Score Sequences for Tournaments

Garth Isaak
Lehigh University
Score Sequences of Round Robin Tournaments

U wins 3 games, V wins 2 games, W wins 2 games, X wins 2 games, Y wins 1 games

Score sequence is (3,2,2,2,1)
Score Sequences of Round Robin Tournaments
U wins 3 games, V wins 2 games, W wins 2 games, X wins 2 games, Y wins 1 game

Score sequence is (3,2,2,2,1)
Is the following sequence of 25 numbers a score sequence?

22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3
Is the following sequence of 25 numbers a score sequence?

22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

Try testing ALL possible tournaments?
Is the following sequence of 25 numbers a score sequence?

22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

Try testing *ALL* possible tournaments?

**UNIVERSE-ALL computer:**
Is the following sequence of 25 numbers a score sequence?
22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

Try testing ALL possible tournaments?

UNIVERSE-ALL computer:

All of the atoms in the known universe checking a billion tournaments per second
Is the following sequence of 25 numbers a score sequence?

22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

Try testing ALL possible tournaments?

UNIVERSE-ALL computer:

All of the atoms in the known universe checking a billion tournaments per second

Still not done checking all possibilities for this instance
Is the following sequence of 25 numbers a score sequence?

22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

Try testing ALL possible tournaments?

UNIVERSE-ALL computer:

All of the atoms in the known universe checking a billion tournaments per second

Still not done checking all possibilities for this instance

Use mathematical tools to make the check faster
For 7 players there are \( \frac{7(7 - 1)}{2} = 21 \) games in a round robin tournament.
For 7 players there are \[
\frac{7(7 - 1)}{2} = 21
\] games in a round robin tournament.

Which of the following are score sequences for a tournament with 7 players?

\[(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)\]

\[(5, 4, 3, 3, 3, 1, 0)\]

\[(3, 3, 3, 3, 3, 3)\]

\[(6, 6, 4, 2, 1, 1, 1)\]

NO - Scores must be non-negative integers.

NO - Total number of wins must be 21 = \[7 \times 6\frac{1}{2}\] wins.

YES
For 7 players there are \( \frac{7(7 - 1)}{2} = 21 \) games in a round robin tournament.

Which of the following are score sequences for a tournament with 7 players?

\((7, 5, 4 \frac{1}{3}, 4, 2 \frac{3}{7}, 0, -2)\) NO - Scores must be non-negative integers

\((5, 4, 3, 3, 3, 1, 0)\)

\((3, 3, 3, 3, 3, 3)\)

\((6, 6, 4, 2, 1, 1, 1)\)
For 7 players there are \( \frac{7(7 - 1)}{2} = 21 \) games in a round robin tournament.

Which of the following are score sequences for a tournament with 7 players?

(7, 5, 4\(\frac{1}{3}\), 4, 2\(\frac{3}{7}\), 0, −2)  NO - Scores must be non-negative integers

(5, 4, 3, 3, 3, 1, 0)  NO - Total number of wins must be 21 = \( \frac{7 \times 6}{2} \)

(3, 3, 3, 3, 3, 3, 3)

(6, 6, 4, 2, 1, 1, 1)
For 7 players there are \( \frac{7(7 - 1)}{2} = 21 \) games in a round robin tournament.

Which of the following are score sequences for a tournament with 7 players?

\((7, 5, 4 \frac{1}{3}, 4, 2 \frac{3}{7}, 0, -2)\) \ NO - Scores must be non-negative integers

\((5, 4, 3, 3, 3, 1, 0)\) \ NO - Total number of wins must be 21 = \( \frac{7 \cdot 6}{2} \)

\((3, 3, 3, 3, 3, 3, 3)\)

\((6, 6, 4, 2, 1, 1, 1)\) \ NO - Last 5 teams must have at least 10 = \( \frac{5 \cdot 4}{2} \) wins
For 7 players there are \( \frac{7(7 - 1)}{2} = 21 \) games in a round robin tournament.

Which of the following are score sequences for a tournament with 7 players?

