Degree Sequences for Edge Colored Graphs

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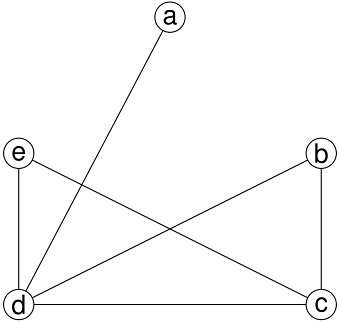
Simple Graph: Vertex set V and edge set E which is a collection of two element subsets of V.

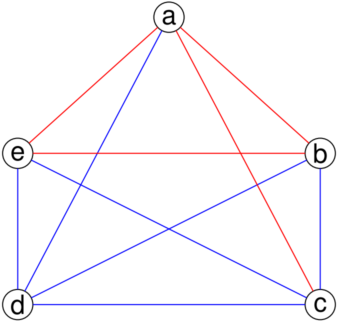
Degree Sequence
$4, 3, 2, 2, 1$



Simple graph: Vertex set V and a partition of the two element subsets of V into edges and non-edges

Simple graph: A two coloring of the edges of a complete graph on V

Degree Sequence
4,3,2,2,1
0,1,2,2,3



Is 13, 13, 12, 10, 10, 9, 8, 7, 6, 5, 3, 3, 2, 2, 1 a degree sequence?

Sierksma and Hoogeveen (1991) review 7 criteria for a graphic sequences:

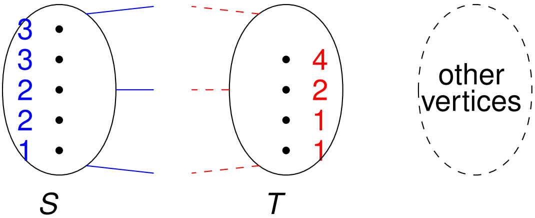
non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ are graphic if and only if:

- (Erdos-Gallai) $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_i\}$ for each $k = 1, 2, \ldots, n$
- (Berge) $\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \overline{d}_i$ where $\overline{d}_i$ for each $k = 1, 2, \ldots, n$ is the corrected conjugate sequence.
- (Fulkerson-Hoffman-McAndrew) $\sum_{i=1}^{k} d_i \leq k(n-m-1) + \sum_{i=n-m+1}^{n} d_i$ for each $k = 1, 2, \ldots, n$ and $m = 0, 1, \ldots, n-k$

Is $13, 13, 12, 10, 10, 9, 8, 7, 6, 5, 3, 3, 2, 2, 1$
a degree sequence?

complementary degrees $n - 1 - d_i$

Is $1, 1, 2, 4, 4, 5, 6, 7, 8, 9, 11, 11, 12, 12, 13$
a degree sequence?

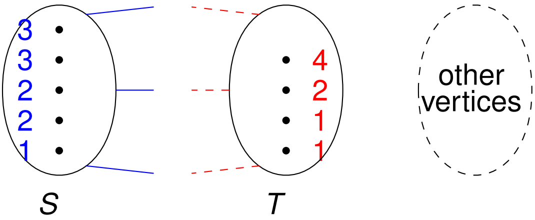


\# edges out of S
$\leq \sum_{i \in S} d_i$

\# non-edges out of T
$\leq \sum_{i \in T}(n - 1 - d_i)$

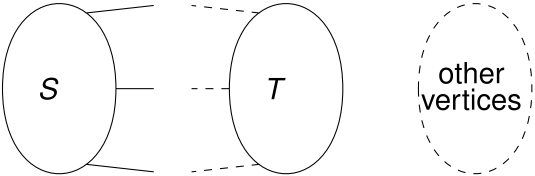
13, 13, 12, 10, 10, 9, 8, 7, 6, 5, 3, 3, 2, 2, 1
1, 1, 2, 4, 4, 5, 6, 7, 8, 9, 11, 11, 12, 12, 13

Is not a degree sequence: At most 11 edges (blue) out of S
and at most 8 non-edges (red) out of T
Not enough to account for coloring $4 \cdot 5 = 20$ edges (edge/blue or non-edge/red) between S and T



edges out of S $\leq \sum_{i \in S} d_i$

non-edges out of T $\leq \sum_{i \in T}(n - 1 - d_i)$



\# edges out of S $\leq \sum_{i \in S} d_i$

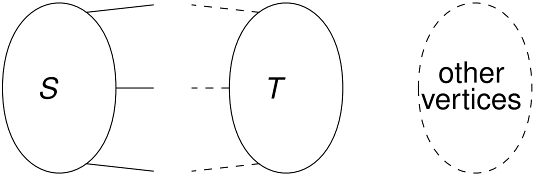
\# non-edges out of T $\leq \sum_{i \in T}(n-1-d_i)$

