Forbidden SEO-Induced Subgraphs

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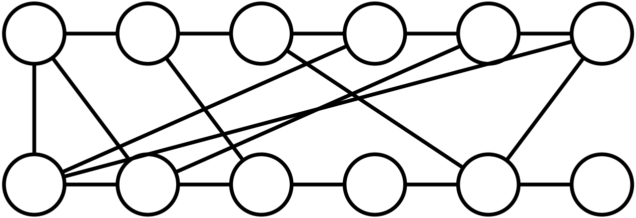
AMS Southeastern Section Meeting
Robert E. Jamison Special Session

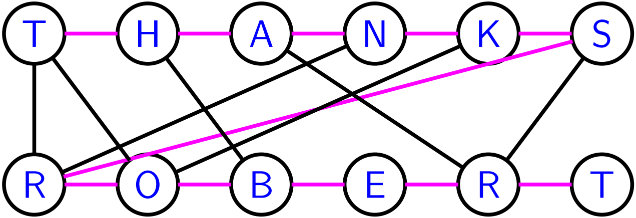
From 2014 talk:

Hamiltonian Path

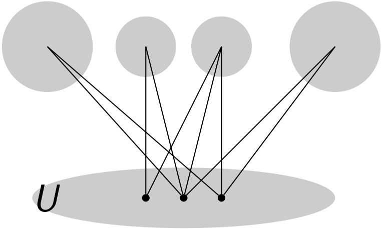


From 2014 Talk

Hamiltonian Path



Reminder of basic necessary condition for Hamiltonian Cycle

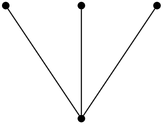


Fact (Well known)

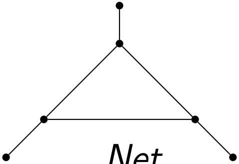
G Hamiltonian $\Rightarrow$ $C(G - U) \geq |U|$ *all non-empty U*
i.e. G 1-Tough is necessary condition

$\Longleftrightarrow$ for trees, interval graphs, co-comparability graphs,...
Nice certificate if NO Hamiltonian cycle in these classes

Reminder of basic sufficient condition for Hamiltonian Cycle



$K_{1,3}$



Net

Theorem (Duffus, Gould, Jacobson 1981)

G Hamiltonian $\Leftarrow$ G is $\{K_{1,3}, Net\}$-free (and 2-connected)

$\Longleftrightarrow$ for ?

Goal - Nice certificate if NO Hamiltonian cycle

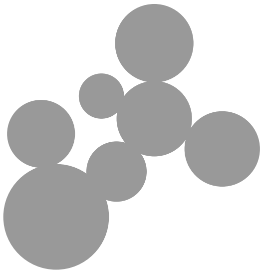
Problem - Add universal vertices ....

(Could we get something like?....)

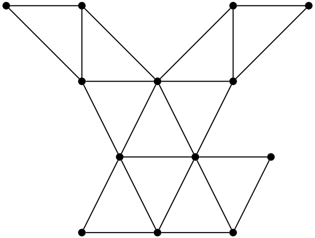
If G is a Jamison graph then
G is Hamiltonian $\Leftrightarrow$ G is $\{K_{1,3}, Net\}$-free

- Problem: G not Hamiltonian, has an induced $K_{1,3}$ or Net
  $G \vee K_t$ is Hamiltonian and still has induced $K_{1,3}$ or Net
- 'Jamison' can't be closed under adding universal vertices
- Problem: Cycles are Hamiltonian but no induced subraph is

two of the simplest graph classes beyond trees:



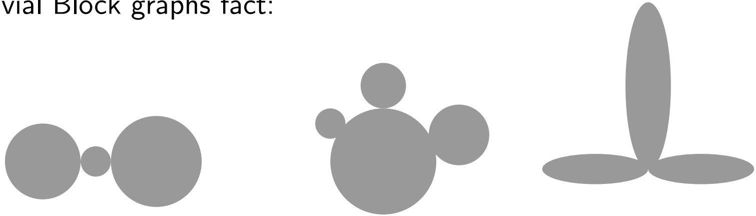
Block Graph



2-Tree

Easy to find efficient algorithm for Hamiltonian cycle, Hamiltonian path, path partition on these classes

Trivial Block graphs fact:



