Integer and Fractional Security in Graphs

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Abstract

Let $G = (V, E)$ be a graph. A subset S of V is said to be secure if it can defend itself from an attack by vertices in $N[S] - S$. In the usual definition, each vertex in $N[S] - S$ can attack exactly one vertex in S, and each vertex in S can defend itself or one of its neighbors in S. A defense of S is successful if each vertex has as many defenders as attackers. We look at allowing an attacking vertex to divide its one unit of attack among multiple targets, or allowing a defending vertex to divide its one unit of defense among multiple allies. Three new definitions of security are given. It turns out that two of the new definitions are the same as the original.

Keywords: Secure sets, Hall's theorem, Integer attack, Integer defense

1. Introduction

All graphs in this paper are finite and simple. For a graph $G = (V, E)$ and $v \in V$ we will follow convention by letting $N(v) = \{u \mid uv \in E\}$, and $N[v] = \{v\} \cup N(v)$. For $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The foundation for the study of security in graphs was laid by Brigham, Dutton, and Hedetniemi in [2]. Given a graph $G = (V, E)$, they define an attack on $S = \{s_1, s_2, ..., s_k\} \subseteq V$ to be a collection of pairwise disjoint sets $A = \{A_1, A_2, ..., A_k\}$ for which $A_i \subseteq N[s_i] - S$, $1 \leq i \leq k$. A defense of S is a collection of pairwise disjoint sets $D = \{D_1, D_2, ..., D_k\}$ such that $D_i \subseteq N[s_i] \cap S$, $1 \leq i \leq k$. An attack is defendable if there is a defense D such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$. In this setting, each vertex in $N[S] - S$ can attack only one of its neighbors in S, and each vertex in S can defend

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itself or one of its neighbors in S.

We consider the cases in which a vertex may send out attack or defense to as many appropriate vertices as it wants, so long as the total amount of attack or defense from that vertex sums to at most one.

2. Definitons

Let $G = (V, E)$ be a graph and $S \subseteq V$. An *attack* on S is a function $A{:}(V - S) \times S \rightarrow [0, 1]$ such that $A(u, v) = 0$ if $uv \notin E$ and for $u \in V - S, \sum_{v \in N(u) \cap S} A(u, v) \leq 1$.

A *defense* of S is a function $D{:}S \times S \rightarrow [0, 1]$ such that $D(u, v) = 0$ if $u \neq v$ and $uv \notin E$ and for $u \in S, \sum_{v \in N[u] \cap S} D(u, v) \leq 1$.

Suppose that A is an attack on S and D is a defense of S. For $u \in S$ let $D^*(u) = \sum_{v \in N[u] \cap S} D(v, u)$ and $A^*(u) = \sum_{v \in N(u) - S} A(v, u)$. An attack A is *defendable* if there exists a defense D such that for each $u \in S$, $D^*(u) \geq A^*(u)$. The set S is *secure* if every attack on S is defendable.

The definition of attack and defense given by Brigham, et al in [2] corresponds to the case $A(u, v) \in \{0, 1\}$ for all $u \in (V - S), v \in S$ and $D(u, v) \in \{0, 1\}$ for all $u, v \in S$. We will refer to these as *integer attack* and *integer defense* respectively. This naturally leads to four scenarios:
a) an integer attack against an integer defense, (I,I);
b) an integer attack against a defense, (I,F);
c) an attack against an integer defense, (F,I);
d) an attack against a defense, (F,F).

For each situation there is a corresponding definition of security. We now explore relationships among them.

3. Results

Lemma. *If $S \subseteq V$ is secure in G in any of the four senses, then for each set $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$.*

Proof. Suppose that, for some $X \subseteq S$, $|N[X] \cap S| < |N[X] - S|$. Make an integer attack A on S by letting each vertex in $|N[X] - S|$ attack any of its neighbors in X with its whole unit of attack. Let every other vertex in $N[S] - S$ attack one of its neighbors in S, or none; it does not matter. Then for any defense D of S,

$$\begin{aligned}\sum_{x \in X} D^*(x) &= \sum_{x \in X} \sum_{u \in N[x] \cap S} D(u, x) = \sum_{u \in N[X] \cap S} \sum_{x \in X} D(u, x) \\ &\leq \sum_{u \in N[X] \cap S} 1 = |N[X] \cap S| < |N[X] - S| \\ &= \sum_{x \in X} A^*(x).\end{aligned}$$