\( (7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2) \) NO - Scores must be non-negative integers

\( (5, 4, 3, 3, 3, 1, 0) \) NO - Total number of wins must be 21 = \( \frac{7 \times 6}{2} \)

\( (3, 3, 3, 3, 3, 3, 3) \) YES

\( (6, 6, 4, 2, 1, 1, 1) \) NO - Last 5 teams must have at least 10 = \( \frac{5 \times 4}{2} \) wins
Landau (1951) considered tournaments in the context of pecking order in poultry populations

A necessary condition for a sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers to be the score sequence of a round-robin tournament:

$$\sum_{i \in I} s_i \geq \frac{|I|(|I| - 1)}{2} \quad \text{for any } I \subseteq \{1, 2, \ldots, n\}$$

with equality when $I = \{1, 2, \ldots, n\}$.
Landau (1951) considered tournaments in the context of pecking order in poultry populations

A necessary condition for a sequence \((s_1, s_2, \ldots, s_n)\) of non-negative integers to be the score sequence of a round-robin tournament:

\[
\sum_{i \in I} s_i \geq \frac{|I|(|I| - 1)}{2} \quad \text{for any } I \subseteq \{1, 2, \ldots, n\}
\]

with equality when \(I = \{1, 2, \ldots, n\}\)

The number of wins for any set of teams must be as large as the number of games played between those teams and the total number of wins must equal the total number of games played
A necessary condition for a sequence \((s_1, s_2, \ldots, s_n)\) of non-negative integers to be the score sequence of a round-robin tournament:

\[
\sum_{i \in I} s_i \geq \frac{|I|(|I| - 1)}{2} \quad \text{for any } I \subseteq \{1, 2, \ldots, n\}
\]

with equality when \(I = \{1, 2, \ldots, n\}\)
A necessary condition for a sequence \((s_1, s_2, \ldots, s_n)\) of non-negative integers to be the score sequence of a round-robin tournament:

\[
\sum_{i \in I} s_i \geq \frac{|I|(|I| - 1)}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\}
\]

with equality when \(I = \{1, 2, \ldots, n\}\)

**Landau’s Theorem:** these necessary conditions are also sufficient

If the conditions hold there is a tournament with the given score sequence
A necessary condition for a sequence \((s_1, s_2, \ldots, s_n)\) of non-negative integers to be the score sequence of a round-robin tournament:

\[
\sum_{i \in I} s_i \geq \frac{|I|(|I| - 1)}{2} \quad \text{for any } I \subseteq \{1, 2, \ldots, n\}
\]

with equality when \(I = \{1, 2, \ldots, n\}\)

**Landau’s Theorem:** these necessary conditions are also sufficient

If the conditions hold there is a tournament with the given score sequence

If not a score sequence then there is a set of teams violating these obvious conditions
The sequence
22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3
can be checked by hand in a few minutes. It is not a score sequence.
The sequence
22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3
can be checked by hand in a few minutes. It is not a score sequence

22, 22, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

Not a score sequence

Last 10 teams have 44 wins in 45 = \( \frac{10.9}{2} \) games
Landau’s Theorem:
A sequence \((s_1, s_2, \ldots, s_n)\) of non-negative integers is a score sequence of a round-robin tournament if and only if

\[
\sum_{i \in I} s_i \geq \binom{|I|}{2}
\]
for any \(I \subseteq \{1, 2, \ldots, n\}\)

with equality when \(I = \{1, 2, \ldots, n\}\)
Landau’s Theorem:
A sequence \((s_1, s_2, \ldots, s_n)\) of non-negative integers is a score sequence of a round-robin tournament if and only if

\[
\sum_{i \in I} s_i \geq \binom{|I|}{2}
\]

for any \(I \subseteq \{1, 2, \ldots, n\}\)

with equality when \(I = \{1, 2, \ldots, n\}\)

What if we allow ties?

What if the score is 3 points for a win, 1 for a tie and 0 for a loss (world cup soccer scoring)?
Landau’s Theorem:
A sequence \((s_1, s_2, \ldots, s_n)\) of non-negative integers is a score sequence of a round-robin tournament if and only if

\[
\sum_{i \in I} s_i \geq \binom{|I|}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\}
\]

with equality when \(I = \{1, 2, \ldots, n\}\)

What if we allow ties?
This problem is not solved

What if the score is 3 points for a win, 1 for a tie and 0 for a loss (world cup soccer scoring)?
This problem is not solved
Landau’s Theorem:
A sequence \((s_1, s_2, \ldots, s_n)\) of non-negative integers is a score sequence of a round-robin tournament if and only if

\[
\sum_{i \in I} s_i \geq \binom{|I|}{2}
\]

for any \(I \subseteq \{1, 2, \ldots, n\}\)

with equality when \(I = \{1, 2, \ldots, n\}\)

