Degrees and Trees

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Acknowledgements to: Kathleen Ryan, ...

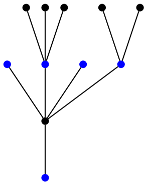
REU Students (Hannah Alpert, Amy Becker, Jenny Iglesius, James Hilbert)

T.S. Michael

Recall degree sequence conditions for trees
Basic exercise in a first graph theory course

- Degrees are positive integers and degree sum is even (always assume this)
- Trees (on n vertices) have $n-1$ edges
  $\Rightarrow$ Degree sum is $2n-2$

Positive integers $d_1, d_2, \ldots, d_n$ are degrees of a tree $\Leftrightarrow$ $\sum d_i = 2n-2$

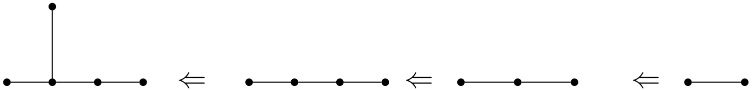


$(5, 4, 3, 1, 1, 1, 1, 1, 1, 1, 1)$

(One) proof (Leaf Removal) of

Positive integers $d_1, d_2, \dots, d_n$ are degrees of a tree $\Leftarrow$ $\sum d_i = 2n - 2$

- $d_1 \geq \cdots \geq d_{n-1} \geq d_n$ with $\sum d_i = 2n - 2$
  $\Rightarrow d_n = 1$ and $d_1 \geq 2$
- By induction, tree with $d_1 - 1, d_2, \dots, d_{n-1}$
- Add edge $v_1 v_n$



$$(3,2,1,1,1) \;\Rightarrow\; (2,2,1,1,\ ) \;\Rightarrow\; (1,2,1,\ ,\ ) \;\Rightarrow\; (1,1,\ ,\ ,\ )$$

- Added edge has degree 1 $\Rightarrow$ no cycle created

Recall degree sequence conditions for (loopless) multigraphs
Another basic exercise in a first graph theory course

- Degrees are positive integers and degree sum is even
- No loops
  $\Rightarrow$ edges from max degree vertex go to other vertices
  $\Rightarrow$ max degree $\leq$ sum of other degrees

Positive integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with even degree sum, are degrees of a loopless multigraph $\Leftrightarrow$ $d_1 \leq \sum_{i=2}^{n} d_i$

(one) proof of

Positive integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with even degree sum, are degrees of a loopless multigraph $\Leftrightarrow$ $d_1 \leq \sum_{i=2}^{n} d_i$

- $d_1 \leq d_2 + \cdots + d_n \Rightarrow d_1 - d_n \leq d_2 + \cdots + d_{n-1}$
- $d_2 \leq d_1$ and $d_n \leq d_{n-1} \Rightarrow d_2 \leq (d_1 - d_n) + d_3 + \cdots + d_{n-1}$
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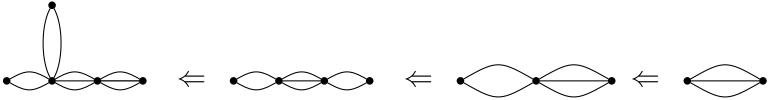


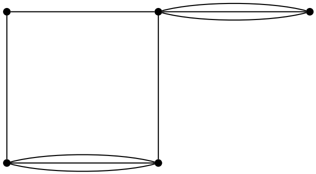
$(7, 5, 2, 2, 2) \Rightarrow (5, 5, 5, 2, \;) \Rightarrow (3, 5, 2, \;, \;) \Rightarrow (3, 3, \;, \;, \;)$

- Underlying added edge has degree 1 $\Rightarrow$ no cycle created

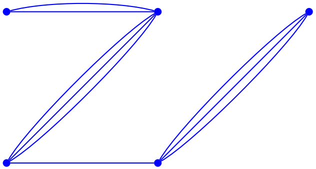
Both proofs added a 'leaf' $\Rightarrow$ no cycles created

Have we just proved?

