Degrees and 2-Multitrees

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Abstract

We examine degree characterizations for 2-multitrees. These are multigraphs with underlying tree structure and at most 2 copies of each edge. We provide characterizations for both when a degree bipartition is given and when it is not given.

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1 Introduction

A 2-multigraph is a graph for which each edge multiplicity is 1 or 2. We will call a 2-multitree or 2-multiforest such a graph in which the underlying graph is a tree or forest. We will characterize degree sequences for these. While the proofs are reasonably straightforward, it is interesting to see how 'standard' proofs for characterizations of degree sequences of trees extend to this setting. Hence we review three different proofs characterizing degree sequences of trees and forests and then provide parallel proofs for 2-multitrees and forests.

Unlike degree sequences for trees we get different conditions depending on whether or not a degree partition is given. Given the degree sequence of a tree and any partition of this sequence into 2 parts with equal sums, there is a tree with a bipartition such that the degrees of the parts correspond to the the given partition. The same does not hold for 2-multitrees. We will also give conditions for a partitioned sequence to be the degrees of a 2-multitree with corresponding bipartition. Also, unlike the case for trees, for a given degree sum we can get 2-multiforest realizations with different numbers of components. We also include discussion of the number of components the forest realization may have.

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1.1 Review of basic definitions

We start with notation and basic definitions including brief comments on basic definitions to make this self contained even with minimal exposure to graph theory. See for example standard introductory texts such as [1], [3] for more details on graph theory.

A k-multigraph is a vertex set V along with a multiset E of pairs of distinct vertices called edges with each edge appearing at most k times. We will not consider loops, which would correspond to an edge which is a size 2 multiset, with the same vertex twice. A graph is the case $k = 1$ and a multigraph is the case where there is no bound on the number repeated edges (i.e., $k = \infty$). The underlying graph for a k-multigraph is the graph obtained by keeping one copy of each edge.

A path is a sequence of distinct vertices $v_1 v_2, \ldots, v_t$ with $v_i v_{i+1}$ an edge for each i. A cycle is as a path except $v_1 = v_t$ and all other vertices distinct. A graph is connected if there is a path between each pair of vertices. Components are maximal connected subgraphs. A forest is a graph with no cycles and a tree is a connected forest. A graph is bipartite if V partitions into two sets with every edge having one end in each part. Forests are bipartite. A k-multibipartite graph, k-multiforest, k-multitree has underlying graph that is bipartite, forest or tree respectively.

The degree of a vertex is the number of edges it is contained in. The degree sequence of a k-multigraph is the sequence of vertex degrees. Usually we will assume nonincreasing order $d_1 \geq d_2 \geq \cdots \geq d_n$. A basic result, often called the handshaking lemma is that degree sequences have even sum (as each edge contributes to 2 vertex degrees). A realization of a sequence is a k-multigraph with the given sequence as its degree sequence.

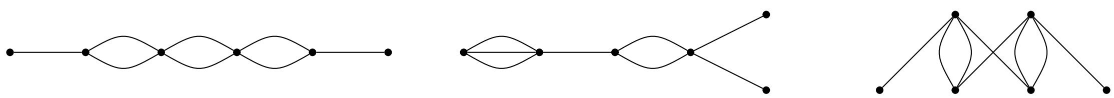
For degree sums we will write $\sum d_i$ for $\sum_{i=1}^{n} d_i$ when there is no chance of confusion. For a sequence we will use the notation n_j to denote the number of $d_i = j$. While technically forests might contain components consisting of a single vertex and no edges, for convenience in notation we will usually omit these trivial cases and assume all vertex degrees are positive.

The degree bipartition of a bipartite graph is the degree sequence partitioned into two parts corresponding to the vertex bipartition. The parts have equal sum. We will refer to a partitioned sequence as a pair of integer sequences with equal sums and an unpartitioned sequence as an arbitrary sequence.

Characterizations of degree sequences for k-multigraphs, k-multibipartite graphs (including the $k = \infty$ cases of multigraphs and bipartite multigraphs) as well as forests and trees (the $k = 1$ case) are well known. In another paper [2] we examine the straightforward characterization of multitrees and multiforests ($k = \infty$ case) and observe that every degree sequence of a multibipartite graph has a realization as a mulitiforest. This paper examines the $k = 2$ case for forests and trees.

1.2 Partitioned vs. unpartitioned sequences

Consider the sequence $4, 4, 3, 3, 1, 1$. This can be realized as a 2 multitree (indeed as a 2-multipath) by taking a path on 6 vertices and making the three internal edges have multiplicity 2.



2-multitree with degree bipartition $(4, 3, 1); (4, 3, 1)$

3-multitree with degree bipartition $(4, 4); (3, 3, 1, 1)$

2-multibipartite graph with degree bipartition $(4, 4); (3, 3, 1, 1)$

Figure 1: Realizations of $(4, 4, 3, 3, 1, 1)$

Note that the degree bipartition has parts $4, 3, 1$ and $4, 3, 1$. Consider the sequence partition with parts $4, 4$ and $3, 3, 1, 1$, having equal sum. It is straightforward to check that there is no 2-multitree or 2-multiforest realization with this equal sum partition. For $4, 4$ and $3, 3, 1, 1$, there is a bipartite 2-multigraph realization with this degree partition and there is a multiforest realization with this degree partition but it uses edges of multiplicity 3. See Figure 1 for examples of these realizations.

In Section 4 we will give necessary and sufficient conditions for realizability of an unpartitioned sequence as a 2-multiforest and in Section 5 we consider the case where the sequence is partitioned into two parts with equal sum.

2 Review of some tree proofs

Note first that as trees on n vertices have $n - 1$ edges the degree sum is $2(n - 1)$ from the handshaking lemma. So

$$\sum d_i = 2n - 2 \tag{1}$$

is a necessary condition for degree sequences of trees. This condition is also sufficient:

Fact 2 (Degree sequences for Trees) *For $n \geq 2$, positive integers $d_1, d_2, \ldots, d_n$ are the degrees of a tree if and only if $\sum d_i = 2n - 2$.*

We will provide quick proof sketches of three standard proofs of sufficiency. Each will then have a parallel proof in the 2-multitree setting.

