Enumerating List Colorings

Garth Isaak   Lehigh University

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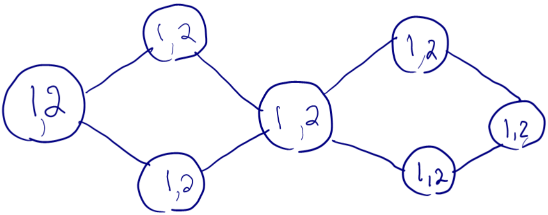
1 minute version of this talk

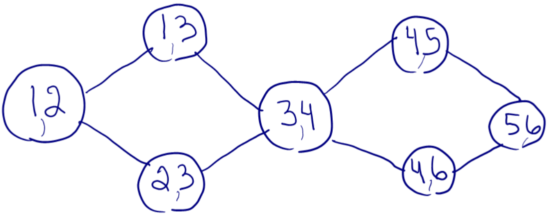
- Recall: Mobius inversion on Lattice of Partitions to enumerate Systems of distinct representatives
- Recall: Chromatic Polynomial via Mobius inversion on Lattice of flats of polygon matroid of a graph
- Observe: List coloring on complete graphs corresponds to systems of distinct representatives
- Combine these ideas!

List coloring examples



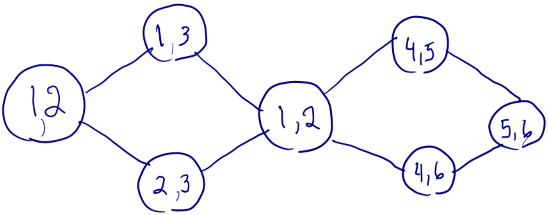
Bipartite graph so two 2-colorings from lists $\{1, 2\}$

List coloring examples



No proper list coloring ...

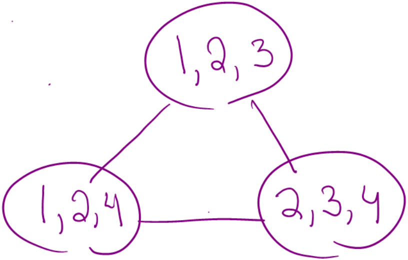
List coloring examples



Straightforward to check: 36 proper list colorings

Aim for systematic approach to count list colorings

List coloring examples



Complete graph so all colors distinct

So just # systems of distinct representatives (SDRs) of $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$

Count SDR's for $\{1,2,3\},\{1,2,4\},\{1,3,4\}$
Start with all triples

111	113	114	121	123	124	141	143	144
211	213	214	221	223	224	241	243	244
311	313	314	321	323	324	341	343	344

'Uncount' *aab* and *aba* and *baa*

111	113	114	121	123	124	141	143	144
211	213	214	221	223	224	241	243	244
311	313	314	321	323	324	341	343	344

111 was 'uncounted' 2 so 'recount' 2 times.

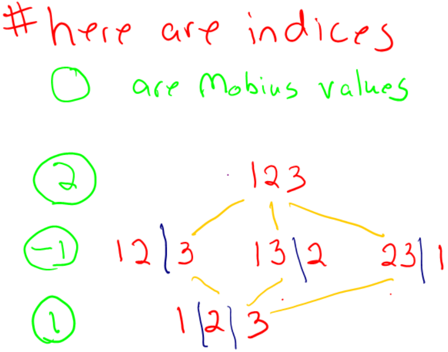
$\#$ SRDs $= 27 - (6+6+6) + 2 \cdot 1 = 11$

Count SDR's for $\{1,2,3\},\{1,2,4\},\{1,3,4\}$ via Mobius Inversion

Let x_I be the size of $\cap A_i$ for $i \in I$ # SDRs

$x_1x_2x_3 - (x_{12}x_3 + x_{13}x_2 + x_{23}x_1) + 2 \cdot x_{123} = 27 - (6+6+6) + 2 \cdot 1 = 11$

Multipliers given by Mobius function on lattice



Count SDRs for 4 sets (of any size)...