Non-Theorem: *Positive integers* $d_1 \geq d_2 \geq \cdots \geq d_n$ *with even degree sum, are degrees of a loopless* *multitree* $\Leftrightarrow d_1 \leq \sum_{i=2}^{n} d_i$
i.e. *Multigraph* $\Rightarrow$ *Multitree with same degrees*



$(5, 4, 4, 3, 2)$



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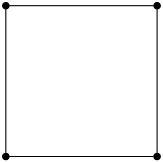
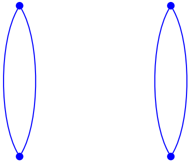


$(2, 2, 2, 2)$



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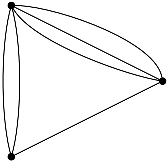

$(5, 4, 3)$

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- $(2, 2, 2, 2)$ and $(5, 4, 3)$ fail
-
-

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i.e. Multigraph $\Rightarrow$ Multitree with same degrees

- $(2, 2, 2, 2)$ and $(5, 4, 3)$ fail
- Forests are bipartite so $d_1 \leq d_2 + \cdots d_n \Rightarrow$ can partition d_i into two parts with equal sum
- Test if given integer list partitions into 2 equal sum parts? NP-hard problem so something is really wrong

What went wrong with multgraph proof?

Positive integers $d_1 \geq d_2 \geq \cdots \geq d_n$ with even degree sum, are degrees of a loopless multigraph $\Leftrightarrow$ $d_1 \leq \sum_{i=2}^{n} d_i$

- $d_1 \leq d_2 + \cdots + d_n \Rightarrow d_1 - d_n \leq d_2 + \cdots + d_{n-1}$
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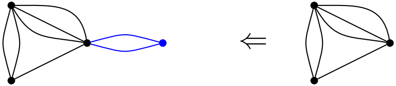

$(6, 5, 3, 2) \quad \Rightarrow \quad (4, 5, 3, \ \ )$

What went wrong with multgraph proof?

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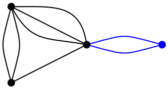
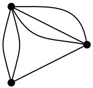
- $d_2 \leq d_1$ and $d_n \leq d_{n-1} \Rightarrow d_2 \leq (d_1 - d_n) + d_3 + \cdots + d_{n-1}$

IF $n \geq 4$

- By induction multigraph with $d_1 - d_n, d_2, \ldots, d_{n-1}$
- Add edges $v_1 v_n$

 $\Leftarrow$ 

$(6, 5, 3, 2) \quad \Rightarrow \quad (4, 5, 3, \ \ )$

With correct basis for $n = 3$ we get

Degrees of a multigraph $d_1 \leq d_2 + \cdots + d_n$ have a realization with underlying graph a forest or a graph with exactly one cycle (which is a triangle)

Note that partitioning integer lists into equal sum parts is NP-hard. So might not detect forest realization if there is one.

Good example why need basis for induction

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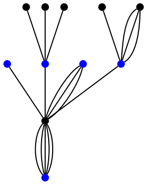
Good example why need basis for induction

- What are conditions for a multiforest?
- What if we want connected? i.e., multitree?

Loopless multitree

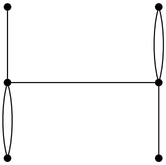


Degree conditions for multitrees?

> *Positive integers $d_1, d_2, \ldots, d_n$ are degrees of a multiforest*
> $\Leftrightarrow$ *degrees partition into two parts with equal sum*
> *I.e., Bipartite multigraph degree sequences have multiforest realizations*

- easy exercise(s), induction; switching, ...
- Get $d_1 \leq \sum_{i=1}^{n} d_i$ and even degree sum for free
- Need a little more for (connected) multitrees

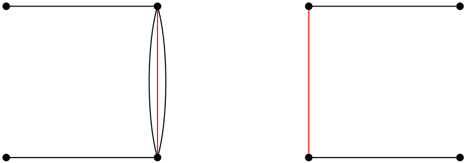
In a multiforest:
If all d_i are even then edge multiplicities are all even