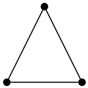
Fact

If G is a block graph

- *G is Hamiltonian $\Leftrightarrow$ G is complete*
- *$\Leftrightarrow$ G is 1-tough*
- *$\Leftrightarrow$ G is P_3-free*

Basic 2-tree fact

Special case of k-tree result of Broersma, Xiong, Yoshimoto 2007



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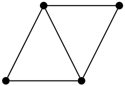

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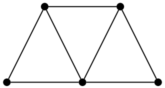

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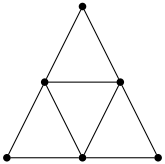


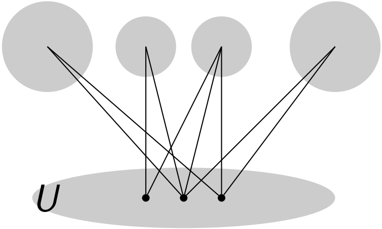
Fact (Special case of Broersma, Xiong, Yoshimoto 2007)

If G is a 2-tree

- *G is Hamiltonian $\Leftrightarrow$ G is 1-tough*
- *$\Leftrightarrow$ every edge is is in 1 or 2 triangles*
- *$\Leftrightarrow$ G is $K_2 \vee 3K_1$-free*

Unique Hamiltonian cycle on 1-edges

Reminder of basic necessary condition for Hamiltonian Path

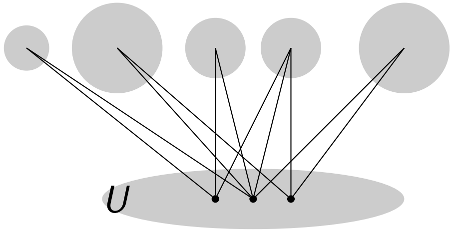


Fact (Well known)

G Hamiltonian $\Rightarrow C(G-U) \geq |U|+1$ *all* U
i.e. Scattering number 1 is necessary for traceable

$\Leftrightarrow$ for trees, interval graphs, co-comparability graphs,...
Nice certificate if NO Hamiltonian path in these classes

Reminder of basic necessary condition for Hamiltonian Path



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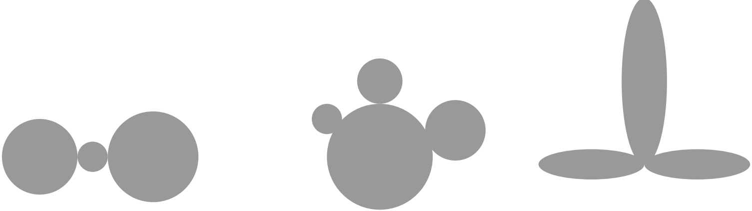
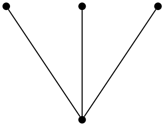


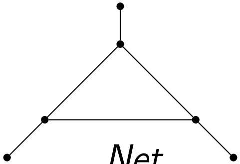
Fact

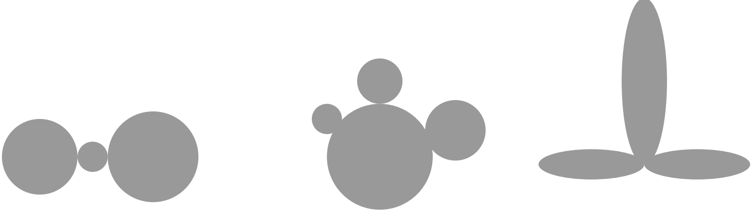
If G is a block graph

G is Traceable (has a Hamiltonian path)
$\Leftrightarrow$ *G has scattering number 1*

FAILS





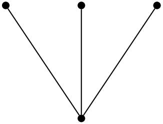


Fact

If G is a (connected) block graph

G is Traceable (has a Hamiltonian path) $\Leftrightarrow$ G is $\{K_{1,3}, Net\}$-free

i.e., Duffus, Gould, Jacobson 1981 sufficient condition for traceability is necessary in block graphs



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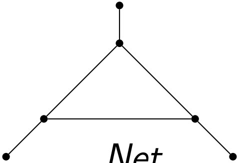
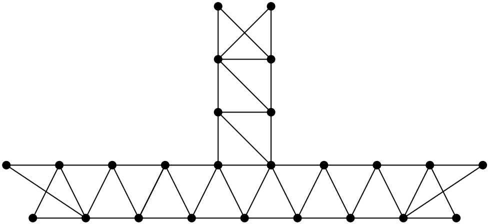