While the third proof below is somewhat longer for trees and is very close to the second but with different bookkeeping, we present it as its analog in the 2-multi-tree setting is fairly short. Each proof makes different additional properties more obvious. As these proofs are elementary and well known, we do not provide details, only a sketch of the ideas.

We will sketch three proofs of sufficiency after observing a rewrite of the condition and a connection to forests.

2.1 Alternative version of characterization equation

We observe an alternate way of writing $\sum d_i = 2n - 2$, that will be useful for two of our proofs. We use the idea that the average degree is about 2. Let N_{2+} denote the set of indices i with $d_i \geq 2$ and let n_1 denote the number of $d_i = 1$. Then the degree condition of equation 1 becomes

$$2 + \sum_{i \in N_{2+}} (d_i - 2) = n_1 \tag{3}$$

The 'excess' for degrees more than 2 equals two less than the number of 1's. We can view this as indicating how many 1's to add to a sequence of numbers greater than 1 to get the degree sequence of a tree.

Observe that this also makes clear the role of degree 2 vertices. To any sequence realizable as a forest we can remove or add an arbitrary number of 2's and still have a realizable sequence. This corresponds in a realization to either replacing an edge with a path, i.e., subdividing an edge, or reversing this process. As there are no 3 cycles, the reverse process does not create multiple edges.

We observe that Equation (3) holds more generally if the sum includes some values equal to 1 and the right side counts the number of 1's not included in the sum. That is, if B is a set of indices that includes all $d_i \geq 2$ and possibly some $d_i = 1$ then Equation (1) becomes

$$2 + \sum_{i \in B} (d_i - 2) = n - |B|. \tag{4}$$

2.2 Forest characterization from tree characterization

It is not difficult to extend the characterization of degree sequences of trees to degree sequences of forests using the same proof ideas. However it is more convenient to give the proofs for trees and note the forest version as an easy corollary of the tree version.

Fact 5 (Degree sequences for Forests) *For $n \geq 2$, $t \geq 0$, positive integers $d_1, d_2, \ldots, d_n$ are the degrees of a forest with $t+1$ components if and only if $\sum d_i = 2n - 2 - 2t$.*

Moreover there is a realization with t components each of which is a single edge.

Proof sketch: If the sequence has sum $2n - 2 - 2t$ then there are at least $2t + 2$ ones. Remove $2t$ ones from the sequence to get a sequence with $n - 2t$ terms and sum $2n - 2 - 2t - 2t = 2(n - 2t) - 2$. By Fact 2 we can construct a tree using the remaining terms. Then add t disjoint edges to realize the original degree sequence. □

2.3 Proof 1 of sufficiency in Fact 2: Leaf removal

Use induction on n, the sequence size. The basis is the sequence $1, 1$ realized by an edge. A sequence of $n \geq 3$ positive integers with sum $2n - 2$ contains at least one 1 and at least one $d_i \geq 2$. Remove a 1 and replace some $d_i \geq 2$ by $d_i - 1$. The new sequence has sum $2(n - 1) - 2$, so by induction can be realized by a tree. Add a new vertex with an edge incident to a vertex with the degree $d_i - 1$ to obtain a tree with the specified degrees. □

We call this leaf removal despite the fact that we actually remove a number, 1, from the sequence, use induction and then add a leaf. We use the term removal as the underlying idea is that the tree we construct is built from one with a leaf removed.

Note that this proof shows that given any degree at least 2, there is a realization in which a vertex with that degree is incident to a leaf. Indeed, by strengthening the hypothesis it is easy to see there is a realization with a vertex of a given degree $d \geq 2$ incident to $d - 1$ leaves.

Finally this proof shows that given any partition of the sequence into two parts with equal sums there is a realization in which the partition corresponds to the degrees in the parts of a bipartition since for any partition with equal sums we can pick a 2 and a 1 from different parts (except for the sequence $1, 1$).

Observe also that if we implement the induction as a recursive algorithm and repeatedly pick the same index until it reduces to 1 we will construct the same tree as in the caterpillar construction below.

2.4 Proof 2 of of sufficiency in Fact 2: Caterpillar construction

A caterpillar is a tree for which deleting the leaves leaves a path.

If n_1 of the d_i are 1, construct a path with $n - n_1$ vertices. Arbitrarily assign each $d_i \geq 2$ to a vertex on the path and add $d_i - 2$ leaves to the corresponding vertex if it is not a path end and add $d_i - 1$ leaves to the ends of the path. This creates a caterpillar with the correct degrees on the path and the correct number of degree 1 vertices $n_1 = 2 + \sum_{i \in N_{2+}} (d_i - 2)$ by Equation (3). $\square$

The word caterpillar used in a different context also relates to leaf removal.

Note that this proof shows that given any two degree that are at least 2, there is a realization in which vertices of these degree are adjacent. It also shows that there is a realization with a vertex of a given degree $d \geq 2$ incident to $d - 1$ leaves by placing such a degree on the end of the initial path in the construction. Finally, it shows, by appropriate placement along the path, that for any partition of the degrees that are at least 2 there is a realization with an edge whose removal creates two components with the degrees (that are at least 2) in the components corresponding to the given partition.

Observe also that instead of starting with a path, we can use any tree on $n - n_1$ vertices with the degree of the i^{th} vertex at most d_i for $i = 1, 2, \ldots, n - n_1$. Using a path to start is convenient as the degree property is obvious.

2.5 Proof 3 of sufficiency in Fact 2: Branch repair

We first give a small example. $5, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1$ satisfies the conditions. Partition the values greater than 1 into 2 nonempty part arbitrarily, say $5, 3, 2$ and $3, 2$. Pick a value in each to reduce by 1, for example to get $4, 3, 2$ and $3, 1$. Then spread the seven 1's to make each a tree sequence, $4, 3, 2, 1, 1, 1, 1, 1$ and $3, 1, 1, 1$. By induction constructs trees with these values then add an edge between the degree 4 vertex and one of the degree 1 vertices in the other part.

Use induction on the number of $d_i \geq 2$. For the basis the sequence is $1, 1$ or $d, 1, 1, \ldots, 1$ where $d \geq 2$. Realize $1, 1$ as an edge and $d, 1, 1, \ldots, 1$ as a star, a tree with a vertex of degree d adjacent to d leaves.