- 'Proof': simple parity argument
- In general edge multiplicities are multiples of $gcd(d_1, \ldots, d_n)$
- For multiforest realizations may as well divide by $gcd(d_1, \ldots, d_n)$

Positive integers $d_1, d_2, \ldots, d_n$ that partition into two parts with equal sum realize a multitree if $\dfrac{\sum d_i}{gcd} \geq 2n - 2$

Proof: Get multiforest and use switching to get multitree



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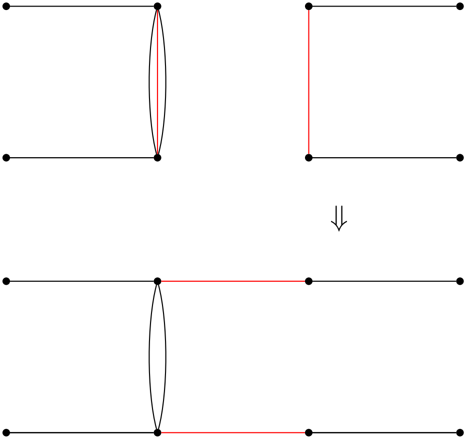

Degrees of a multigraph $d_1 \leq d_2 + \cdots + d_n$ have a realization with underlying graph a forest or a graph with exactly one cycle (which is a triangle)

Alternate Proofs:

- Induction
- Switching (Will and Hulett 2004)
- Split one degree to get degree partition $\Rightarrow$ forest $\Rightarrow$ merge to get one cycle

Positive integers $d_1, d_2, \ldots, d_n$ that partition into two parts with equal sum realize a multitree if $\frac{\sum d_i}{gcd} \geq 2n-2$

Alternate Proofs:

- Switching
- Induction with careful choice of values to reduce

Multigraph degrees result
$\Rightarrow$ Realization with at most n underlying edges

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Question

What is range of number of underlying edges for multigraph sequences?

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$\Rightarrow$ Realization with at most n underlying edges

Question

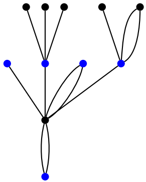
What is range of number of underlying edges for multigraph sequences?

- Realization to minimize number of underlying edges is NP-hard (Hulett, Will, Woeginger 2008)
- Realization to maximize number of underlying edges: Minimize number of 2's to add to degree sequence to get (simple) graph (Owens and Trent 1967)

Question

What are Degree Sequences of 2-multitrees ?
Each edge multiplicity 1 or 2

2-multitree



2-multiforest conditions, $d_1 \geq \ldots, \geq d_n$ with even degree sum

- If all d_i even $\Rightarrow$ edge multiplicities all 2 $\Rightarrow \frac{d_1}{2}, \frac{d_2}{2}, \ldots, \frac{d_n}{2}$ are degrees of a forest
  i.e., sum is a multiple of 4 and at most $2(2n-2) = 4n-4$

- At most 2 edges to each vertex $\Rightarrow d_1 \leq 2(n-1)$
- At least 2 'leaves' $\Rightarrow$ at least two d_i are 1 or 2
- At most $2(n-1)$ edges $\Rightarrow$ degree sum at most $4n-4$

These 3 will be implied by further conditions

More 2-multiforest conditions

- Each odd degree vertex adjacent to edge with multiplicity 1
  $\Rightarrow$ degree sum $\leq 4n - 4 - \#$odd degrees
- Remove degree 1 vertices
  $\Rightarrow$ what is left can't have too large a degree sum
  $\Rightarrow$ degree sum $\leq 4n - 4 - 2 \cdot (\#$degree 1 vertices$)$

Conditions are also sufficient

Positive integers $d_1, d_2, \ldots, d_n$ with even degree sum are degrees of a 2-multiforest $\Leftrightarrow$

- *When all d_i even:* $\sum d_i \leq 4n - 4$ *and a multiple of 4*
- *Some d_i odd:* $\sum d_i \leq 4n - 4 - \max\{n_{odd}, 2n_1\}$

