 1
- (2) sum 2: 2,11
- (3) sum 3: 12,21,111
- (5) sum 4: 22,112,121,211,1111
- (8) sum 5: 122,212,1112,221,1121,1211,2111,11111
- (13) sum 6: 222,1122,1212,2112,11112, 1221,2121,11121,2211,11211,12111,21111,111111

Recall the Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ and the golden ratio $(1+\sqrt{5})/2$. How are Fibonacci numbers related to powers of the golden ratio?

- $F_n = F_{n-1} + F_{n-2}$
- If $F_n = x^n$ Then $x^n = x^{n-1} + x^{n-2} \Rightarrow x^2 - x - 1 = 0$

Recall the Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ and the golden ratio $(1+\sqrt{5})/2$. How are Fibonacci numbers related to powers of the golden ratio?

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- If $F_n = x^n$ Then
  $x^n = x^{n-1} + x^{n-2} \Rightarrow x^2 - x - 1 = 0$
- Roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$
- $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$

Recall the Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ and the golden ratio $(1+\sqrt{5})/2$. How are Fibonacci numbers related to powers of the golden ratio?

- $F_n = F_{n-1} + F_{n-2}$
- If $F_n = x^n$ Then
  $x^n = x^{n-1} + x^{n-2} \Rightarrow x^2 - x - 1 = 0$
- Roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$
- $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n .$
- F_n is closest integer to $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$

What is $\lim_{n\to\infty}(1+\frac{1}{n})^n$ and why does your bank care care?

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e \approx 2.718\ldots$$

100% interest compounded annually yields $(1+1)^1$

100% interest compounded monthly yields $(1+\frac{1}{12})^{12}$

100% interest compounded daily yields $(1+\frac{1}{365})^{365}$

100% interest compounded continuously yields
$\lim_{n\to\infty}(1+\frac{1}{n})^n = e \approx 2.718\ldots$