Necessary Condition
for $(d_1, d_2, \ldots, d_n)$ to be a Degree Sequence
For disjoint S, T (and $\sum d_i$ even)

$$\sum_{i \in S} d_i + \sum_{j \in T}(n-1-d_j) \geq |S|\,|T|$$

This is also sufficient

Edge Count Criterion (ECC) for degree sequences



$(d_1, d_2, \ldots, d_n)$ is NOT a degree sequence if
degrees in S plus non-degrees in T
too small to account for edges between S and T

Edge Count Criterion (ECC)
$(d_1, d_2, \ldots, d_n)$ with $\sum d_i$ is a Degree Sequence
if and only if

$$\sum_{i \in S} d_i + \sum_{j \in T} (n - 1 - d_j) \geq |S|\,|T|$$

for all disjoint S, T

- ECC is given by Koren (1973) and is close to Fulkerson-Hoffman-McAndrew condition
- Koren's version is used by Peled and Srinivasan (1989) to characterize the polytope of degree sequences

A proof (sketch by example) that ECC is sufficient for degree sequences :

Use the following elementary lemma(a special case of lemma of Fulkerson-Hoffman-McAndrew): If a sequence is graphic there is a realization in which a vertex with maximum degree is adjacent to an no-isolated vertex. (simple pigeonhole proof)
Use induction on the blue sum:
If the sequence satisfies ECC

$$\begin{array}{l} 13, 13, 12, 10, 10, 9, 8, 7, 6, 5, 3, 3, 2, 2, 1 \\ 1, 1, 2, 4, 4, 5, 6, 7, 8, 9, 11, 11, 12, 12, 13 \end{array}$$

look at

$$\begin{array}{l} 12, 13, 12, 10, 10, 9, 8, 7, 6, 5, 3, 3, 2, 2, 0 \\ 2, 1, 2, 4, 4, 5, 6, 7, 8, 9, 11, 11, 12, 12, 14 \end{array}$$

Show that the new sequence satisfies ECC then apply lemma to get a red edge between vertex of red degree 14 and red degree 2 and switch this to blue
[redacted]ofs this still takes some work for

A proof (sketch) that ECC is sufficient for degree sequences :

Use the following elementary lemma(a special case of lemma of Fulkerson-Hoffman-McAndrew): If a sequence is graphic there is a realization in which a vertex with maximum degree is adjacent to an no-isolated vertex. (simple pigeonhole proof)
Use induction on the blue sum:
If the sequence satisfies ECC

$$b_1, b_2, \ldots, b_n \quad r_1, r_2, \ldots, r_n$$

look at

$$b_1 - 1, b_2, \ldots, b_n - 1 \quad r_1 + 1, r_2, \ldots, r_n + 1$$

Show that the new sequence satisfies ECC then apply lemma to get a red edge between vertex of red degree 14 and red degree 2 and switch this to blue
Like most degree sequence proofs this still takes some work for cases but its still quite short.

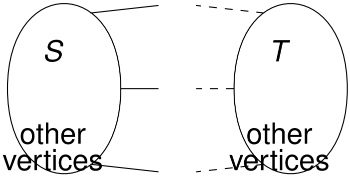
Bipartite graphs:

Non-negative integers $r_1 \geq r_2 \geq \cdots \geq r_m$ and $s_1 \geq s_2 \geq \cdots \geq s_n$ are bipartite graphic if and only if

- (Gale-Ryser) $\sum_{i=1}^{m} \min\{r_i, k\} \geq \sum_{j=1}^{k} s_j$ for each $k = 1, 2, \ldots, n$.
- (Ford-Fulkerson) $\sum_{i \in I} r_i + \sum_{j \in J}(m - s_j) \geq |I||J|$

The Ford-Fulkerson criterion is the bipartite version of the edge count criterion.
Can give a proof that is 'the same' as ECC for graphs