Proof Version 1 Idea: Leaf Removal

Some d_i odd: $\sum d_i \le 4n - 4 - \max\{n_{odd}, 2n_1\} \Rightarrow$ *2-Multitree*

- Remove 1 or 2 from list and reduce another term
- Multiple cases to consider

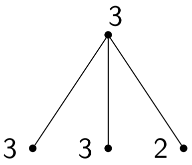
Proof Version 2 Idea: Caterpillar Construction

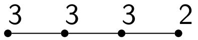
Some d_i odd: $\sum d_i \leq 4n - 4 - \max\{n_{odd}, 2n_1\} \Rightarrow$ *2-Multitree*

For Trees: Dominated Subtree on degree ≥ 2 vertices

$\Rightarrow$ add leaves

$$(3, 3, 3, 2, 1, 1, 1, 1, 1)$$





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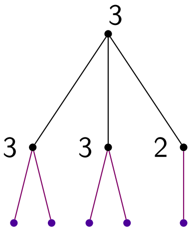

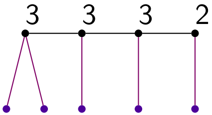

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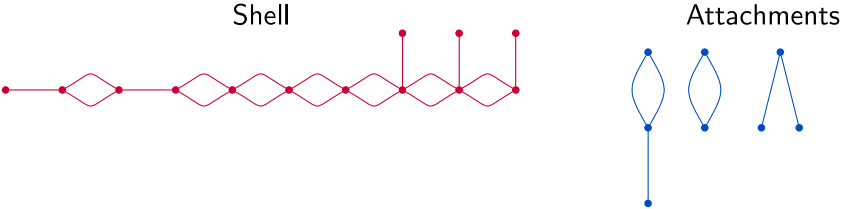
For 2 MultiTrees:



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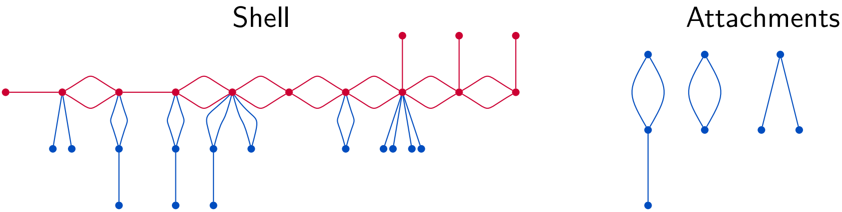

Proof Version 3 Idea: Branch Repair

Some d_i odd: $\sum d_i \leq 4n - 4 - \max\{n_{odd}, 2n_1\} \Rightarrow$ *2-Multitree*

$$(5, 4, 4, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

$4, 4, 3, 3, 2, 2$ $\quad$ $5, 2$

Proof Version 3 Idea: Branch Repair

Some d_i odd: $\sum d_i \leq 4n - 4 - \max\{n_{odd}, 2n_1\} \Rightarrow$ *2-Multitree*

$$(5, 4, 4, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

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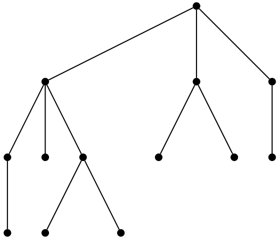
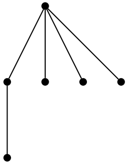


$5, 2$

$4, 2, 1, 1, 1, 1$



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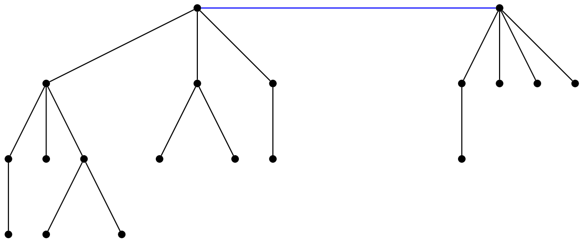
4, 4, 3, 3, 2, 2 5, 2