Therefore, for any such D, there must be some $x \in X$ such that $D^*(x) < A^*(x)$. Thus A is not defendable, and so S is not secure. $\square$

Brigham et al [2] gave necessary and sufficient conditions for a set to be (I,I)-secure. We shall refer to this as the BDH theorem. As an aside, we give a short proof of this result using Hall's Theorem:

Theorem HRHV ([4], [6], [5]). *Suppose $P_1, ..., P_n$ are sets and $k_1, ..., k_n$ are non-negative integers. There exist pairwise disjoint sets $D_1, ..., D_n$ such that $D_i \subseteq P_i$ and $|D_i| = k_i$ for $1 \leq i \leq n$ if, and only if, for each $J \subseteq \{1, ..., n\}$, $|\cup_{j \in J} P_j| \geq \sum_{j \in J} k_j$.*

Theorem BDH. *A set $S \subseteq V$ is (I,I)-secure if and only if $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$.*

Proof. The necessity of the condition follows from the Lemma. Let $G = (V, E)$ be a graph. Let $S = \{s_1, s_2, ..., s_n\} \subseteq V$ be a set such that $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$. Let $A = \{A_1, ..., A_n\}$ be an (integer) attack on S, by the original definition in [2]. Define $P_i = N[s_i] \cap S$ for $1 \leq i \leq n$. Thus P_i is the set of potential defenders of s_i. Let $k_i = |A_i|$ for $1 \leq i \leq n$; k_i is the number of attackers of s_i. For any $J \subseteq \{1, 2, ..., n\}$, let $X_J = \{s_j \mid j \in J\}$. Then we have $\sum_{j \in J} k_j = \sum_{j \in J} |A_j| \leq |N[X_J] - S| \leq |N[X_J] \cap S| = \left|\bigcup_{j \in J} P_j\right|$. By Theorem HRHV we can find $D_i \subseteq P_i$ for $1 \leq i \leq n$ such that the D_i are pairwise disjoint and $|D_i| = k_i$. Thus $D = \{D_1, ..., D_n\}$ is an integer defense that thwarts the attack, and S is secure. $\square$

As remarked in [2], Theorem BDH shows that the problem of deciding whether or not $S \subseteq V$ is (I,I)-secure is in co-NP: S is not (I,I)-secure if and only if there is a certificate proving that it is not, a set $X \subseteq S$ such that $|N[X] \cap S| < |N[X] - S|$. Reportedly, Dutton [3] has recently shown that the problem is co-NP-complete.

In order to prove the next result, we need the following analogue of Hall's theorem due to Bollobás and Varopoulos [1].

Theorem BV. *Suppose that (X, μ) is an atomless measure space, $M_1, ..., M_n$ are subsets of X of finite measure, and $r_1, ..., r_n$ are non-negative real numbers. There exist pairwise disjoint sets $C_1, ..., C_n$ such that $C_i \subseteq M_i$ and $\mu(C_i) = r_i, 1 \leq i \leq n$, if, and only if, for each $J \subseteq \{1, ..., n\}$ we have $\mu(\bigcup_{j \in J} M_j) \geq \sum_{j \in J} r_j$.*

Theorem. *Let $G = (V, E)$ be a graph and $S \subseteq V$. Then (a) S is (I,I)-secure $\Leftrightarrow$ (b) S is (I,F)-secure $\Leftrightarrow$ (c) S is (F,F)-secure.*

Proof. We will show (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (c) . If S is (F,F)-secure then all attacks on S are defendable, so all integer attacks on S are defendable. Thus S is (I,F)-secure. Now suppose S is (I,F)-secure. By the Lemma, for any $X \subseteq S$ it must be that $|N[X] \cap S| \geq |N[X] - S|$. Therefore, by the BDH Theorem, S is also (I,I)-secure.

