Star Avoiding Ramsey Numbers

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Graph Ramsey Numbers

Example

\[ R(\mathcal{C}_5, \mathcal{K}_4) = 13 \]

- There exists a 2-coloring of \( K_{12} \) with no red \( \mathcal{C}_5 \) and no blue \( \mathcal{K}_4 \).
- Every 2-coloring of \( K_{13} \) has a red \( \mathcal{C}_5 \) or a blue \( \mathcal{K}_4 \).
Graph Ramsey Numbers

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There are 6 critical colorings (later)
Questions

- When can we classify all sharpness examples for $R(G, H) = r$?
- What are all good colorings of $K_{r-1}$ (critical colorings)?
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- When can we classify all sharpness examples for $R(G, H) = r$?
  - What are all good colorings of $K_{r-1}$ (critical colorings)
- How many edges to the $r^{th}$ vertex must be colored before a red $G$ or blue $H$ is forced?
A second look at our problem:

- **Graph Ramsey**: smallest $r$ with no good coloring
  \[ \ldots \quad K_{r-1}, \quad K_r, \quad K_{r+1}, \quad \ldots \]
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- **Size Ramsey**: smallest $s$ with no good coloring for *some* $F$
  
  \[ \ldots \quad |E(F)| = s - 1, \quad |E(F)| = s, \quad |E(F)| = s + 1, \quad \ldots \]
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- **Upper and lower Ramsey for $R(G, H) = r$**:
  
  - **Lower**: smallest $s$ with no good coloring for some $F$
  
  - **Upper**: smallest $s$ with no good coloring for every $F$

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  *Restrict to $|V(F)| = r$*
A second look at our problem:

- **Graph Ramsey**: smallest $r$ with no good coloring
  
  ... $K_{r-1}$, $K_r$, $K_{r+1}$, ...

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- **Upper and lower Ramsey for $R(G, H) = r$**:  
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  *Restrict to $|V(F)| = r$*

- **Star avoiding Ramsey for $R(G, H) = r$**:  
  smallest $r - 1 - t$ with no good coloring
  
  ... $K_{r-1} \setminus S(1, t - 1)$, $K_{r-1} \setminus S(1, t)$, $K_{r-1} \setminus S(1, t + 1)$, ...

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Star Avoiding Ramsey Numbers
Star avoiding Ramsey:

\[ R(G, H) = r \] add/color edges to \( K_{r-1} \) one at a time:

When is a red \( G \) or blue \( H \) forced?

\[ K_{r-1} \]
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When is a red \( G \) or blue \( H \) forced?

Proofs: First classify sharpness examples
- Good colorings of \( K_{r-1} \)
- Examples with ‘few’ extra edges needed and with ‘many’ extra edges needed
Example

- $R(K_m, K_n) = r$: must add all $r - 1$ edges (Chvatal 1974) even though we do not know what $r$ is
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- make a copy of a vertex.
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- make a copy of a vertex
- similar for $R(mK_3, mK_3) = 5m$
Example \(R(P_n, P_3) = n\) (Gerencser and Gyrafas 1967)

- \(R(P_n, P_3) = n\)

- \(K_{n-1} \setminus tK_2\)
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- Red edge \( \Rightarrow \) red \( P_n \)
Example \((R(P_n, P_3) = n \text{ (Gerencser and Gyrafas 1967)})\)

- \(R(P_n, P_3) = n\)
- Sharpness examples: Blue graph is a matching plus isolated vertices
- Can only add one edge to \(K_{n-1}\) before a red \(P_n\) or blue \(P_3\) is forced.

![Graph](image)

- Red edge \(\Rightarrow\) red \(P_n\)
- Two Blue edges \(\Rightarrow\) blue \(P_3\)
Example ($R(P_n, P_m)$ (Gerencser and Gyrafas 1967))

- $R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ for $n \geq m \geq 4$
- Sharpness examples for $n \geq m + 2$. Black graph is arbitrary. Red clique can have one blue edge for odd $m$
- 3 other families when $n = m$ or $n = m + 1$
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- Red or Blue edge to red \(K_{n-1}\) forces red \(P_n\) or blue \(P_m\)
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- Sharpness examples for \(n \geq m + 2\). Black graph is arbitrary. Red clique can have one blue edge for odd \(m\)

- 3 other families when \(n = m\) or \(n = m + 1\)

- (only) add all red edges to \(A_{\left\lfloor \frac{m}{2} \right\rfloor - 1}\)
Example \( R(T_n, K_m) = (n - 1)(m - 2) + 1 \) (Chvatal 1977)

- Unique sharpness example:
  - Red graph is \((m - 1)K_{n-1}\)
  - Blue graph is \(K_{n-1,n-1,...,n-1}\)
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- Blue edges to all parts \(\Rightarrow\) blue \(K_m\)
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- Unique sharpness example:
  - Red graph is \((m - 1)K_{n-1}\)
  - Blue graph is \(K_{n-1,n-1,\ldots,n-1}\)
- (only) add all \((n - 1)(m - 2)\) blue edges avoiding one part
Example ( $R(C_5, K_4) = 13$ )

- Exactly 6 good colorings of $K_{12}$ (Jayawardene and Rousseau 2000)
- Ends must be different (or same) for 3 extra red edges
- Extends to $R(C_n, K_4) = 3n - 2$ (but not $n = 4$)
Example \( (R(C_5, K_4) = 13) \)

- Exactly 6 good colorings of \( K_{12} \) (Jayawardene and Rousseau 2000)
- Ends must be the same for 3 extra red edges for \( n \geq 6 \)
- Extends to \( R(C_n, K_4) = 3n - 2 \)
## Summary of Results

<table>
<thead>
<tr>
<th>Ramsey number</th>
<th>Minimum Number of edges to force bad coloring</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(mK_2, mK_2) = 3m - 1$ [L 1984]</td>
<td>$m$</td>
</tr>
<tr>
<td>$R(mK_3, mK_3) = 5m$ [BES 1975]</td>
<td>$5m$</td>
</tr>
<tr>
<td>$R(T_n, K_m) = (n - 1)(m - 1) + 1$ [C 1977]</td>
<td>$(n - 1)(m - 2) + 1$</td>
</tr>
<tr>
<td>$R(C_n, K_3) = 2n - 1$ [FS 1974]</td>
<td>$n + 1$</td>
</tr>
<tr>
<td>$R(C_n, K_4) = 3n - 2$ [SRM 1999]</td>
<td>$2n$</td>
</tr>
<tr>
<td>$R(P_n, P_3) = n$ [GG 1967]</td>
<td>$2$</td>
</tr>
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<td>$R(P_n, P_4) = n + 1$ [GG 1967]</td>
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</tr>
<tr>
<td>$R(P_n, P_5) = n + 1$ [GG 1967]</td>
<td>$3$</td>
</tr>
<tr>
<td>$R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1$ [GG 1967]</td>
<td>$\frac{m}{2}$</td>
</tr>
</tbody>
</table>

for all $n \geq m \geq 2$