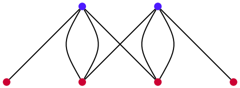
3, 4, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1 4, 2, 1, 1, 1, 1



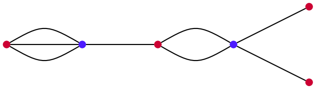
With 2-multitrees split degree $\geq$ 4 and distribute 3,2,1's

For 2-multitrees the degree partition matters

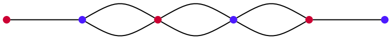
- Degree partition does not matter for trees and multitrees
- Degree partition matters for 2-multitrees and bipartite
- Similar conditions for parititon lists and 2-multitrees



$(4,4);(3,3,1,1)$
with degree bipartition
2-multibipartite graph



bipartition $(4,4);(3,3,1,1)$
3-multitree with degree

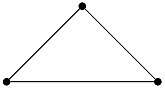


bipartition $(4,3,1);(4,3,1)$
2-multitree with degree

Question

What are Degree Sequences of 2-trees

'Build' by repeatedly attaching a 'pendent' vertex to an edge



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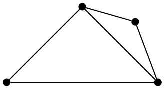

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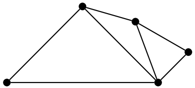

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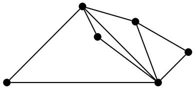

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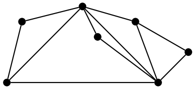

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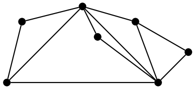

Necessary Conditions for degrees of a 2-tree

- degree sum is $4n - 6$
- $n - 1 \geq d_1 \geq \ldots \geq d_n \geq 2$
- There are at least two $d_i = 2$

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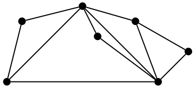
What are Degree Sequences of 2-trees

'Build' by repeatedly attaching a 'pendent' vertex to an edge



Necessary Conditions for degrees of a 2-tree

- degree sum is $4n - 6$
- $n - 1 \geq d_1 \geq \ldots \geq d_n \geq 2$
- There are at least two $d_i = 2$
- sequence is not $\left\langle \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, 2, 2, \ldots, 2 \right\rangle$
- All d_i even $\Rightarrow (\# \ d_i = 2) \geq \frac{n+3}{3}$

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Theorem (Bose, Dujmovic, Kriznac, Langerman, Morin, Wood, Wuher 2008)

Necessary and sufficient for degree sequences of 2-trees

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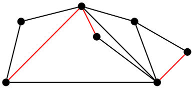
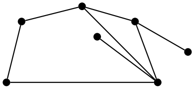
Necessary and sufficient for degree sequences of 2-trees

- If some d_i is odd 'almost always' works if degree sum is $4n - 6$
- If all d_i even need 'about' $1/3$ of the d_i to be 2

Partial 2-tree: subgraph of a 2-tree



Partial 2-tree: subgraph of a 2-tree



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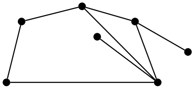

- K_4 minor free graphs
- series-parallel graphs construction :
  add pendent edge; replace edge with a path, add parallel edges

Necessary conditions for degrees of a partial 2-tree

g is the number of 'missing' edges $\Rightarrow \sum d_I = 4n - 6 - 2g$

- When $g = 0$ sequence is not $\left\langle \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, 2, 2, \ldots, 2 \right\rangle$
- $d_n \leq n - 1$
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Theorem (Ryan 2013)

Necessary and sufficient for degree sequences of partial 2-trees

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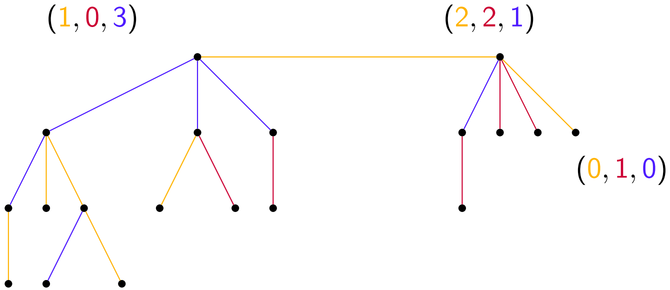
Theorem (Ryan 2013)

Necessary and sufficient for degree sequences of partial 2-trees

- When some d_i is odd condition is essentially $(\# \ d_i = 1) \leq g$
- If all d_i even $(\# \ d_i = 2) \geq \frac{n+3-2g}{3}$ holds whenever $\sum d_i \leq \frac{18}{5}(n-1)$

Question

What are degee sequences of edge colored trees?