What is \(\binom{|I|}{2}\)?
\[ \binom{13}{2} = \frac{13 \cdot 12}{2} = 8 \text{ choose } 2 \]

Number of 2 element subsets of \{1, 2, \ldots, 13\}

Binomial coefficients \( \binom{n}{k} = n \text{ choose } k \)

number of \( k \) elements subsets of \{1, 2, \ldots, n\}

\[ \binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2} \]

\[ \binom{13}{5} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2} \]
Binomial coefficients - ‘Pascal’s Triangle’

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Hayluda Bhattotpala (India around 1000)
Al-Karaji and Kayyam (Persia around 1050)
Yang Hui (China around 1350)
Tartaglia (Italy around 1550)
Pascal (France around 1650)
Binomial coefficients - ‘Pascal’s Triangle’

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Hayluda Bhattotpala (India around 1000)
Al-Karaji and Kayyam (Persia around 1050)
Yang Hui (China around 1350)
Tartaglia (Italy around 1550)
Pascal (France around 1650)
Binomial identity: \( \binom{7}{3} = \binom{6}{2} + \binom{6}{3} \)

1 1 1 1 2 1 1 3 3 1 1 4 6 4 1 1 5 10 10 5 1 1 6 15 20 15 6 1 1 7 21 35 35 21 7 1

"Proof":

The \( \binom{7}{3} = 35 \) size 3 subsets of \( \{A, B, C, D, E, F, G\} \)

= The \( \binom{6}{2} = 15 \) subsets including A + The \( \binom{6}{3} = 20 \) subsets avoiding A
\begin{array}{ccccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
8 &=& 1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
\end{array}
1 = 1
2 = 1 1
4 = 1 2 1
8 = 1 3 3 1
16 = 1 4 6 4 1
32 = 1 5 10 10 5 1
64 = 1 6 15 20 15 6 1
128 = 1 7 21 35 35 21 7 1

Row sums are powers of 2

"Proof": 128 = 2^7, number of subsets of \{1, 2, \ldots, 7\}
row sums over choices of subset size
Row sums are powers of 2
Row sums are powers of 2

"Proof": $128 = 2^7$, number of subsets of $\{1, 2, \ldots, 7\}$
row sums over choices of subset size
Diagonal sums are binomial coefficients:

1 + 3 + 6 + 10 + 15 = 35
Diagonal sums are binomial coefficients:

\[ 1 + 3 + 6 + 10 + 15 = 35 \]
Diagonal sums are binomial coefficients:

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
\end{array}
\]
Diagonal sums are binomial coefficients:

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
\]

"Proof":

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
\]

=}
Diagonal sums are binomial coefficients:

Proof:

\[
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
= \\
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
= \\
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
\]
\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
3 &= 1 & 3 & 3 & 1 \\
& & 4 & 6 & 4 & 1 \\
& 1 & 5 & 10 & 10 & 5 & 1 \\
& 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
& & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}
\]
<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>3</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>20</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>7</td>
<td>21</td>
<td>35</td>
<td>35</td>
<td>21</td>
</tr>
</tbody>
</table>
1
1 1
1 2 1
3  = 1 3 3 1
5  = 1 4 6 4 1
8  = 1 5 10 10 5 1
13 = 1 6 15 20 15 6 1
   1 7 21 35 35 21 7 1
1 = 1
1 = 1 1
2 = 1 2 1
3 = 1 3 3 1
5 = 1 4 6 4 1
8 = 1 5 10 10 5 1
13 = 1 6 15 20 15 6 1
21 = 1 7 21 35 35 21 7 1
34 = 1 8 28 56 70 56 28 8 1
Anti-diagonal sums are Fibonacci numbers

\[ F_n = \sqrt{5} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \sqrt{5} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

\[ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \text{ with } F_0 = 0, F_1 = 1. \]

"Proof": Use binomial identity \((n k) = (n-1 k-1) + (n-1 k)\).
Anti-diagonal sums are Fibonacci numbers

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n. \]
Anti-diagonal sums are Fibonacci numbers

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n. \]

\[ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \text{ with } F_0 = 0, F_1 = 1. \]
Anti-diagonal sums are Fibonacci numbers

\[
F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

\[F_n = F_{n-1} + F_{n-2}\] for \(n \geq 2\) with \(F_0 = 0, F_1 = 1\).