Partition N_{2+}, the indices of degrees that are at least 2, into two nonempty parts L and R. Add $2 - 1 + \sum_{i \in L} (d_i - 2)$ ones to L and replace some d_i with $d_i - 1$ and similarly for R. This is a partition of the original list with 2 of the $d - i$ reduced by 1 as, by equation (3), $n_1 = 2 + \sum_{i \in N_{2+}} (d_i - 2) = (1 + \sum_{i \in L} d_i) + (1 + \sum_{i \in R} d_i)$. Each sequence clearly satisfies the degree condition (4) so inductively realize as a tree and add an edge between a vertex in each part having the degree that was reduced by 1. □

Note that this proof shows that for any partition of all degrees into parts with equal sum there is a realization with an edge whose removal creates two components with the degrees (that are at least 2) in the components corresponding to the given partition.

3 2-Multiforests when all degrees are even

For degree sequences of 2-multiforests we will get different conditions depending on whether or not some degree is odd. When all degrees are even a characterization follows immediately from characterizations for forests. In this case there is no difference between unpartitioned and partitioned sequences. In addition, the degree sum determines exactly the number of components in a 2-forest realization. When some degree is odd there will be a range for the number of components and as observed earlier, realizability may depend on the partition.

We start by observing an elementary condition when all degrees are even. A general version for degrees of multitrees with arbitrary multiplicity using the greatest common divisor of the degrees is also noted in [2].

Fact 6 *If G is a 2-multiforest with all degrees even then all edge multiplicities are two.*

Proof: If all degrees are even then the edge multiplicities for edges adjacent to degree 1 vertices in the underlying forest are 2. Removing all such we get a forest in which all degrees are even. Inductively all remaining edge multiplicities are even. □

Thus a sequence of even integers is the degree sequence of a 2-multitree or 2-multiforest if and only if the sequence obtained by halving each term is the degree sequence of a tree or forest. As degree sequences of trees and forests have even sum, sequences of even integers are degree sequences of 2-multiforests when the degree sum is at most $4n - 4$ and is a multiple of 4:

Fact 7 *For $n \geq 2$, $t \geq 0$, positive even integers $d_1, d_2, \ldots, d_n$ are the degrees of a 2-multiforest with $t + 1$ components if and only if $\sum d_i = 4n - 4 - 4t$.*

Moreover there is a realization with t components each of which is a pair of vertices joined by 2 parallel edges.

Proof: Immediate from Facts 5 and 6. □

4 2-Multiforests when some degree is odd

When some degree is odd, we start by noting some obvious necessary conditions. We will need additional conditions which will turn out to imply the obvious conditions. However, it is worth commenting on the obvious necessary conditions.

Once we establish the slightly less obvious necessary conditions we will give three proofs which will parallel the three tree proofs, albeit with more cases for the 2-multitree case.

For sequences with some term odd we start by noting three obvious conditions for degree sequences of multiforests.

As the underlying graph has at most $n-1$ edges there are at most $2n-2$ edges in the multigraph so the degree sum is at most $2(2n-2) = 4n-4$. As each vertex is adjacent to at most $n-1$ other vertices with edge multiplicity at most 2, the maximum degree is at most $2(n-1) = 2n-2$. As the underlying forest has at least two vertices of degree 1 and these have degree 1 or 2 in the multiforest, there are at least 2 degrees that are 1 or 2.

Observe that the analogs of the second two conditions for trees, the maximum degree is at most $n-1$ and at least two degrees are 1 do not need to be stated in the degree conditions for trees as they follow from being a sequence of positive integers with sum at most $2n-2$. For 2-multitrees the corresponding conditions do not immediately follow from the upper bound on the degree sum but it is reasonably straightforward to show that they follow from the more general necessary conditions that we will now describe.

Fact 8 *Let G be a 2-multiforest on $n \geq 3$ vertices with degrees $d_1 \geq d_2 \cdots d_n \geq 1$ at least one of which is odd. Let n^o denote the number of odd d_i and n_1 denote the number of $d_i = 1$. Then*

1. $\sum_{i=1}^n d_i \leq 4n-4-2n_1$
2. $\sum_{i=1}^n d_i \leq 4n-4-n^o$

Proof: For the first condition, when some $d_i \geq 2$, the subgraph induced by vertices with degree at least two is a 2-multiforest on $n-n_1$ vertices. This has at most $2(n-n_1-1)$ edges. There are at most n_1 edges with one end a vertex with degree 1. Thus there are at most $2(n-n_1-1)+n_1 = 2n-2-n_1$ edges and the degree sum is twice the number of edges. If all $d_i = 1$ it is easy to check that the inequality holds.

For the second condition observe that every odd degree vertex must be incident to at least one edge with multiplicity one and each such single edge is incident to at most

2 odd degree vertices. Hence at least $\frac{n^o}{2}$ edges are 'missing' from the possible $2n-2$ edges and the degree sum is twice the number of edges, which is at most $2n-2-\frac{n^o}{2}$. $\square$

In the case that $n=2$, the sequence 1,1 is realizable (as a single edge) even though the second condition in Fact 8 fails. This will play a role when we determine the number of possible components for realizations.

Recall that Fact 7 covers the case when all d_i are even. The necessary conditions above are sufficient when some d_i is odd.

Theorem 9 *For $n \geq 3$, positive integers $d_1, d_2, \ldots, d_n$ with even sum which are not all even are the degrees of a 2-multiforest if and only if*

$$\sum_{i=1}^{n} d_i \leq 4n-4-\max\{n^o, 2n_1\} \tag{10}$$

where n^o denotes the number of odd d_i and n_1 denotes the number of $d_i = 1$. Moreover, if $\sum_{i=1}^{n} d_i \geq 2n-2$ there is a 2-multitree realization.

Observe that these conditions do imply that at least 2 values are 1 or 2. Using n_j for the number of $d_i = j$: $n_1 + 2n_2 + 3n_3 + 4(n-n_1-n_2-n_3) \leq \sum_{i=1}^{n} d_i \leq 4n-4-n^o \leq 4n-4-(n_1+n_3) \Rightarrow n_1+n_2 \geq 2$.