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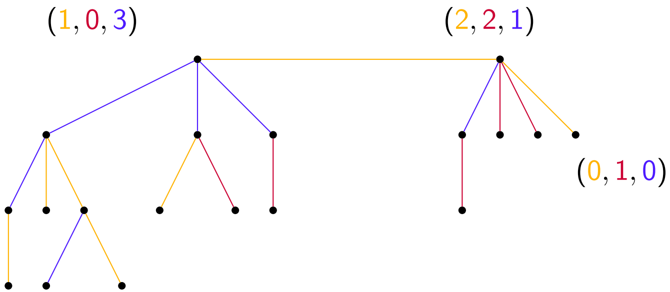
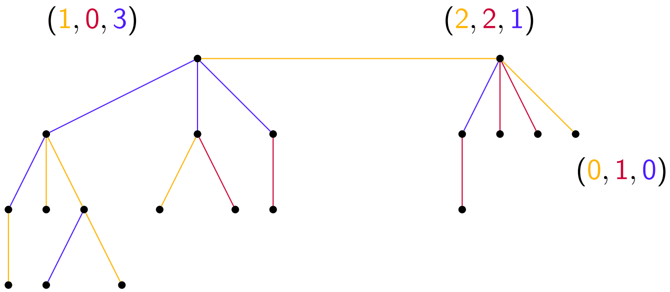


Necessary Condition:
'Collapse' each subset of colors ⇒ forest realizable

Degree sequence of edge colored tree
$\Leftrightarrow$ each subset of colors realizable as a forest



- Carroll and Isaak 2008 - inductive proof
- Alpert, Becker, Iglesius, Hilbert 2010 - extremal and switching proof
- Hillebrand and McDiarmid 2015 - extend to unicyclic with extra condition

Degree sequences of 2-edge colored graphs (degree sequence packing): a hint of some results

Assume both sequences and their sum realizable

- Realize if one color sequence has all degrees $\in \{k, k+1\}$ (Kundu's Theorem, 1973)
- Realize if both sequences and their sum can be realized by forests (Kleitman, Koren and Li, 1977)
- Realize if $\Delta_2 \geq \Delta_1$, $\delta_1 \geq 1$ and $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ (Diemunsch, Ferrara, Jahanbekam, Shook 2015)
- Realize if sequences are identical (switch to get 'nice' Eulerian cycle in these colors then alternate) (Alpert, Becker, Iglesius, Hilbert 2010)
- Checking is NP-hard (Durr, Guinez, Matamala 2009)
- ....

Degree sequences of k-edge colored graphs $k \geq 3$ (degree sequence packing): a hint of some results

Assume all sums of subsets of colors realizable

- Polynomial for fixed k and fixed maximum degree (Alpert, Beck, Hilbert, Iglesius 2010)
- n even, total degree sum is $\leq \frac{n}{2} + 1$ and all but one color constant (Busch, Ferrara, Hartke, Jacobson, Kaul, West 2012)
  e.g., Realization of the sum with all but one color a 1-factor
- Realize if complete bipartite and each color constant on one part: next ...
- k-edge colored general graphs $= k + 1$ coloring of complete graph

Is there a complete bipartite graph with given color vectors?

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

$(2, 1, 1, 0, 0)$

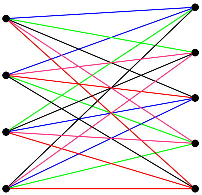
$(0, 1, 1, 0, 2)$

$(2, 0, 1, 1, 0)$

$(0, 2, 0, 0, 2)$

$(0, 0, 1, 3, 0)$

Is there a complete bipartite graph with given color vectors?