Net

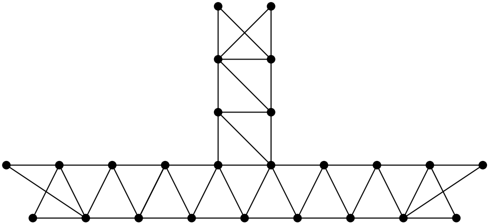
Finally to something that is not elementary



A 2-tree with scattering number 1 that is not traceable

- Find 'nice' $\Leftrightarrow$ conditions for Hamiltonian Paths in 2-trees?
- Possibly forbidden subgraphs?
-

Finally to something that is not elementary



A 2-tree with scattering number 1 that is not traceable

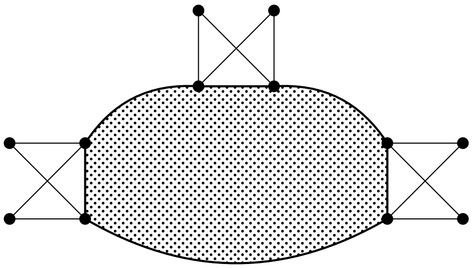
- Find 'nice' $\Leftrightarrow$ conditions for Hamiltonian Paths in 2-trees?
- Possibly forbidden subgraphs?
- Yes but a long list ....

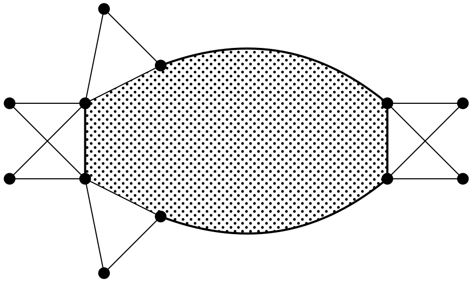
Theorem

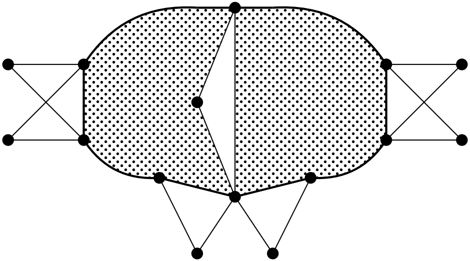
If G is a 2-tree then G has a Hamiltonian Path $\Leftrightarrow$

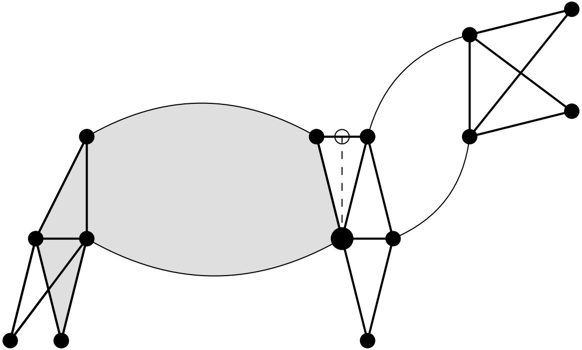
- *G has scattering number 1 and*
- *G has no induced (sub 2-tree) $H_a, H_b, \ldots, H_g$ (7 infinite families)*

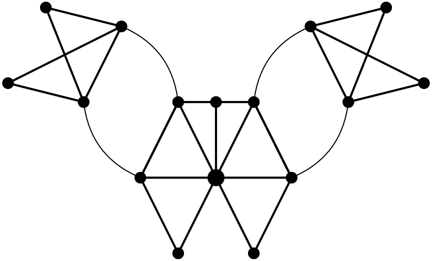
- Proof uses related results for 2-HP (Hamiltonian path is fixed endpoints)
- If traceable there can induced non-traceable subgraphs, $K_{1,3}$, *Nets* for example
- 2-tree with a non-traceable induced subgraph that is a 2-tree is not traceable
- Suggests ideas for SEO-induced subgraphs that will apply in more general setting of Chordal graphs