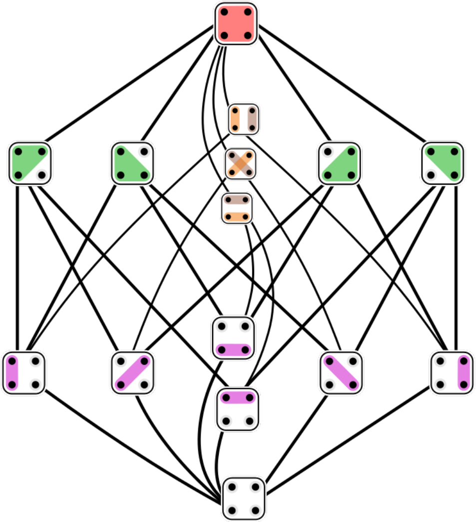
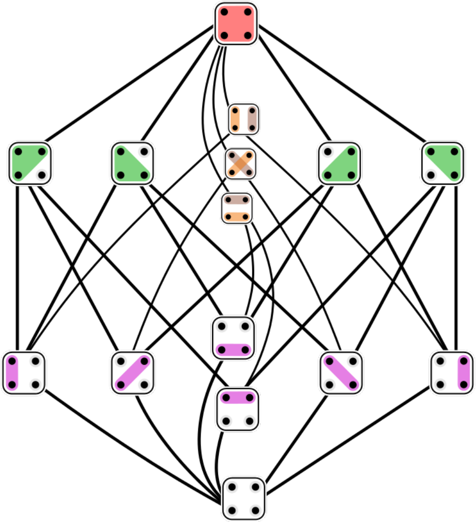


Figure source wikimedia commons, Tilman Piesk

For list coloring graph on 4 vertices
omit elements where underlying graph is disconnected
including in Mobius computation

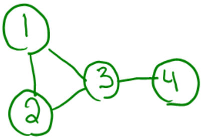


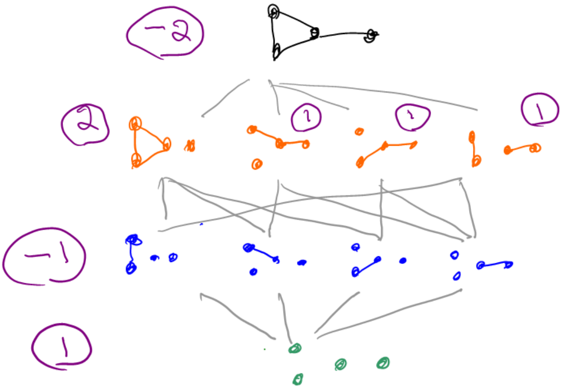
More formal version for counting SDRs
(well known, not sure of history)

Given a collection $\mathcal{C} = A_1, A_2, \ldots, A_n$ of sets and a partition $\gamma = S_1 \cup S_2 \cup \cdots S_k$ of $[n]$, let $f(\gamma, \mathcal{C}) = \Pi_{i=1}^{k} |\cap_{j \in S_i} A_j|$. Let L be the lattice of partitions of $[n]$ and $\mu(\gamma)$ the Mobius function on this lattice evaluated on the interval from the minimal element (the partition of $[n]$ into n parts of size 1) to γ. The number of systems of distinct representatives of $\mathcal{C}$ is $\sum_{\gamma} f(\gamma, \mathcal{C})\mu(\gamma)$ where the sum is over all partitions of $[n]$.

Chromatic polynomial via Mobius inversion

$t^4 - 4t^3 + 5t^2 - 2t$



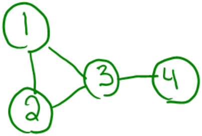


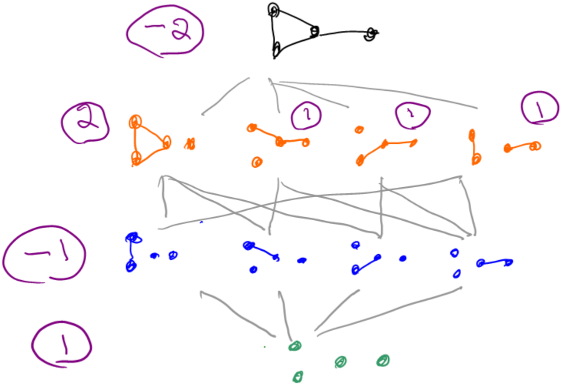
More formal version of chromatic polynomial via Mobius inversion (well known, not sure of history)

Let L_G be the lattice of flats of the polygon matroid of a graph. Equivalently, if G has vertex set $[n]$, the elements of L_G are partitions γ of $[n]$ such that the subgraph induced by each part is connected. Let $\mu_G(\gamma)$ denote the Mobius function on this lattice evaluated on the interval from the minimal element to γ. Let $g(\gamma)$ denote the number of parts in γ. The chromatic polynomial is then $\sum_\gamma \mu_G(\gamma) x^{g(\gamma)}$ where the sum is over all elements of L_G.