Bipartite Edge Count Criterion (BECC) for degree sequences



$(r_1, r_2, \ldots, r_m)$ and $(s_1, s_2, \ldots, s_n)$ is NOT a bipartite degree sequence if

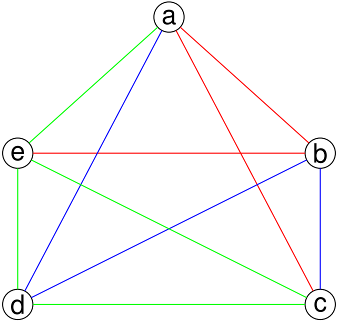
r_i degrees in S plus bipartite complement s_i degrees in T too small to account for edges between S and T

Other degree conditions also can be stated in the pattern of ECC. For example p-graphs; i.e., partitions into 2 colors of 'complete graph' with p multiple edges between each pair of vertices.

Edge Colored Graphs

Look at (non-proper) edge colorings of complete graphs

	blue	green	red
a	1	1	2
b	2	0	2
c	1	2	1
d	2	2	0
e	0	3	1



Which sequences of vectors can be realized as degrees of an edge colored complete graph?

Columns - color sequences

Rows - color degrees

Which sequences of vectors can be realized as degrees of an edge colored complete graph?

Graphs with (not necessarily proper) edge colorings arise in many contexts

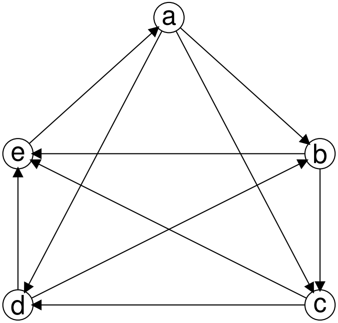
- Distinguishing colorings
- Ramsey theory
- Paths with alternating color
- Etc.

Which sequences of vectors can be realized as degrees of an edge colored complete graph?

- Not really a new question, Rao and Rao (1969) asked:
  - When is a degree sequence realizable by a graph with a k-factor?
- In edge colored graph terminology
  - What are degree sequences of three edge colored graphs if one color sequence is constant?
- Necessary condition: Each color sequence must be realizable as a graph
- Kundu (1973): necessary condition is also sufficient (for three edge colors and one color sequence constant)

Which sequences of vectors can be realized as degrees of an edge colored complete graph?

- About a dozen papers written on the 3 color version in 1970's and early 1980's
- Can be viewed as a version of graph factorization and of potentially graphic sequences but questions in those settings are usually slightly different
- If the input graph is arbitrary (not complete) then the problem is NP-hard (since: Is a 3 regular graph 3-edge colorable is NP-hard)



Score sequence: $(3, 2, 2, 2, 1)$

Tournament (complete directed graph)

Score $=$ outdegree (the number of wins)

Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Is the following sequence of 25 numbers a score sequence?

22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

Landau's Theorem: The number of wins for any set of teams must be as large as the number of games played between those teams

and

the total number of wins must equal the total number of games played

Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Landau's Theorem: The number of wins for any set of teams must be as large as the number of games played between those teams

and

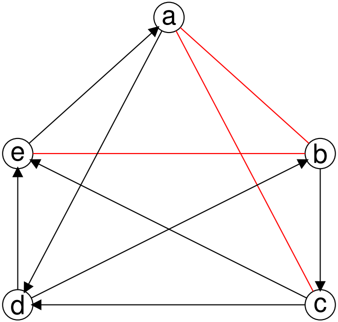
the total number of wins must equal the total number of games played

The sequence is not a score sequence: 9 teams with lowest scores have a total of 44 wins but play $\binom{9}{2} = 45$ games.

What if ties are allowed?

Can we tell if a given set of triples (win,loss,tie) arises from a round robin tournament?

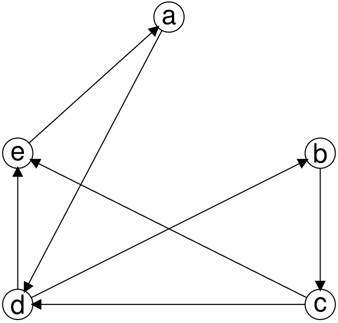
	win	loss	tie
a	1	1	2
b	1	1	2
c	2	1	1
d	2	2	0
e	1	2	1



What if ties are allowed?