"Proof": Use binomial identity \(\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}\)
Landau’s Theorem via systems of linear inequalities

Possible score sequence \((s_1, s_2, \ldots, s_n)\)

For each integral pair \(1 \leq i < j \leq n\) define a variable \(x_{i,j}\) with
\[ x_{i,j} = 1 \text{ if } i \text{ beats } j \text{ and } x_{i,j} = 0 \text{ if } i \text{ losses to } j. \]

There is a tournament with the given score sequence if and only if the following has a solution:
\[
\sum_{i < j} (1 - x_{i,j}) + \sum_{j < k} x_{j,k} = s_j \text{ for } j = 1, 2, \ldots, n
\]

Relax to \(0 \leq x_{i,j} \leq 1\)

For scores \((6, 6, 4, 2, 1, 1, 1)\) equation for \(s_3 = 4\) is
\[
(1 - x_{1,3}) + (1 - x_{2,3}) + x_{3,4} + x_{3,5} + x_{3,6} + x_{3,7} = 4
\]
Landau’s Theorem via systems of linear inequalities

- Possible score sequence \((s_1, s_2, \ldots, s_n)\)
- For each integral pair \(1 \leq i < j \leq n\) define a variable \(x_{i,j}\) with \(x_{i,j} = 1\) if \(i\) beats \(j\) and \(x_{i,j} = 0\) if \(i\) losses to \(j\)
- There is a tournament with the given score sequence if and only if the following has a solution:

\[
\sum_{i<j}(1 - x_{i,j}) + \sum_{j<k} x_{j,k} = s_j \text{ for } j = 1, 2, \ldots, n
\]

\[
x_{i,j} \in \{0, 1\} \text{ for all } i < j
\]

Relax to \(0 \leq x_{i,j} \leq 1\)
Landau’s Theorem via systems of linear inequalities

- Possible score sequence \((s_1, s_2, \ldots, s_n)\)
- For each integral pair \(1 \leq i < j \leq n\) define a variable \(x_{i,j}\) with \(x_{i,j} = 1\) if \(i\) beats \(j\) and \(x_{i,j} = 0\) if \(i\) losses to \(j\)
- There is a tournament with the given score sequence if and only if the following has a solution:

\[
\sum_{i<j}(1-x_{i,j}) + \sum_{j<k}x_{j,k} = s_j \text{ for } j = 1, 2, \ldots, n
\]
\[
x_{i,j} \in \{0, 1\} \text{ for all } i < j
\]

Relax to \(0 \leq x_{i,j} \leq 1\)

For scores \((6, 6, 4, 2, 1, 1, 1)\) equation for \(s_3 = 4\) is
\[
(1 - x_{1,3}) + (1 - x_{2,3}) + x_{3,4} + x_{3,5} + x_{3,6} + x_{3,7} = 4
\]
Circulation in a network: flow between lower and upper bounds satisfying flow conservation at each node
Circulation in a network: flow between lower and upper bounds satisfying flow conservation at each node

\[
\begin{array}{ccc}
[1,3,3] & [2,3,6] & [0,0,2] \\
[1,1,2] & [1,1,3] & [1,2,6] \\
[0,1,5] & [3,4,5] & [5,6,8] \\
\end{array}
\]
Circulation in a network: flow between lower and upper bounds satisfying flow conservation at each node
Requirements in $= 3 + 4 = 7 > 6 = 4 + 2 = \text{capacity out}$
Requirements in \( = 3 + 4 = 7 > 6 = 4 + 2 = \) capacity out

No circulation
Hoffman (1956)

A necessary condition for a circulation:

for every set of nodes:

capacities out $\geq$ the requirements in

$(\text{sum of upper bounds}) \geq (\text{sum of lower bounds in})$
Hoffman (1956)

A necessary condition for a circulation:
for for every set of nodes:
capacities out $\geq$ the requirements in
(sum of upper bounds) $\geq$ (sum of lower bounds in)

Hoffman’s Circulation Theorem (1956): These necessary conditions are also sufficient

If the conditions hold there is a circulation
Hoffman (1956)

A necessary condition for a circulation:
for every set of nodes:
capacities out $\geq$ the requirements in
(sum of upper bounds) $\geq$ (sum of lower bounds in)

Hoffman’s Circulation Theorem (1956): These necessary conditions are also sufficient
If the conditions hold there is a circulation
If there is no circulation there is some set with capacities out $<$ requirements in
Hoffman’s Circulation Theorem via systems of linear inequalities

