Hamiltonicity of digraphs for universal cycles of permutations

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Abstract

The digraphs $P(n,k)$ have vertices corresponding to length k permutations of an n set and arcs corresponding to $(k+1)$ permutations. Answering a question of Starling, Klerlein, Kier and Carr we show that these digraphs are Hamiltonian for $k \leq n-3$. We do this using restricted Eulerian cycles and the fact that $P(n,k)$ is nearly the line digraph of $P(n,k-1)$. We also show that the digraphs $P(n,n-2)$ are not Hamiltonian for $n \geq 4$ using a result of Rankin on Cayley digraphs.

1 Introduction

For $1 \leq k < n$ let $P(n,k)$ be the digraph with vertices corresponding to k permutations of $[n] = \{1,2,\ldots,n\}$ and arcs corresponding to $k+1$ permutations of $[n]$. The arc corresponding to $\sigma_1\sigma_2\ldots\sigma_k\sigma_{k+1}$ is $(\sigma_1,\sigma_2,\ldots,\sigma_k) \to (\sigma_2,\sigma_3,\ldots,\sigma_k,\sigma_{k+1})$. Our aim is to show that $P(n,k)$ contains a directed Hamiltonian cycle for all n and $k \leq n-3$ and to show that $P(n,n-2)$ is not Hamiltonian for all $n \geq 4$. This answers a question of Starling, Klerlein, Kier and Carr [7] who showed Hamiltonicity for the case $k=2$ and asked about the general case.

For $k \leq n-3$, the method will be to observe that $P(n,k)$ is closely related to the line digraph $L(P(n,k-1))$ and that certain restricted Eulerian chains in $L(P(n,k-1))$ will correspond to Hamiltonian cycles in $P(n,k)$. Then we will find such restricted Eulerian chains. This approach will also indicate a straightforward but more general method of finding Eulerian chains in digraphs subject to restrictions on which arcs can appear consecutively. For the cases $P(n,n-2)$ the method will be to note that these are Cayley digraphs [1] and then apply a result of Rankin [6] about necessary conditions for Cayley digraphs with two generators to be Hamiltonian.

The digraphs $P(n,n-1)$ consist of disjoint cycles and hence contain neither Hamiltonian cycles nor Hamiltonian paths except the trivial case $P(2,1)$ which has a Hamiltonian path consisting of a single arc. Consider the digraphs $P(n,n-2)$. As noted above we will show that these are not Hamiltonian for $n \geq 4$. $P(3,1)$ has 3 vertices with arcs in both directions between each pair of vertices and is Hamiltonian. Using the correspondence to restricted Eulerian cycles described below it is straightforward to check that $P(4,2)$ contains a Hamiltonian path. Klerlein, Carr and Starling [5] report that a computer search has also

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shown that $P(5,3)$ contains a Hamiltonian path but not a Hamiltonian cycle. In general, we do not know whether or not the digraphs $P(n,n-2)$ contain Hamiltonian paths.

Some algebraic properties of the digraphs $P(n,k)$ have been studied in [1], which also contains a discussion and references to related digraphs with vertices corresponding to permutations which have come up in various engineering applications. The digraphs $P(n,k)$ have also been studied (with different notation) in [4] in connection with universal cycles of permutations. A universal cycle for a $k-$permutation of n is a cyclic listing of of $n!/(n-k)!$ symbols from $[n]$ such that each $k-$permutation appears exactly once as a string of length k in the listing. These are exactly Hamiltonian cycles in the line digraph $L(P(n,k-1))$. Recalling the correspondence between Eulerian chains in a digraph and Hamiltonian cycles in its line digraph we see that finding these universal cycles is equivalent to finding Eulerian cycles in $P(n,k-1)$.

Each vertex in $P(n,k)$ has indegree and outdegree equal to $n-k$. So $P(n,k)$ will be Eulerian if it is strongly connected. This is straightforward to show for $k \leq n-2$ (see for example [4]). This will also follow inductively for $k \leq n-3$ from the results below as we will be showing that $P(n,k)$ is Hamiltonian and hence strongly connected.

We have noted that a Hamiltonian cycle in $L(P(n,k-1))$ corresponds to a universal cycle for $k-$permutations. We are interested in Hamiltonian cycles in $P(n,k)$. These also correspond to universal cycles for $k-$permutations with the additional restriction that the length $k+1$ strings are also permutations. We see this as follows. Note that the line digraph $L(P(n,k-1))$ has the same vertex set as $P(n,k)$ and contains each arc of $P(n,k)$ but in addition contains some extra arcs of the form $(\sigma_1,\sigma_2,\ldots,\sigma_k) \to (\sigma_2,\sigma_3,\ldots,\sigma_k,\sigma_1)$. Thus, we can show that $P(n,k)$ is Hamiltonian by finding a Hamiltonian cycle in $L(P(n,k-1))$ avoiding these extra arcs. We do this by finding an Eulerian chain in $P(n,k-1)$ with certain restrictions on which arcs can appear consecutively.