Can we tell if a given set of triples (win,loss,tie) arises from a round robin tournament?

	win	loss
a	1	1
b	1	1
c	2	1
d	2	2
e	1	2



Equivalently - Is there a digraph with specified indegrees and outdegrees?

Is there a digraph with specified indegrees and outdegrees?

- Gale (1957) and Ryser (1957) give conditions (equivalent to degree sequences of bipartite graphs) - but this allows loops and 2-cycles
- Ore (1957) and Fulkerson (1960) give conditions with no loops - but still allows 2-cycles

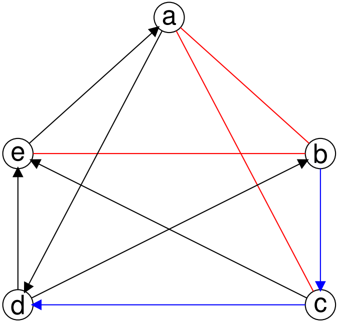
Surprisingly the (win,loss,tie) version appears to be unsolved

- Consider instead: indegree minus outdegree - conditions given by Avery (1991) for tournaments and Mubayi, West and Will (2001) for simple digraphs (this is equivalent to a total score of 2 points for a win, 1 for a tie and 0 for a loss), one solution is via network flows.

If the score total is 3 points for a win, 1 for a tie and 0 for a loss appears to be unsolved

What if we in addition specify shutout (or big) wins?

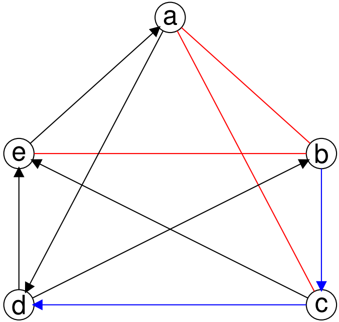
	big win	win	tie	loss	big loss
a	0	1	2	1	0
b	1	0	2	1	0
c	1	1	1	0	1
d	2	1	0	1	1
e	0	1	1	2	0



Can we tell if specified 5-tuples (big win, win, tie, loss, big loss) arise from a round robin tournament?

What if we in addition specify shutout (or big) wins?

	big win	win	tie	loss	big loss
a	0	1	2	1	0
b	1	0	2	1	0
c	1	1	1	0	1
d	2	1	0	1	1
e	0	1	1	2	0



Can we tell if specified 5-tuples (big win, win, tie, loss, big loss) arise from a round robin tournament?

If big win = +2, win = +1, tie = 0, loss = -1, big loss = -2 then can get conditions for total score via Hoffman's circulation theorem

Edge Colored Graphs

Look at (non-proper) edge colorings of complete graphs

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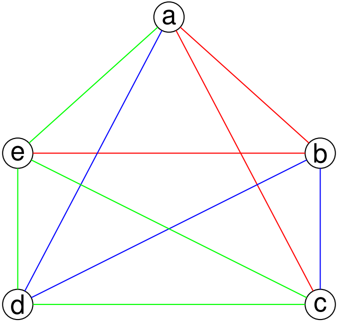

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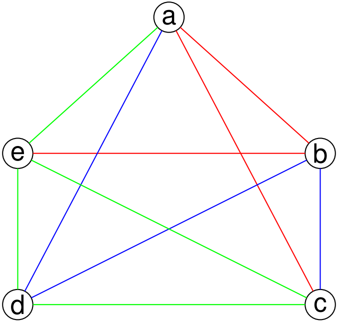


Necessary Condition - each color sequence is a degree sequence

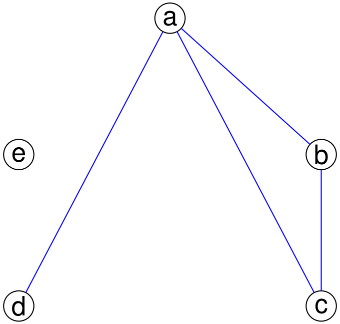
$\geq$ 4 colors: sums of color sequences must be degree sequences

Necessary condition is sufficient (3 colors) when:

- One color sequence has all degrees $\in \{k, k+1\}$ (Kundu's Theorem, 1973)
  - extends to two outlying degrees and ...
- Two color sequences and their sum can be realized by forests (Kleitman, Koren and Li, 1977)
  - also a broader condition on the sum; characterize when both colors can be forests
- Minimum degree for one color is $> (n-1) - \sqrt{2n}$ (Busch and Ferrara, 2007+)