- Network with upper bounds $u_{i,j}$ and lower bounds $l_{i,j}$ for arcs $i,j$
- For each arc $i,j$ define a variable $x_{i,j}$ which will correspond to the amount of flow.
- There is a circulation if and only if the following has a solution:

$$\sum_{i,j \in A} x_{i,j} = \sum_{j,k \in A} x_{j,k} \text{ for each node } j$$

$$l_{i,j} \leq x_{i,j} \leq u_{i,j} \text{ for each arc } i,j$$

Equations force flow conservation
inequalities enforce lower and upper bounds
Hoffman’s Circulation inequalities

\[
\sum_{i,j \in A} -x_{i,j} + \sum_{j,k \in A} x_{j,k} = 0 \text{ for each node } j
\]

\[l_{i,j} \leq x_{i,j} \leq u_{i,j} \text{ for each arc } i,j\]

Landau’s score sequence inequalities

\[-(s_j + j - 1) + \sum_{i<j} -x_{i,j} + \sum_{j<k} x_{j,k} = 0 \text{ for } j = 1, 2, \ldots, n\]

\[0 \leq x_{i,j} \leq 1 \text{ for all } i < j\]
Hoffman’s Circulation inequalities

\[ \sum_{i,j \in A} -x_{i,j} + \sum_{j,k \in A} x_{j,k} = 0 \] for each node \( j \)

\( l_{i,j} \leq x_{i,j} \leq u_{i,j} \) for each arc \( i, j \)

Landau’s score sequence inequalities

\[ -(s_j + j - 1) + \sum_{i<j} -x_{i,j} + \sum_{j<k} x_{j,k} = 0 \] for \( j = 1, 2, \ldots, n \)

\( 0 \leq x_{i,j} \leq 1 \) for all \( i < j \)

Almost the same: Create new vertex with flows to \( j \) forced to be \( s_j - j + 1 \)

\( \Rightarrow \)

Landau’s Theorem as a special case of Hoffman’s Circulation Theorem
Yellow arcs left to right, lower bound 0, upper bound 1
Yellow arcs left to right, lower bound 0, upper bound 1
Requirements in $= 11 > 10 = $ capacities out
Yellow arcs left to right, lower bound 0, upper bound 1
Requirements in  = 11 > 10 = capacities out

No circulation
Yellow arcs left to right, lower bound 0, upper bound 1
Requirements in = 11 > 10 = capacities out

No circulation
Corresponds to (6, 6, 4, 2, 1, 1, 1)
Not a score sequence
Do these have nonnegative solutions?

\[
\begin{align*}
  x + 2y &= 3 \\
  4x + 5y &= 6 \\
  4x + 8y &= 12 \\
  4x + 8y &= 6 \\
\end{align*}
\]
Do these have nonnegative solutions?

\[
\begin{align*}
 x + 2y &= 3 \\
 4x + 5y &= 6
\end{align*}
\]

\[
\begin{align*}
 x + 2y &= 3 \\
 4x + 8y &= 12 \\
 4x + 8y &= 6
\end{align*}
\]

\[
\begin{align*}
 x &= -1, \ y &= 2 \\
 \text{line } x + 2y &= 3
\end{align*}
\]

Has no solution

Why not?

no

yes
Do these have *nonnegative* solutions?

\[ \begin{align*}
  x + 2y &= 3 \\
  4x + 5y &= 6 \\
  x + 2y &= 3 \\
  4x + 8y &= 12 \\
  x + 2y &= 3 \\
  4x + 8y &= 6
\end{align*} \]

\[ \begin{align*}
  x &= -1, \ y = 2 & \text{line } x + 2y &= 3 & \text{Has no solution} \\
  \text{no} & & \text{yes} & \text{Why not?}
\end{align*} \]

Intersection of two lines
May be a point, a line or parallel lines
Do these have nonnegative solutions?

\[
\begin{align*}
  x + y + 2z &= 3 \\
  5x + 8y + 13z &= 21 \\
  x - y + z &= 0 \\
  5x + 8y + 13z &= 21 \\
  x - 3y - 3z &= 1
\end{align*}
\]
Do these have nonnegative solutions?