2 $P(n,k)$ for $k \leq n-3$

It will be notationally convenient to express Eulerian chains in terms of arcs. In $P(n,k-1)$ consider the following pairing of arcs: let $f((\sigma_1,\sigma_2,\ldots,\sigma_k)) = (\sigma_2,\sigma_3,\ldots,\sigma_k,\sigma_1)$. These are exactly the pairs in $P(n,k-1)$ that give rise to arcs in the line digraph $L(P(n,k-1))$ that are not in $P(n,k)$. Thus an Eulerian chain in $P(n,k-1)$ that avoids an arc a followed by $f(a)$ will correspond to a Hamiltonian cycle in $L(P(n,k-1))$ that only uses arcs that are also present in $P(n,k)$. That is, it corresponds to a Hamiltonian cycle in $P(n,k)$. It is also not difficult to see that each Hamiltonian cycle in $P(n,k)$ comes from such a restricted Eulerian chain.

We will start with an Eulerian chain in $P(n,k-1)$ and modify it in such a way that the result has the property described above. The modification will require that the common indegree and outdegree in $P(n,k-1)$ is at least 4. Thus, even though there are Eulerian cycles in $P(n,n-2)$ and $P(n,n-3)$ this method will not be able to show that $P(n,n-1)$ and $P(n,n-2)$ are Hamiltonian. Indeed, as noted above they are not. In the proof below induction is used only to show that $P(n,k-1)$ is strongly connected. As noted above, this also can easily be checked directly as is done, for example in [4].

Theorem 1 *For $1 \leq k \leq n-3$ the digraphs $P(n,k)$ contain a Hamiltonian cycle.*

Proof: Using the correspondence noted above we will construct Eulerian chains in $P(n,k-1)$ with the restriction that each arc a is not followed by $f(a)$. The proof will be by induction on k. For the basis for the induction note that $P(n,1)$ has n vertices and an arc in each direction between each pair of vertices. Hence $P(n,1)$ is Hamiltonian (except for $n=2$). Consider $2 \leq k \leq n-3$. By induction, $P(n,k-1)$ is Hamiltonian and hence strongly connected. Since also each vertex has indegree equal to outdegree, $P(n,k-1)$ is Eulerian. Let $d = n-k+1$ denote the common indegree and outdegree.

Construct an Eulerian chain in $P(n,k-1)$. In general this chain will not have the necessary property that arc a is not followed by $f(a)$. The chain passes through each vertex d times. Pick a vertex v and let $S_1, S_2, \ldots, S_d$ be the segments from one appearance of v to the next. Create a new digraph D with vertex set corresponding to $S_1, S_2, \ldots, S_d$ and an arc from S_i to S_j for $i \neq j$ if S_j can follow S_i in a 'good' Eulerian chain. That is, if a is the last arc of S_i then $f(a)$ is not the first arc of S_j. Note that the outdegree and indegree of each S_i is at least $d-2$. (S_i has indegree and outdegree $d-1$ if the last arc of S_i is a and the first is $f(a)$, these degrees are $d-2$ otherwise.) For $d \geq 4$ we have $d-2 \geq d/2$. Since $k \leq n-3$, we have $d = n-k+1 \geq 4$. Thus each vertex of D has indegree and outdegree at least half the number of vertices in D and hence D is Hamiltonian (by a well known theorem of Ghouila-Houri). Use a Hamiltonian cycle in D to order the S_i to yield a new Eulerian chain for which we do not have $f(a)$ following a at vertex v. Note also that the order of two consecutive arcs does not change except at v. Repeat this process at each vertex to yield a 'good' Eulerian chain, that is, one which corresponds to a Hamiltonian cycle in $P(n,k)$. $\square$

The process described above can be used in a broader context. For each vertex v and each arc a entering v specify a set of forbidden arcs which cannot follow a. If d is the common indegree and outdegree at v and as long as at most $(d-2)/2$ arcs are forbidden for each entering arc a and each arc leaving v is on at most $(d-2)/2$ forbidden lists we can find an Eulerian chain avoiding forbidden pairs. As a special case, if we color the arcs of an Eulerian digraph in such a way that if a vertex has indegree and outdegree d then there are at most $(d-2)/2$ arcs of each color entering the vertex and similarly for arcs leaving the vertex then we can find an Eulerian chain for which no two consecutive arcs have the same color.

3 $P(n, n-2)$

We now consider the digraphs $P(n,n-2)$. As noted above these digraphs are Cayley digraphs by a result of [1]. Let G be a group with generating set S. The Cayley digraph $C(G,S)$ has vertex set G and arcs $(\boldsymbol{\pi}, \boldsymbol{\pi\alpha})$ for $\boldsymbol{\pi} \in G$ and $\boldsymbol{\alpha} \in S$. (See [3] and [9] for surveys of results about Hamiltonicity in Cayley digraphs.) We describe here an explicit representation of $P(n,n-2)$ as a Cayley digraph for group A_n, the alternating group on $[n]$ with generating set $\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$ where for even n

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-2 & n-1 & 1 & n \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-2 & n & n-1 & 1 \end{pmatrix}$$

and for n odd

$$\boldsymbol{\alpha} = \begin{pmatrix} 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-2 & n-1 & n & 1 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-2 & n & 1 & n-1 \end{pmatrix}.$$