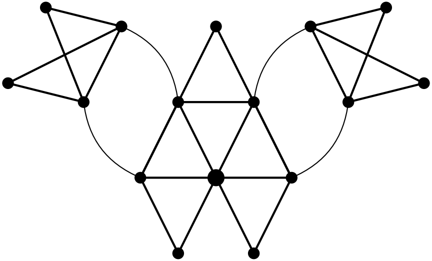


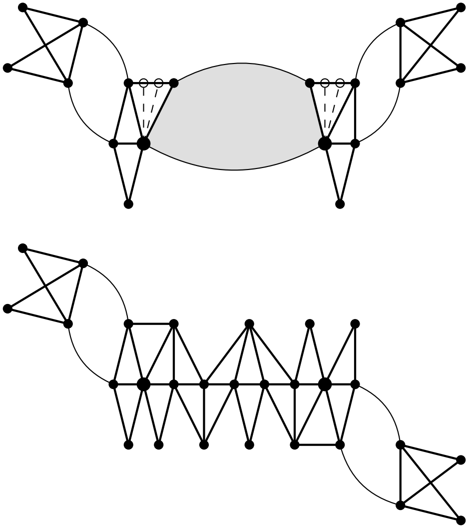








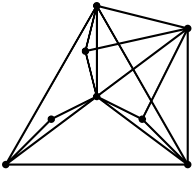




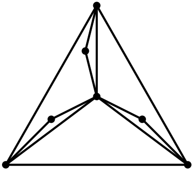
Finally to the title

H - NOT Hamiltonian induced subgraph of Hamiltonian G

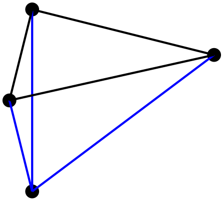
G



H



Build G with reverse of Simplicial Elimination Ordering (SEO):



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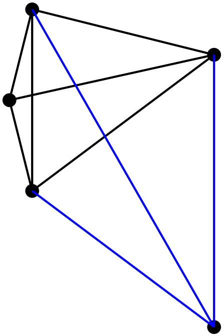

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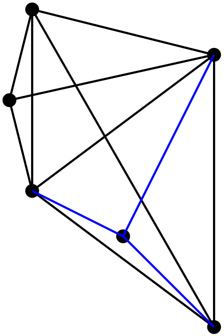

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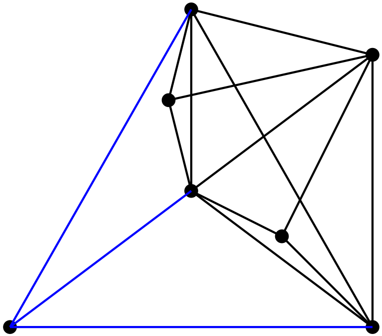

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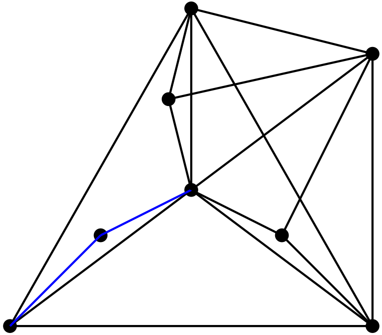

G is a 2,3-Tree (induced subgraph of a treewidth 3 chordal graph)

Built joining new vertices to 2 or 3 cliques

Consider only subgraphs in a building sequence for forbidden list

Fact (Elementary)

Let G be a chordal graph built with a reverse SEO ordering as
$G_0, G_1, G_2, \ldots, G_t = G$
Then G Hamiltonian $\Rightarrow$ each G_i is Hamiltonian

Constructing G by adding simplicial vertices cannot 'patch' a non-Hamiltonian graph into a Hamiltonian graph

Fact (Restated)

If Chordal G is not Hamiltonian then some G_i
in a reverse SEO building sequence is not Hamiltonian
and minimal with respect to this property.

If $\mathcal{G}$ is a family of chordal graphs
(closed under SEO induced)
then hope for a nice theorem like:
G Hamiltonian $\Leftrightarrow$ no SEO induced graph from $\mathcal{F}$

- Produces a certificate when G is not Hamiltonian
- SEO induced subgraph idea behind proof for forbidden subgraphs for traceable 2-trees
- same idea works for traceable, 1HP, path partition
- Find such for cases where there is an efficient algorithm but no nice characterization/certificates
  e.g., Ptolemaic graphs, 1HP in interval graphs, ...