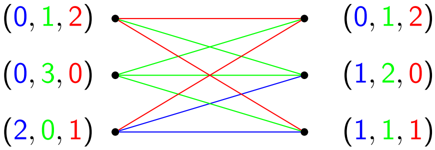
$(1,1,1,1,1)$		$(2,1,1,0,0)$
$(1,1,1,1,1)$		$(0,1,1,0,2)$
$(1,1,1,1,1)$		$(2,0,1,1,0)$
$(1,1,1,1,1)$		$(0,2,0,0,2)$
		$(0,0,1,3,0)$

YES for this instance

In general checking is NP-hard

If all $(1,1,\cdots,1)$ in one part then always a solution
i.e. a proper edge coloring in one part

Is there a complete bipartite graph with given color vectors?



Fill array to get specified margins?

R	G	R	(0, 1, 2)
G	G	G	(0, 3, 0)
R	B	B	(2, 0, 1)
(0, 1, 2)	(1, 2, 0)	(1, 1, 1)	

Fill array to get specified margins?

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$(0, 1, 2)$	$(1, 2, 0)$	$(1, 1, 1)$	

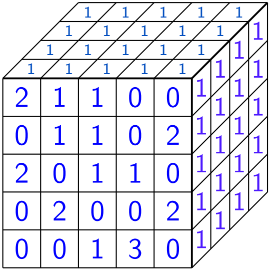
- 2-colors $=$ degree sequences of bipartite graph
- 3-colors: NP-hard (Durr et al 2009) ‘discrete tomography’
- test for degree sequence of oriented bipartite graph is NP-hard

Fill array to get specified margins?

R	G	R	$(0, 1, 2)$
G	G	G	$(0, 3, 0)$
R	B	B	$(2, 0, 1)$
$(0, 1, 2)$	$(1, 2, 0)$	$(1, 1, 1)$	

Use variable $x_{i,j,k}$

1 if entry i, j is color k
0 if not



- Contingency table - fill with $0, 1$'s to meet specified marginals
  Assume 'obvious' sum conditions
- Arbitrary marginals encodes all integer linear programming problems (DeLoera and Onn 2006)
- One face all 1's: Discrete Tomography, edge colored complete bipartite graphs ... NP-hard
- Two faces all 1's (or constant rows) then easy ....

a	b	c	e	d	(1,1,1,1,1)
a	e	d	b	c	(1,1,1,1,1)
b	c	a	e	d	(1,1,1,1,1)
c	e	a	b	d	(1,1,1,1,1)
(2,1,1,0,0)	(0,1,1,0,2)	(2,0,1,1,0)	(0,2,0,0,2)	(0,0,1,3,0)	

Question

Discrete Tomography - Can we fill array with specified margins when rows are permutations?

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

$(1, 1, 1, 1, 1)$

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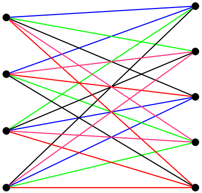


$(2, 1, 1, 0, 0)$

$(0, 1, 1, 0, 2)$

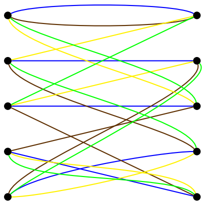
$(2, 0, 1, 1, 0)$

$(0, 2, 0, 0, 2)$

$(0, 0, 1, 3, 0)$

Question

Does a complete bipartite graph have an edge coloring with one side proper?



Array specifies edge multiplicities $\begin{array}{|ccccc|} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{array}$

Question

Does a regular bipartite multigraph have a proper coloring?