We will first note an alternative way to write Inequality (10) that will be useful for our proofs. We will then observe how we can get the result in general from the result when equality holds in Inequality (10) and discuss the range of numbers of components that can be realized. Finally we provide three proofs sufficiency of this condition for a 2-multitree realization that parallel the proofs sketched for trees.

4.1 Alternate version of characterization inequality

We will use the following notations. N_{4^+} denotes the set of indices i with $d_i \geq 4$, $n^o_{k^+}$ denotes the number of indices i with $d_i \geq k$ and odd and $n^e_{k^+}$ denotes the number of indices i with $d_i \geq k$ and even.

It is convenient to rewrite the Inequality (10) in Theorem 9 in a manner similar to what we did for degree sequences of trees in Equation (3). This is motivated by the the fact that the average degree is about 4. An easy computation shows that the following is equivalent to Inequality (10) in Theorem 9.

$$4 + \sum_{i \in N_{4+}} (d_i - 4) + \max\{n_3 + n_{5+}^o, n_1\} \leq n_3 + 2n_2 + 2n_1 \tag{11}$$

Hence we have two inequalities depending on which term yields the maximum.

$$4 + \sum_{i \in N_{4+}} (d_i - 4) + n_{5+}^o \leq 2n_2 + 2n_1 \text{ if } n_1 \leq n_{3+}^o \tag{12}$$

$$4 + \sum_{i \in N_{4+}} (d_i - 4) \leq n_3 + 2n_2 + n_1 \text{ if } n_1 \geq n_{3+}^o \tag{13}$$

Observe that in the case that we include some values that are 2 or 3 in the sum on the left and adjust the right side values to count the number of 2's and 3's not included in the sum these are still valid.

We can view Inequalities (12) and (13) as indicating how many 'small' (at most 3) terms are necessary for the sequence to be that of a 2-multiforest.

Observe also that the inequalities are valid when all d_i are even. Indeed if all d_i are even and equality holds in (12) and (13) then $\sum d_i = 4n - 4$ and the sequence is realizable as a 2-multitree by Fact 7. We will make use of this in our proofs to avoid the need to check that a 'new' sequence has odd terms.

The rewritten inequality also makes clear the role of degree 4 vertices in 2-multitrees. To (almost) any sequence realizable as a 2-multiforest we can remove or add an arbitrary number of 4's and still have a realizable sequence. This corresponds in a realization to either replacing an edge with multiplicity 2 with a path having all edges with multiplicity 2 or reversing this process. If there are no multiplicity 2 edges replace a vertex with degree at least 2 with an edge with multiplicity 2 having one end adjacent to two edges (creating the new degree 4 vertex) and the other end adjacent to the remaining edges. If all degrees are 1, we can replace a pair of edges by a star with a central vertex adjacent to 4 leaves to add a degree 4 vertex. Each of these can also be reversed. The only case where we cannot add a degree 4 vertex is the sequence 1,1 corresponding to an edge which fails the necessary conditions which apply when $n \geq 3$.

In addition, as long as $n_1 \leq n_{3+}^o$ we see that degree 3 vertices (in pairs to maintain an even number of odd degree vertices) can added or removed as such vertices play no role in Equation (12). An edge of multiplicity 1, which exists since some degrees are odd, can be replaced with a path which alternates edge multiplicity 1 and 2 having multiplicity 1 edges on both ends. Reversing, i.e., removing pairs of degree 3 vertices if $n_1 \leq 2 + n^o_{3+}$ does not have an obvious construction but from equation (12) we do see that we can delete 3's from a degree sequence of a 2-multitree with $n_1 \leq n^o_{3+}$ and still have a 2-multitree realizable sequence.

4.2 2-Multiforests and numbers of components

As noted above we will, in later sections, give proofs for the case where equality holds in Inequality (10). For now we use Fact 7 and assume that if equality holds in Inequality (10) there is a connected realization except for the case $n = 4$ with the sequence $(1, 1, 1, 1)$.

We will perform reductions on the initial sequence until we get a new sequence where equality holds. Thus along with the assumption of the previous paragraph we will be able to get a connected realization of the new sequence except possibly when the new sequence is $(1, 1, 1, 1)$. However, it is straightforward to check that the reductions will not result in this sequence.

If there is a strict inequality in Inequality (10), we proceed as follows. Let $g = 4n - 4 - \max\{n^o, 2n_1\} - \sum_{i=1}^{n} d_i$ denote the gap in the inequality. Note that both sides of the inequality in the condition are even so the gap g is even. Let $t = \lfloor \frac{g}{4} \rfloor$.

Removing a 2 from the sequence or removing a 1 and a 3 together reduces the gap by 2. If $n_1 > n^o_{3+}$ (there are more 1's than larger odd terms), then removing a pair of 1's reduces the gap by 2.

Case 1: $n_1 \leq n^o_{3+}$. Remove 2's and 3,1 pairs as described above until the gap is 0. Observe that if $n_2 = n_3 = 0$ and Inequality (12) holds then, as the left side is at least $4 + 2n^o_{5+}$ and the right side is $2n_1$, we would have $n_1 > n^o_{3+}$, a contradiction. So we can remove 2's and 3,1 pairs to get equality. By removing 2's first it is straightforward to check that we will not end with a sequence having all terms even. The number of removals is $\frac{g}{2} \geq 2t$.

Realize the remaining sequence as a 2-multitree. Then, for any given $s \in \{0, 1, \ldots, t\}$ we construct a realization of the original sequence with $s + 1$ components as follows. Add s components that have one of the degree sequences $(3, 2, 1)$, $(3, 3, 1, 1)$, $(2, 2)$ corresponding to degrees from removal of a 2 and a 3,1 pair, two 3,1 pairs and two 2's respectively. These are trivial to realize. For any remaining 2's and $3, 1$ pairs we replace an edge with multiplicity 1 (which exists as some vertex degree is odd) with a path having as many internal vertices as the number of 2's and 3's then add pendent edges to a vertex of degree 1 for those corresponding to a 3.

Thus we can get realizations with from 1 to $t + 1$ components in this case.

Case 2: If $n_1 > n^o_{3+}$, let $2r = n_1 - n^o_{3+}$.