\[ x + y + 2z = 3 \quad x + y + 2z = 13 \]
\[ 5x + 8y + 13z = 21 \quad 5x + 8y + 13z = 21 \]
\[ x - y + z = 0 \quad x - 3y - 3z = 1 \]

\[
\begin{align*}
x &= 0, \quad y = z = 1 \\
yes \\
no \quad Why \ not?
\end{align*}
\]
This system has no nonnegative solution

\[ x + y + 2z = 13 \]
\[ 5x + 8y + 13z = 21 \]
\[ x - 3y - 3z = 1 \]
This system has no nonnegative solution

\[
\begin{align*}
-2 & \quad x + y + 2z = 13 \\
1 & \quad 5x + 8y + 13z = 21 \\
2 & \quad x - 3y - 3z = 1
\end{align*}
\]

Multiply equations by (-2), 1, 2 respectively

Every solution has at least one of \( x, y, z \) negative

Farkas’ Lemma: Either a linear system has a nonnegative solution or there are multipliers showing an inconsistency.
This system has no nonnegative solution

\[-2 \quad x + \quad y + \quad 2z = 13\]
\[1 \quad 5x + \quad 8y + \quad 13z = 21\]
\[2 \quad x - \quad 3y - \quad 3z = 1\]

Multiply equations by (-2), 1, 2 respectively
Add resulting equations

\[-2x - 2y - 4z = -26\]
\[5x + 8y + 13z = 21\]
\[2x - 6y - 6z = 2\]

Every solution has at least one of \(x\), \(y\), \(z\) negative

Farkas' Lemma: Either a linear system has a nonnegative solution or there are multipliers showing an inconsistency
This system has no nonnegative solution

\[-2 \quad x + \quad y + \quad 2z \quad = \quad 13\]
\[1 \quad 5x + 8y + 13z \quad = \quad 21\]
\[2 \quad x - 3y - 3z \quad = \quad 1\]

Multiply equations by (-2), 1, 2 respectively
Add resulting equations

\[-2x - 2y - 4z = -26\]
\[5x + 8y + 13z = 21\]
\[2x - 6y - 6z = 2\]

Result is

\[5x + 0y + 3z = -3\]

Every solution has at least one of \(x, y, z\) negative
This system has no nonnegative solution

\[-2 \quad x + \quad y + \quad 2z = 13\]
\[1 \quad 5x + 8y + 13z = 21\]
\[2 \quad x - 3y - \quad 3z = 1\]

Multiply equations by (-2), 1, 2 respectively
Add resulting equations

\[-2x - 2y - 4z = -26\]
\[5x + 8y + 13z = 21\]
\[2x - 6y - 6z = 2\]

Result is

\[5x + 0y + 3z = -3\]

Every solution has at least one of \(x, y, z\) negative

Farkas’ Lemma: Either a linear system has a nonnegative solution or there are multipliers showing an inconsistency
Farkas’ Lemma

Either a linear system has a nonnegative solution

OR

There are multipliers showing inconsistency

\[
\begin{align*}
-2 & \quad x + y + 2z = 13 \\
1 & \quad 5x + 8y + 13z = 21 \\
\Rightarrow & \quad 3x + 6y + 9z \leq -5
\end{align*}
\]
Rewrite

\[-2 \begin{array}{c} x \\ 5 \end{array} + \begin{array}{c} y \\ 8 \end{array} + 2z = 13 \quad \Rightarrow \quad 3x + 6y + 9z \leq -5\]

as

\[
\begin{pmatrix} \frac{1}{5} \\ \frac{1}{8} \end{pmatrix} x + \begin{pmatrix} \frac{1}{8} \\ \frac{2}{13} \end{pmatrix} y + \begin{pmatrix} \frac{2}{13} \end{pmatrix} z = \begin{pmatrix} \frac{13}{21} \end{pmatrix}
\]
Farkas’ Lemma

Either \( \begin{pmatrix} 13 \\ 21 \end{pmatrix} \) is in the cone generated by 
\[ \left\{ \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 13 \end{pmatrix} \right\} \]

OR

There is a separating hyperplane

Multipliers showing inconsistency provide the normal to the hyperplane forming an angle less than 90 degree with the ‘columns’ and greater than 90 degrees with the right hand side

\[-\frac{2}{1} x + y + \frac{2}{13} z = \frac{13}{21} \Rightarrow 3x + 6y + 9z \leq -5\]
Set up systems for circulations and score sequences. If no solution, the ‘multipliers’ are 0, 1 and produce violations of necessary conditions.
Set up systems for circulations and score sequences. If no solution, the ‘multipliers’ are 0, 1 and produce violations of necessary conditions.

Farkas’ Lemma for nonnegative solutions to linear systems of equations

↓

Hoffman’s Circulation Theorem

↓

Landau’s Theorem for Score Sequences