Let S be (I,I)-secure. Then $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$. Let A:$(N(S) - S) \times S \to$ [0,1] be an attack on S. (We restrict the domain of the attackers to $N(S) - S$ because all vertices in $V - N(S)$ will have no attack.) Recall that for $v \in S$, $A^*(v) = \sum_{u \in N(v) - S} A(u, v)$. Let $\{I(v) | v \in S\}$ be an indexed family of pairwise disjoint unit intervals in the real numbers. These are the defense reservoirs of the vertices of S. For $v \in S$ let $M(v) = \bigcup_{w \in N[v] \cap S} I(w)$. This is the total defense available to v. So we have another indexed family $\{M(v) | v \in S\}$. Let λ denote the Lebesgue measure. To achieve a successful fractional defense against the attack A, it suffices to find an indexed family $\{C(v) | v \in S\}$ of pairwise disjoint Lebesgue measurable sets such that for all $v \in S$, $C(v) \subseteq M(v)$ and $\lambda(C(v)) = A^*(v)$. If such a family is found, define a defense D:$S \times S \to$[0,1] by $D(w, v) = \lambda(I(w) \cap C(v))$.

Then for v $\in S$ we would have

$$
\begin{aligned}
D^*(v) &= \sum_{w\in S} D(w,v)\\
&= \sum_{w\in S} \lambda(I(w)\cap C(v))\\
&= \lambda(\bigcup_{w\in S}(I(w)\cap C(v))) \qquad \text{[the intervals } I(w), w\in S \text{ are pairwise disjoint]}\\
&\geq\ \lambda(M(v)\cap C(v)) = \lambda(C(v)) = A^*(v).
\end{aligned}
$$

Also for each $w \in S$,

$$
\begin{aligned}
\sum_{v\in S} D(w,v) &= \sum_{v\in S}\lambda(I(w)\cap C(v))\\
&= \lambda(\bigcup_{v\in S}(I(w)\cap C(v)) \leq \lambda(I(w)) = 1.
\end{aligned}
$$

So D is a defense of S, and it defends against A.

Now we will show that $|N[X]\cap S| \geq |N[X]-S|$ for all $X \subseteq S$ implies the existence of such a family $\{C(v) \mid v \in S\}$. By Theorem BV, it is sufficient to show that for all $X \subseteq S$, $\sum_{v\in X} A^*(v) \leq \lambda(\bigcup_{v\in X} M(v))$. Suppose $X \subseteq S$. We have

$$
\begin{aligned}
\lambda(\bigcup_{v\in X} M(v)) &= \lambda(\bigcup_{v\in X}(\bigcup_{w\in N[v]\cap S} I(w)))\\
&= \lambda(\bigcup_{w\in N[X]\cap S} I(w)) = \sum_{w\in N[X]\cap S}\lambda(I(w))\\
&= \sum_{w\in N[X]\cap S} 1 = |N[X]\cap S| \geq |N[X]-S|\\
&\geq \sum_{u\in N[X]-S}\sum_{v\in S} A(u,v) \geq \sum_{u\in N[X]-S}\sum_{v\in X} A(u,v)\\
&= \sum_{v\in X}\sum_{u\in N[X]-S} A(u,v) = \sum_{v\in X} A^*(v). \quad \square
\end{aligned}
$$

This leaves the question of how (F,I)-security relates to (I,I)-security. Clearly, (F,I)-security implies (I,I)-security, but we will show the converse does not hold.

Example. In [2] it is seen that any $\lceil \frac{n}{2} \rceil$ vertices of K_n form an (I,I)-secure set; however, $n-1$ vertices are needed to be (F,I)-secure. Let $S = \{s_1, ..., s_k\} \subseteq V(K_n)$ such that $k \leq n-2$. Then $|V(K_n) - S| \geq 2$. Let $v_1, v_2 \in V(K_n) - S$. Let $A(v_1, s_i) = \frac{1}{|S|}$ for $1 \leq i \leq k$ and $A(v_2, s_1) = 1$ and $A(v_2, s_i) = 0$ for $2 \leq i \leq k$. A successful integer defense of this attack requires $|S| + 1$ defenders, so S is not (F,I)-defendable. So, in order for S to be (F,I)-secure, $|S| \geq n-1$. Any set S such that $|S| = n-1$ is (F,I)-secure.

So in some sense a (F,I)-secure set has a greater security than a set that is only (I,I)-secure. We give a necessary and sufficient condition, in the spirit of Theorem BDH, for (F,I)-security in another paper, currently in preparation. It may be possible to obtain results similar to Theorem BDH and the main theorem of this paper when attack and defense capabilities are extended to general values, and are not necessarily constant from vertex to vertex.

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