Case 2a: If $g \leq 2r$ then remove g 1's. Realize the remaining sequence as a 2-multitree. This is possible even if the remaining sequence has all even terms as in this case the sum is 4 times the number of terms in the new sequence minus 4. Add $\frac{g}{2}$ isolated edges to get a realization with $\frac{g}{2}+1$ components. Note that this is $2t$ or $2t+1$ components depending on g modulo 4.

Case 2b: If $g > 2r$ then remove $2r$ 1's to get a new sequence with $n_1 = n^o_{3+}$. If some odd terms remain, proceed as in Case 1, using a gap of $g' = g - 2r$ and $t' = \lfloor \frac{g'}{4} \rfloor$. We can construct 2-multitrees with $s \in \{1, 2, \ldots, t'+1\}$ components from Fact 7 or Case 1 and add r isolated edges to get 2-multitrees with between $r+1$ and $r+t'+1$ components.

In this case we might have a sequence with all even terms after removing the 1's. We will remove $\frac{g'}{2}$ 2's and realize the remaining sequence as a 2-multitree by Fact 7. We add r isolated edges and then proceed with the $\frac{g'}{2}$ 2's as in Case 1, using one of the r edges as an edge with multiplicity 1.

To complete Case 2 we need to consider minimizing the number of components. If $\sum d_i \geq 2n-2$ we will be able to get 1 component and any number up to the maximums noted above. If $\sum d_i = 2n-2-2u$ we need at least $u+1$ components as this is the minimum number of components on any graph with this degree sum. We will be able to get $u+1$ components and any number up to the maximums noted above. We can easily get $u+1$ realizing as a forest by Fact 5 but will need to do a little more to get all values in the range we have described.

If there is an isolated edge uv and an edge xy with multiplicity 2 (in some other component). Then removing edges uv and xy and adding edges ux and vy produces a new 2-multitree with one less component. Call this an edge switch. We can repeatedly perform edge switches until either there are no remaining isolated edges or there are no remaining edges of multiplicity 2. It is straightforward to check using the handshaking lemma that the numbers of edges with multiplicity 2 are correct to obtain the ranges described in the previous paragraph.

To summarize informally, we have described how to construct realizations where the number of components is 1 plus the one fourth the gap plus up to an additional one fourth the gap depending on the number of 'excess' 1's. Below we will show that it is impossible to have more components. The minimum number of components is 1, i.e., there is a connected realization, if the degree sum is at least $2n-2$ and if the degree sum is smaller, the minimum number of components is the same as the number of components for a regular forest. Note that this is only when some degree is odd. If all degrees are even there is only one possible value for the number of components as in Fact 7. We state this as follow. The details for converting the values from the cases omitted above are omitted and straightforward.

So assuming the results we will show below, that there is a connected realization when equality holds in the condition for Theorem 10, we have already described all but the upper bounds on components in the following.

Fact 14 *Let positive integers $d_1, d_2, \ldots, d_n$, not all even, satisfy the conditions of Theorem 9 to be realized as a 2-multiforest. There are 2-mulitforest realizations with the number of components between the minimum and maximum values noted below.*

If $\sum d_i \geq 2n-2$ there is a connected realization. The minimum number of components is 1.
If $\sum d_i = 2n-2-2t$ the minimum number of components is $t+1$.

Let $g = 4n-4-\max\{n^o, 2n_1\} - \sum_{i=1}^{n} d_i$ and let n^o_{3+} denote the number of odd $d_i \geq 3$ and n_1 denote the number of $d_i = 1$. Also let $\delta = 1$ if both g and $n_1 - n^0_{3+}$ are 2 modulo 4 and $\delta = 0$ otherwise.

The maximum number of components is:
$1 + \lfloor \frac{g}{4} \rfloor$ *if* $n_1 \leq n^o_{3+}$.
$1 + \lfloor \frac{g}{4} \rfloor + \lfloor \frac{n_1 - n^0_{3+}}{4} \rfloor + \delta$ *if* $n_1 > n^o_{3+}$ *and* $n_1 - n^0_{3+} < g$.
$1 + \frac{g}{2}$ *if* $n_1 > n^o_{3+}$ *and* $n_1 - n^o_{3+} \geq g$.

We outline a proof to show the maximum bounds on number of components.

Consider first a realization T with c components $F_1, F_2, \ldots, F_c$, none of which is a single edge with degrees $1, 1$. Each component has a vertex with degree at least 2. Add a 2-edge between a pair of vertices with degree at least 2 for each F_i, F_{i+1} to form G'. G' is a 2-multitree with the same n, n^o, n_1 as G and $\sum d'_i = \sum d_i + 4(c-1)$. Since $4(c-1) + \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i \leq 4n - 4 - \max\{n^o, 2n_1\}$ we have $g \geq 4(c-1)$. So $c \leq \lfloor \frac{g}{4} \rfloor + 1$.

When $n_1 \leq n_{3^+}$ the previous construction and bound hold even if some of the F_i are a single edge as including these only decreases n_1 and thus does not change $\max\{n^o, 2n_1\}$.

If $n_1 > n^0_{3+}$ consider any realization and let c_1 be the number of components that are a single edge. The subgraph after omitting these edges has gap $g' = g - 2c_1$ if $2c_1 \leq n_1 - n^o_{3+}$. So by the previous paragraph this subgraph has at most $\lfloor \frac{g-2c_1}{4} \rfloor + 1$ components and the original graph has at most $c_1 + \lfloor \frac{g-2c_1}{4} \rfloor + 1 = \lfloor \frac{g+2c_1}{4} \rfloor + 1$ components. As the gap must stay positive $2c_1 \leq g$ so $c_1 \leq \frac{g}{2}$ and we get the bound $1 + \frac{g}{2}$ when $n_1 > n^o_{3+}$ and $n_1 - n^o_{3+} \geq g$.

If $n_1 > n^o_{3+}$ and $n_1 - n^0_{3+} < g$ we still have the bound $\lfloor \frac{g+2c_1}{4} \rfloor + 1$ as long as $2c_1 \leq n_1 - n^o_{3+}$.

This matches the stated bound for this case. If $2c_1 > n_1 - n^o_{3+}$ the subgraph obtained by omitting exactly $\frac{n_1 - n_{3+}}{2}$ of the single edge components has its max term attained by the new n_1. So the bound of the first paragraph applies and we again get at most $\lfloor \frac{g+2c_1}{4} \rfloor + 1 = \lfloor \frac{g+n_1-n^o_{3+}}{4} \rfloor + 1$ which is the given bound.