Necessary condition is not sufficient

	blue	green	red
a	3	1	0
b	2	0	2
c	2	0	2
d	1	2	1
e	0	1	3



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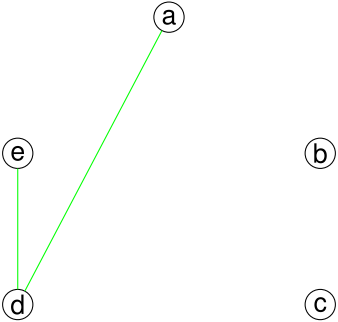

Necessary condition is not sufficient

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a	3	1	0
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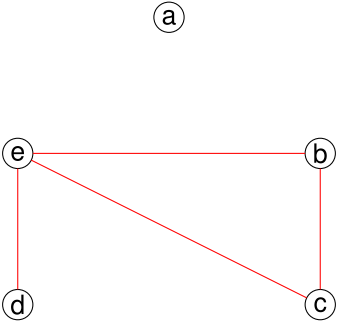
c	2	0	2
d	1	2	1
e	0	1	3



Each color is a degree sequence
Problem: de must be both green and red
(these realizations are unique)

Generalization of ECC works to show non realizability in this case.
Does it work always? No
Fractional version works for bipartite graphs

If necessary conditions are not sufficient?

- (3 colors) Koren characterized conditions when one color is a star (with possibly isolated vertices)
- (3 colors) If any color vector is uniquely realizable reduces to factor problem in that graph. Uniquely realizable sequences are threshold graphs
- (4 colors) If any sum of 2 colors is uniquely realizable reduces to factor problems in two graphs.

When is there an edge colored tree (forest) with given color degrees?

(Joint work with Jim Carroll)

The obvious necessary condition:

sums of any subset of colors are forest realizable

are also sufficient

example:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}1\\0\\1\end{pmatrix} \begin{pmatrix}2\\2\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$$

cannot ‘reduce’ to:

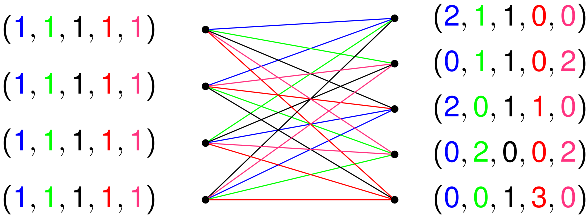
$$\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}2\\2\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$$

but there is always a good choice ...

Is there a bipartite graph with color vectors?

$(1, 1, 1, 1, 1)$ •	• $(2, 1, 1, 0, 0)$
$(1, 1, 1, 1, 1)$ •	• $(0, 1, 1, 0, 2)$
$(1, 1, 1, 1, 1)$ •	• $(2, 0, 1, 1, 0)$
$(1, 1, 1, 1, 1)$ •	• $(0, 2, 0, 0, 2)$
	• $(0, 0, 1, 3, 0)$

Is there a bipartite graph with color vectors?



YES

If all vertices one part have score $(1, 1, \ldots, 1)$ and the number of colors in other part is 'right'

This is just the Birkhoff-VonNeumann Theorem in disguise

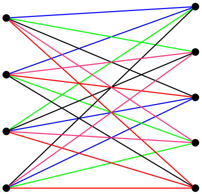
$(1, 1, 1, 1, 1)$ $\quad$ $(2, 1, 1, 0, 0)$

$(1, 1, 1, 1, 1)$ $\quad$ $(0, 1, 1, 0, 2)$

$(1, 1, 1, 1, 1)$ $\quad$ $(2, 0, 1, 1, 0)$

$(1, 1, 1, 1, 1)$ $\quad$ $(0, 2, 0, 0, 2)$

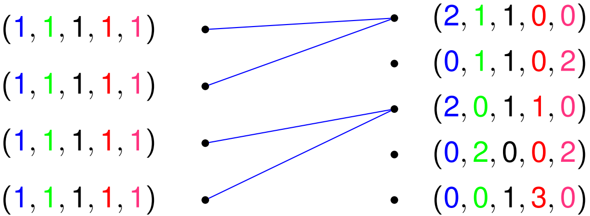
$\quad$ $(0, 0, 1, 3, 0)$



$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This always works when one part degrees are 1's vectors



$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

There are some interesting problems with degree sequences of edge colored graphs