Observe that in both the n even and n odd cases $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are even permutations. For each $n-2$ permutation $(\pi_1,\pi_2,\ldots,\pi_{n-2})$ of $[n]$ (corresponding to vertices of $P(n,n-2)$) there are exactly two permutations in the symmetric group S_n for which the images of $1,2,3\ldots n-2$ are $\pi_1,\pi_2,\ldots,\pi_{n-2}$. Since these differ by a transposition, exactly one of these is an even permutation. Let $h(\pi_1,\pi_2,\ldots,\pi_{n-2})$ be the even permutation. Then h is an isomorphism from $P(n,k)$ to $C(A_n,\{\boldsymbol{\alpha},\boldsymbol{\beta}\})$. To see this note that h is one-to-one and onto from its definition. Consider vertex $(\pi_1,\pi_2,\ldots,\pi_{n-2})$ in $P(n,n-2)$. Let π_{n-1} and π_n be the two elements of $\{1,2,\ldots,n\}$ distinct from $\{\pi_1,\pi_2,\ldots,\pi_{n-2}\}$ labelled so that

$$\boldsymbol{\pi} = \begin{pmatrix} 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \\ \pi_1 & \pi_2 & \cdots & \pi_{n-3} & \pi_{n-2} & \pi_{n-1} & \pi_n \end{pmatrix}$$

is even and thus $h(\pi_1,\pi_2,\ldots,\pi_{n-2}) = \boldsymbol{\pi}$. In $P(n,n-2)$ we have arcs $(\pi_1,\pi_2,\ldots,\pi_{n-2}) \to (\pi_2,\pi_3,\ldots,\pi_{n-2},\pi_{n-1})$ and $(\pi_1,\pi_2,\ldots,\pi_{n-2}) \to (\pi_2,\pi_3,\ldots,\pi_{n-2},\pi_n)$ and it is straightforward to check that $h(\pi_2,\pi_3,\ldots,\pi_{n-2},\pi_{n-1}) = \boldsymbol{\pi\alpha}$ and $h(\pi_2,\pi_3,\ldots,\pi_{n-2},\pi_n) = \boldsymbol{\pi\beta}$ (using the fact that $\boldsymbol{\alpha},\boldsymbol{\beta}$ and $\boldsymbol{\pi}$ are all even). So arcs in $P(n,n-2)$ correspond to arcs in $C(A_n,\{\boldsymbol{\alpha},\boldsymbol{\beta}\})$.

Now, the fact that $P(n,n-2)$ is not Hamiltonian for $n \geq 4$ will follow as a corollary of the following theorem of Rankin [6]. See also [8] for a short proof of Rankin's theorem.

Theorem 2 (Rankin) *Let $C(G,\{\boldsymbol{\alpha},\boldsymbol{\beta}\})$ be a Cayley digraph with two generators and let m_α, m_β and m_γ be the orders of $\boldsymbol{\alpha},\boldsymbol{\beta}$ and $\boldsymbol{\gamma} = \boldsymbol{\alpha}^{-1}\boldsymbol{\beta}$ respectively. If m_γ is odd, a necessary condition for $C(G,\{\boldsymbol{\alpha},\boldsymbol{\beta}\})$ to be Hamiltonian is that $|G|/m_\alpha$ and $|G|/m_\beta$ are odd.*

Note that in Rankin's paper there is no explicit mention of digraphs and the notation there would correspond to digraphs with arc $\boldsymbol{\alpha\pi}$ corresponding to generator $\boldsymbol{\alpha}$. We use digraphs with arcs $\boldsymbol{\pi\alpha}$ to follow the notation of [3] and [9] and to simplify our notation. The result is the same for both cases since a Cayley digraph defined using left multiplication can also be defined using the same generators by replacing the group element for each vertex with its inverse.

Corollary 1 *The digraphs $P(n,n-2)$ are not Hamiltonian for $n \geq 4$.*

Proof: Use the isomorphism between $P(n,n-2)$ and $C(G,\{\boldsymbol{\alpha},\boldsymbol{\beta}\})$ described above. Then when n is even

$$\boldsymbol{\gamma} = \boldsymbol{\alpha}^{-1}\boldsymbol{\beta} = \begin{pmatrix} 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \\ 1 & 2 & \cdots & n-3 & n & n-2 & n-1 \end{pmatrix}$$

and when n is odd

$$\boldsymbol{\gamma} = \boldsymbol{\alpha}^{-1}\boldsymbol{\beta} = \begin{pmatrix} 1 & 2 & \cdots & n-3 & n-2 & n-1 & n \\ 1 & 2 & \cdots & n-3 & n-1 & n & n-2 \end{pmatrix}.$$

In both cases the order of $\boldsymbol{\alpha}^{-1}\boldsymbol{\beta}$ is 3 which is odd. When n is even the orders of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $n-1$ and so $|G|/m_\alpha = |G|/m_\beta = (n!/2)/(n-1)$ which is even for $n \geq 4$. When n is odd the orders of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are n and so $|G|/m_\alpha = |G|/m_\beta = (n!/2)/n$ which is even for $n \geq 5$. $\square$

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