4.3 All degrees at most 3

In each of the three alternate proofs below we will need a basis involving cases when all degrees are at most 3. For convenience we do this separately.

If all $d_i \leq 3$ and equality holds in Inequality (12) or (13), the left side is 4. It is straightforward to check that the sequence must be one of $(3, 3, \dots, 3, 2, 2)$ or $(3, 3, \dots, 3, 2, 1)$ or $(3, 3, \dots, 3, 1, 1)$ or $(3, 1, 1, 1)$. The first three are realized by paths with edges alternating multiplicity 1 and 2 with the multiplicities of first and last edge determining whether the end vertex degrees are $2, 2$ or $2, 1$ or $1, 1$. The sequence $(3, 1, 1, 1)$ is realized by a vertex adjacent to 3 leaves.

4.4 Proof 1 of 'if' in Theorem 9: Leaf removal

We prove 'if' in Theorem 9 for the case that equality holds in the condition of Inequality (10) parallel to the leaf removal proof for trees.

Note first that at least two of the d_i are 1 or 2: $n_1 + 2n_2 + 3n_3 + 4(n - n_1 - n_2 - n_3) \leq \sum_{i=1}^{n} d_i \leq 4n - 4 - n^o \leq 4n - 4 - (n_1 + n_3) \Rightarrow n_1 + n_2 \geq 2$.

Proceed by induction on n. The basis has been noted above.

We consider several cases. In each case we create a new sequence, form a new 2-multitree by induction and add a new vertex with either 1 or 2 edges to an existing vertex. In the underlying graph the new vertex is incident to the inductive tree and has degree 1 so the new underlying graph is connected and has no cycles, hence we get a 2-multitree.

Induction case 1: If $n_2 \geq 1$ and some $d_i \geq 4$ then create a new sequence with one less 2 and some $d_i \geq 4$ replaced with $d_i - 2$. The new sequence has degree sum 4 less than the original. The new sequence has the same n_1 and n^o and as it has $n - 1$ terms the right side in the condition is 4 less than the original. By induction the conditions hold for the new sequence. Add a new vertex with 2 edges to a vertex with degree $d_i - 2$.

Induction case 2: If $n_1 \geq 1$, some odd $d_i \geq 3$ and $n^o \geq 4$ then create a new sequence with one less 1 and some odd $d_i \geq 3$ replaced with $d_i - 1$. The new sequence still has at least one odd entry and has degree sum 2 less than the original. The new sequence has one less 1 and 2 less odd entries and $n-1$ terms so the right side of the condition is 2 less than the original. By induction the conditions hold for the new sequence. Add a new vertex with 1 edge to a vertex with degree $d_i - 1$.

Induction case 3: If $n_1 = n^o$ (all odd entries are 1's) and some $d_i \geq 4$ then create a new sequence with one less 1 and some $d_i \geq 4$ replaced with $d_i - 1$. The new sequence has degree sum 2 less than the original. The new sequence has one less entry that is 1 and the same n^o. However as $n_1 = n^{odd}$ and $n_1 \geq 2$ the max in the right side is attained by $2n_1$ and is 2 smaller than the original. As also the new sequence has $n-1$ terms the sum in right side of the condition is 2 less than the original (4 less for the sum and 2 more for the max). By induction the conditions hold for the new sequence. Add a new vertex with 1 edge to a vertex with degree $d_i - 1$.

If none of the cases can be applied and some entry is at least 4 then there are no 2's, exactly 2 odd entries, a 1 and a 3. By the condition, the sum must be at most $4n-6$ but this is impossible as there is one 1, no 2's and one 3. $\Box$

4.5 Proof 2 of 'if' in Theorem 9: Lobster construction

We prove 'if' in Theorem 9 for the case that equality holds in the condition of Inequality (10) parallel to the caterpillar construction proof for trees.

Recall that a caterpillar is a tree for which removal of leaves leaves a path. That is, a caterpillar is a path with pendent edges attached. A lobster is a tree for which removal of leaves leaves a caterpillar. That is, the process of removing leaves twice leaves a path. We will construct a 2-multitree with underlying tree that is a special type of lobster, a path with pendent edges and pendent paths of length 2.

The cases where all degrees are at most 3 have been noted above so we assume some term is at least 4 and hence the shell we describe below is nonempty.

Before the more formal description we note Figure 2. We begin with a 'shell' which is an underlying path along with some pendent edges and will attach three different types of pendent graphs. We use 3 parameters for the shell, r, s, t which correspond to the number of degree 3, degree 4 and degree 5 vertices in the shell (except each end of the path which will have degree 1, 2, or 3 treated as degree 3, 4 or 5 respectively for internal vertices). Always r will be even. Even though we could avoid pendent edges in the shell and add single pendent edge in the construction we do it as below for convenience in keeping track that parities are correct. What we need to check is that we can pick the sizes r, s, t, x, y, z and make the attachments to get the desired degree sequence.

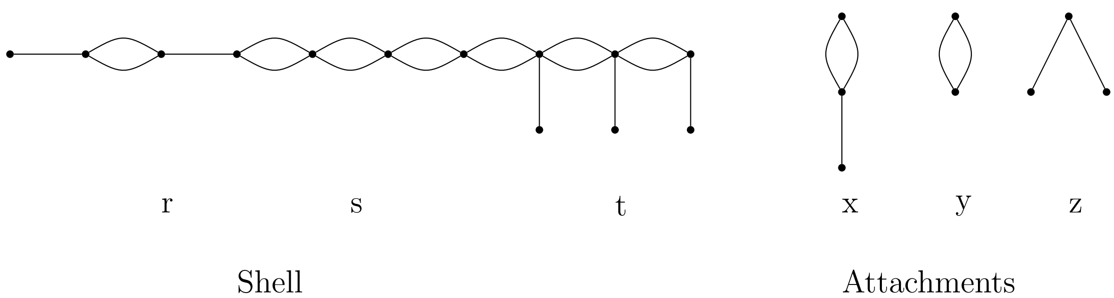


Figure 2: Lobster Construction Example

Use R, S, T to denote the vertices of the path on the shell with degrees 3,4,5 respectively so $|R| = r, |S| = s, |T| = t$. Specifically we have vertices $v_1, v_2, \ldots, v_{r+s+t}$ with edges $v_{2i-1}v_{2i}$ having multiplicity 1 for $i = 1, \ldots \frac{r}{2}$ and all other edges v_jv_{j+1} having multiplicity 2. In addition we add new vertices and pendent edges to these vertices for $v_{r+s+1}, \ldots, v_{r+s+t}$. That is, every other edge has multiplicity 1 in R and pendents are attached to T. We will further split $r = r_1 + r_2$ with r_1 the number of 3's assigned to vertices in R and r_2 the number of odds at least 5. The degree sum in the shell is $3r + 4s + 5t - 4$.