List Chromatic polynomial

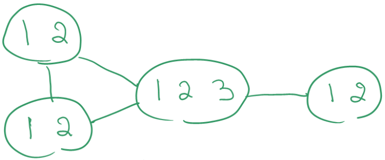
$$x_1x_2x_3x_4 - (x_{12}x_3x_4 + x_{13}x_2x_4 + x_{23}x_1x_4 + x_{34}x_1x_2)$$
$$+(2x_{123}x_4 + x_{134}x_2 + x_{234}x_1 + x_{12}x_{34}) - 2x_{1234}$$





Count list colorings

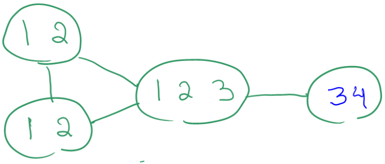
$$x_1x_2x_3x_4 - (x_{12}x_3x_4 + x_{13}x_2x_4 + x_{23}x_1x_4 + x_{34}x_1x_2)$$
$$+(2x_{123}x_4 + x_{134}x_2 + x_{234}x_1 + x_{12}x_{34}) - 2x_{1234}$$



$$2^3 \cdot 3 - (2^2 \cdot 3 + 2^3 + 2^3 + 2^3) + (2 \cdot 2^2 + 2^2 + 2^2 + 2^2) - 2 \cdot 2 = 4$$

Count list colorings

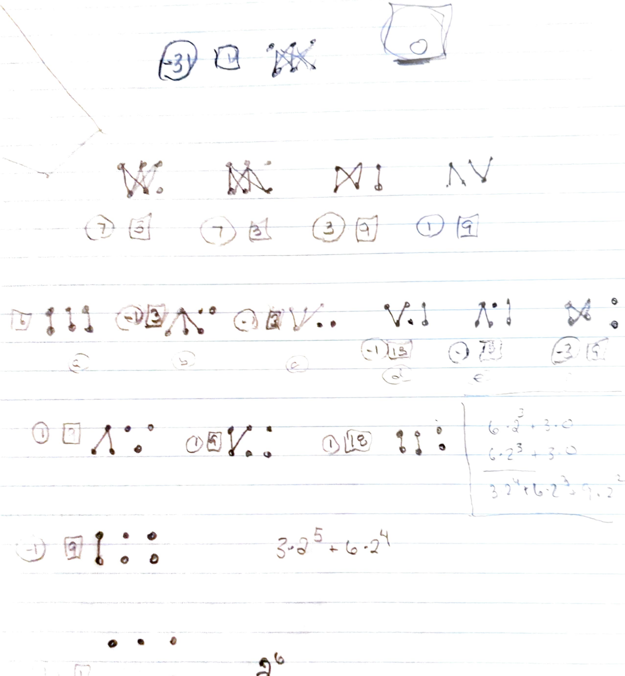
$$x_1x_2x_3x_4 - (x_{12}x_3x_4 + x_{13}x_2x_4 + x_{23}x_1x_4 + x_{34}x_1x_2)$$
$$+(2x_{123}x_4 + x_{134}x_2 + x_{234}x_1 + x_{12}x_{34}) - 2x_{1234}$$



$$2^3 \cdot 3 - (2^2 \cdot 3 + 2^3 + 2^3 + 2^2) + (2 \cdot 2^2 + 0 + 0 + 2) - 2 \cdot 0 = 2$$

Combining standard ideas for a 'list chromatic polynomial' to count list colorings

- Assume vertex set of G is $[n]$.
- Variables are x_T for $T \subseteq [n]$
- Let $\chi_l(G) = \sum_\gamma \mu_G(\gamma)\Pi_{T\in\gamma}x_T$
  where the sum is over all elements of L_G
  and the product is over all parts in γ
- For a given collection of lists $\mathcal{C}$ and $T \subseteq [n]$ let
  $h(T) = |\cap_{j\in T} A_j|$
- Evaluating $\chi_l(G)$ at $x_T = h(T)$
  gives the number of proper list colorings of G using the lists $\mathcal{C}$



$6 \cdot 2^3 + 3 \cdot 0$

$6 \cdot 2^3 + 3 \cdot 0$

$3 \cdot 2^4 + 6 \cdot 2^3 + 9 \cdot 2^2$

$3 \cdot 2^5 + 6 \cdot 2^4$

2^6