5 candidates, 4 votes rank all candidates

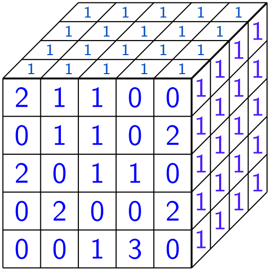
Voter 1: B, C, K, T, R
Voter 2: B, T, R, C, K,
Voter 3: C, K, B, T, R
Voter 4: K, T, B, C, R

Candidate Profile

	B	C	K	R	T
$1st$	2	1	1	0	0
$2nd$	0	1	1	0	2
$3rd$	2	0	1	1	0
$4th$	0	2	0	0	2
$5th$	0	0	1	3	0

Question

Are there votes to realize any possible Candidate profile?


1 1 1 1 1
1 1 1 1 1
1 1 1 1 1
1 1 1 1 1
2 1 1 0 0
0 1 1 0 2
2 0 1 1 0
0 2 0 0 2
0 0 1 3 0
1 1 1 1 1
1 1 1 1 1
1 1 1 1 1
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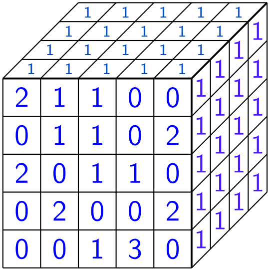

Birkhoff - Von Neumann Theorem

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} +$$

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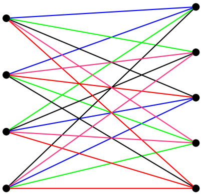

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$(0, 1, 1, 0, 2)$

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$(0, 2, 0, 0, 2)$

$(0, 0, 1, 3, 0)$

$(1, 1, 1, 1, 1)$

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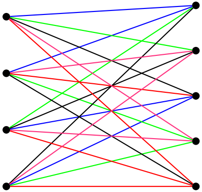

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$(0, 0, 1, 3, 0)$

$$\boxed{\begin{matrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{matrix}} = \boxed{\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}} + \boxed{\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}} + \boxed{\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}} + \boxed{\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}}$$

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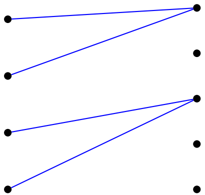

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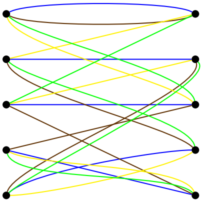

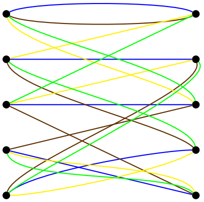

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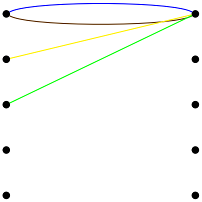

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5 candidates, 4 votes rank all candidates

Voter 1: B, C, K, T, R
Voter 2: B, T, R, C, K,
Voter 3: C, K, B, T, R
Voter 4: K, T, B, C, R

Candidate Profile

$$
\begin{array}{c|ccccc|}
 & B & C & K & R & T \\
\hline
1st & 2 & 1 & 1 & 0 & 0 \\
2nd & 0 & 1 & 1 & 0 & 2 \\
3rd & 2 & 0 & 1 & 1 & 0 \\
4th & 0 & 2 & 0 & 0 & 2 \\
5th & 0 & 0 & 1 & 3 & 0 \\
\hline
\end{array}
$$

$$
\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
+
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}
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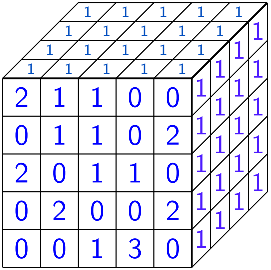
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
+
\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$



Same Problem

Different Notation

- Can we decompose an integer matrix with constant row/colum sums into permutation matrices?
- Can we fill in a 3-dimensional contingency table with 0/1's when marginals in 2 dimensions are 1's?
- Discrete Tomography - Can we fill array with specified margins when rows are permutations?
- Does a complete bipartite graph have an edge coloring with one side proper?
- Does a regular bipartite multigraph have a proper coloring?
- Are there votes to realize any possible Candidate profile?