We describe three types of pendent graphs. These are: vertices a, b, c with ab having multiplicity 2 and bc with multiplicity 1; vertices a, b with edge ab with multiplicity 2; vertices a, b, c with edges ab and ac each having multiplicity 1. We will use x, y, z to denote the number of each used. In each case we will attach with vertex a corresponding to a vertex in the path of the shell.

There will be 3 cases. In each we need to check that once we have specified r, s, t, x, y, z and the attachments, the number of degree 1,2,3 vertices is correct and that the additional degrees from the pendent graphs are correct to get the correct degrees on the shell. The difference between the 'target' degree of vertices on the shell and their degree on the shell will be even and as attachments increase degree by 2 we do not need to check parity.

Correct number of degree 1 vertices: $n_1 = x + 2z + t$
Correct number of degree 2 vertices: $n_2 = y$
Correct number of degree 3 vertices: $n_3 = r_1 + x$
Correct degree requirements: $2x+2z+3r+5t = 4n^o_{5+}+3r_1+n_3+2n_1-\max\{n_3+n^o_{5+}, n_1\}$.

To see the degree requirement equation, the degree requirements on shell match those provided by attachments: $\sum_{R \cup S \cup T} d_i-(3r+4s+5t-4) = 2x+2y+2z$. Using Inequality

(11) at equality, noting that in each case we will have $R \cup S \cup T$ equal to all values at least 4 plus r_1 3's we get $\sum_{R \cup S \cup T} d_i = 4n^e_{4+} + 4n^o_{5+} + 3r_1 + n_3 + 2n_2 + 2n_1 - 4 - \max\{n_3 + n^o_{5+}, n_1\}$. Then noting that in each case we will assign the even values at least 4 to vertices of S (so $s = n^e_{4+}$) and we will set $y = n_2$ we get the degree requirement equation noted above.

We consider 3 cases. In each case checking that the 3 degree requirement equation above holds is straightforward.

Case 1: $n_1 \geq n^o_{3+}$. Let $r = 0$, $s = n^e_{4+}$ and $t = n^o_{5+}$ assigning $N^o_{5+} = T$ in any order. Also $x = n_3$, $y = n_2$ and $z = \frac{n_1 - n^o_{5+} - n_3}{2}$ (the numerator here is even as there are an even number of odd values).

Case 2: $n_1 \leq n^o_{3+}$ and $n_1 \leq n^o_{5+}$. Let $r = n^o_{5+} + n_3 - n_1$ with $r_1 = n_3$, $s = n^e_{4+}$ and $t = n_1$ assigning any n_1 values from N^o_{5+} to T in any order and the rest of N^o_{5+} along with all 3's to R. Also $x = 0$, $y = n_2$ and $z = 0$.

Case 3: $n_1 \leq n^o_{3+}$ and $n_1 \geq n^o_{5+}$. Let $r = r_1 = n^o_{5+} + n_3 - n_1$, $s = n^e_{4+}$ and $t = n^o_{5+}$ assigning all n_1 of N^o_{5+} to T in any order and $n^o_{5+} + n_3 - n_1$ 3's to R. Also $x = n_1 - n^o_{5+}$, $y = n_2$ and $z = 0$. □

4.6 Proof 3 of 'if' Theorem 9: Branch repair

We prove 'if' in Theorem 9 for the case that equality holds in the condition of Inequality (10) parallel to the branch repair proof for trees.

Use induction on the number of $d_i \geq 4$. The basis is a sequence with (at most) one value $d \geq 4$ and n_3 threes, n_2 twos and n_1 ones. If there is a $d \geq 4$ start with a vertex v and add n_2 vertices adjacent to v with an edge of multiplicity 2. Additional edges will depend on whether $n_1 > n_3$ or $n_1 \leq n_3$.

If $n_1 > n_3$ then we have $d = n_3 + 2n_2 + n_1$, add n_3 pairs of vertices x, y with x adjacent to v with an edge of multiplicity 2 and x adjacent to y with an edge of multiplicity 1, and add $n_1 - n_3$ leaves adjacent to v. This gives the correct degrees.

If $n_1 \leq n_3$ then we have $d + \delta = 2n_2 + 2n_1$ where δ is 1 if d is odd and 0 otherwise. Add $n_1 - \delta$ pairs of vertices x, y with x adjacent to v with an edge of multiplicity 2 and x adjacent to y with an edge of multiplicity 1 and add a vertex adjacent to v with an edge of multiplity 1 if $\delta = 1$. Finally replace some edge of multiplicity 1 with a path having $n_3 - n_1 + \delta$ (which is even) vertices and every other edge having multiplicity 2 with the end edges having multiplicity 1. This gives the correct degrees.

Let N_{4+} denote the set of indices i with $d_i \geq 4$. Partition N_{4+} into two nonempty parts L and R. Pick an entry in each sequence and decrease it by 2. We will distribute the

1's, 2's and 3's to L and R so that each part satisfies the conditions for realizability. By induction realize the parts and add an edge of multiplicity 2 between vertices corresponding to the values that were decreased by 2 to get a realization.

We observe that when inequalities (12) and (13) hold with equality and there are no odd d_i, we have $\sum d_i = 4n - 4$ which is realizable as a 2-multitree. Hence we do not need ensure that each part of the split has an odd value. We use $n', d'_i, \ldots$ and $n'', d''_i, \ldots$ for the values in the parts L and R.

Case 1: $n_1 \geq n^o_{3+}$. We have $4 + \sum_{i \in N_{4+}}(d_i - 4) = n_3 + 2n_2 + n_1$ and need to choose $n'_1 + n''_1 = n_1$, $n'_2 + n''_2 = n_2$, $n'_3 + n''_3 = n_3$ so that $4 - 2 + \sum_{i \in N_{4+}}(d'_i - 4) = n'_3 + 2n'_2 + n'_1$ and $4 - 2 + \sum_{i \in N_{4+}}(d'_i - 4) = n''_3 + 2n''_2 + n''_1$. The -2 in each case is as we have reduced one value in L and R by 2. We get the equalities immediately but also need that $n'_1 \geq n'^o_{3+}$ and $n''_1 \geq n''^o_{3+}$ to be in the correct case for induction. Initially place n'^o_{5+} 1's in L and n''^o_{5+} 1's in R. This is possible as $n_1 \geq n_3 + n^o_{5+}$ and the left side of each equation is at least the number of odds that are at least 5. There are still at least n_3 unplaced 1's so we can ensure that for each 3 placed in L or R we also place another 1 to $n'_1 \geq n'^o_{3+}$ and $n''_1 \geq n''^o_{3+}$.

Case 2: $n_1 \leq n^o_{3+}$. We have $4 + \sum_{i \in N_{4+}}(d_i - 4) + n^o_{5+} = 2n_2 + 2n_1$ and need to choose $n'_1 + n''_1 = n_1$, $n'_2 + n''_2 = n_2$, $n'_3 + n''_3 = n_3$ so that $4 - 2 + \sum_{i \in N_{4+}}(d'_i - 4) + n'^o_{5+} = 2n'_2 + 2n'_1$ and $4 - 2 + \sum_{i \in N_{4+}}(d''_i - 4) + n''^o_{5+} = 2n''_2 + 2n''_1$. The -2 in each case is as we have reduced one value in L and R by 2. We get the equalities immediately but also need that $n'_1 \leq n'^o_{3+}$ and $n''_1 \leq n''^o_{3+}$ to be in the correct case for induction. The left side of each equation is at least twice the number of odds that are at least 5. So we initially place 1's so that at most n'^o_{5+} are in L and at most n'^o_{5+} are in R. For each remaining 1, if any, when we place it in L or R we also place 3 in the same set. This is possible as $n_1 \leq n^o_{3+}$ and 3's do not play a role in the equation. Hence we can place the 1's, 2's and 3's so that we get equality and maintain $n'_1 \leq n'^o_{3+}$ and $n''_1 \leq n''^o_{3+}$. □

5 Partitioned degree sequences for 2-multiforests

Recall that forests are bipartite, the vertices can be partitioned in two parts so that all edges are between the parts. Thus the degree sequence partitions into two parts with equal sum. For trees and forests it is easy to see that given any partition of the degree sequence into two parts with equal sum there is a tree/forest realization for which the bipartite degree sequences corresponds to the given degree partition. For example, the standard proof using induction, removing one 1 from the sequence, reducing by 1 an entry that is at least 2 an using induction can easily be modified so the the 1 and the larger entry are from different parts.

We have already observed in Section 1.2 examples where the partition matters for 2-multiforests.

Our goal now is to describe conditions on a partitioned sequence which characterize when there is a 2-multiforest realization. We have already noted in Section 3 that in the case that all degrees are even realizability does not depend on the partition. For a partitioned sequence if all values are even the overall sum is a multiple of 4 and there is a realization with this (indeed any) partition.

In the following we need at least 3 terms as the sequence $1, 1$ fails the conditions but is realized as a single edge.

Theorem 15 *For $n = r+s \geq 3$, positive integer sequences $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ with at least one of the a_i or b_j odd having $\sum a_i = \sum b_j = e$ (the number of edges) are the partitioned degrees of a 2-multiforest if and only if $e \leq 2n - 2 - \max\{n_a^o, n_b^o, n_1\}$ where n_a^o denotes the number of odd a_i, n_b^o denotes the number of odd b_j and n_1 denotes the total number of 1's in both sequences.*

Proof: The necessity of the conditions parallels Fact 8;

To show $e \leq 2n - 2 - n_1$, consider $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ as an unpartitioned sequence and use Fact 8 and $\sum a_i = \sum b_j$.

To show $e \leq 2n - 2 - n_a^o$ (and similarly $e \leq 2n - 2 - n_b^o$) note that each odd degree vertex among the a_i is incident to at least one edge with multiplicity one and each such single edge is incident to at most one odd degree among the a_i. Hence at least n_a^o edges are 'missing' from the possible $2n - 2$ edges.

For sufficiency of the conditions we outline a proof along the lines of the leaf removal proof of Theorem 9. As it is similar we omit some details. This proof will include the strict inequality portion and use induction on e rather than n.

For the basis, if $e \leq n - 1$ then we can realize as a forest by Fact 5. Observe that this includes the case that all values (in both parts) are at most 2. In addition, if all values are 2's there is a realization as a disjoint union of edges of multiplicity 2.

In each of the following cases removing the smaller value and reducing the other by this smaller value produces new partitioned sequence that satisfies conditions as can easily be checked.

(a) Remove a 1 in one part and reduce an odd value that is at least 3 in the other part by 1.

(b) Remove a 1 in one part and reduce an even value that is at least 4 in the other part by 1 if the maximum is not the number of odds in the part not containing the 1 (in particular if the only odd values in that part, if any, are 1's).

(c) Remove a 2 in one part and reduce a value that is at least 4 in the other part by 2.

(d) Remove a 2 in one part and reduce a value that is exactly 3 in the other part if n_1 is not the maximum (in particular if there are no 1's in the part with the 2).

Note that there is some value at most 2 as the degree sum is at most $4n - 4$. We may assume that some $a_i \leq 2$. We can use induction, removing some a_i using one of (a) - (d) unless all $b_j \leq 2$. Then, again using one of (a) - (d) we can remove some $b_j \leq 2$ unless all $a_i \leq 2$. So we can assume all $a_i, b_j \leq 2$, which is covered by the basis. □

6 Conclusion

Informally, the underlying forest has at most $n - 1$ edges, so $2n - 2$ edges if each has multiplicity 2. We subtract the number that are forced to have multiplicity 1 by parity conditions in each part or by having degree 1.

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