Synthesis of Controllers for Constrained Systems

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Current Support Structure for the Center

**Liason members:**
- Air Products and Chemical Inc.
- Petrobras, Brazil

**Industrial Projects:**
- Praxair
- Consolidated Edison Company of NY
- Pittsburgh Digital Greenhouse
- IBM

**Federal/PA Projects:**
- National Science Foundation
- Sandia National Laboratories
- PITA

**Other:**
- Rossin Endowment, Hook Professorship
- McCann Professorship
Current Research Projects

**Control of Periodic/Cyclic Systems:** PSA, bio-rhythm, SMB

**Hardware Embedded MPC:** Controller on chip technology

**Distributed Micro-chemical Systems:** Micro/nano systems

**Model Predictive Control:** Nonlinear, robust, repetitive

**Abnormal Systems Diagnosis:** Identification of sleep disorders

**Agile Plant Start-Up and Operation**

**Design and Control in Retrofit Projects**

**Robust Operation through Redesign**

**Boiler Control Systems in Power Plants**
Research Program
M. V. Kothare

Graduate Students: L. Bleris, S. Mukherjee, M. Medgaarden, P. Tiwari, P. Vouzis
Graduated Students: E. F. Mulder (Ph.D., 2003), L. Ozkan (Ph.D., 2003),
S. V. Karnik (Ph.D., 2003), Z. Wan (Ph.D., 2003), A. Pattekar

Systems and control
Control of microreactors
Microchemical Systems

Theory
• LMIs, SDP
• nonlinear MPC
• multi-model/switched control
• constrained control

Software
• efficient robust MPC
• efficient nonlinear MPC
• anti-windup synthesis

Applications
• benchmark CSTR control
• solution co-polymerization control
• steam generator level control
• load following in pressure swing adsorption (PSA)

Mathematical modeling
• microfluidic models
• reduced order models
• computation/software

Dynamics and control
• distributed controllability/observability
• distributed boundary control formulation
• choice of inputs/measurements
• robustness/impact on design

Prototyping/demonstration
• microfabrication
• microfuel processing
• micro-packed bed reactors

Microchemical sensors

Microchemical Systems
• microchemical sensors

Chemical Process Modeling and Control Research Center
• practical relevance
• graduate/undergraduate internships

Pittsburgh Digital Greenhouse
• industrial/practical relevance

Sandia National Laboratories
• microfabrication
• computational sci. & math. program
• graduate/undergraduate internships

Collaborators
• M. Arnold (CSE, Lehigh)
• M. Hatalis (EE, Lehigh)
• L. Biegler (CHE, Carnegie Mellon)
Microchemical Systems Research

Miniature Fuel Processor for $H_2$ delivery in micro-fuel cells

**Experiments**
- Catalytic reforming microreactor
- Hybrid microreactor/separator
- Integration of resistive heaters
- Microfluidic interfacing

**Theory**
- Microfluidic transport modeling
- Optimization of microsystem
- Dynamic analysis/feedback control
- Systems level integration

**Computation**
Simulation using Computational Fluid Dynamics (CFD)
Background:  
Linear Matrix Inequalities (LMIs)

\[ \text{F(x)} = F_0 + x_1 F_1 + \ldots + x_p F_p > 0 \]

- \( x \in \mathbb{R}^p \) is the variable
- \( F_i = F_i^T \) are given matrices
- \( F(x) > 0 \implies F \) is positive-definite

LMIs are convex constraints on \( x \)

\[ F(x) > 0, \quad F(y) > 0 \]

\[ \implies F(\lambda x + (1 - \lambda)y) = \lambda F(x) + (1 - \lambda)F(y) > 0 \]

for all \( \lambda \in [0,1] \)
Typical LMI problems

**Feasibility:** Find \( x \in \mathbb{R}^n \) such that \( F(x) \geq 0 \)

or show that none exists

**Linear objective minimization:**

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad F(x) \geq 0
\end{align*}
\]

These are all convex problems
Significance of LMI problems

- No “analytic solution”
- Global solution “computable” efficiently

Implication:
Reducing a problem to LMI problem ⟷ solving it

Analogy:
Reducing a problem to linear program (LP) ⟷ solving it
Many problems in systems and control can be cast as LMI problems. Some of them were intractable in their original form.

- Lyapunov-based controller synthesis
- multi-criterion controller synthesis
- optimal filtering problems

See
(1) Boyd et al. (1994), *Linear Matrix Inequalities in Systems & Control*, SIAM.
(2) Gahinet et al. (1995), *MATLAB LMI Toolbox*. 
Long History of LMI’s

Aleksandr M Lyapunov

• Born: June 6, 1857
• Died Nov 3, 1918
• Friend of Markov
• Student of Chebyshev
• Lyapunov method, 1899
  • Stability of ODE’s
Lyapunov Stability

Given an ODE: \[ \dot{x} = f(x) \]

\[ V = g(x) \]

Such That: \[ V > 0, \quad \forall x \neq 0 \]
\[ \dot{V} < 0, \quad \forall x \neq 0 \]
\[ V = 0 \rightarrow x = 0 \]
Lyapunov Stability

Example

\[
\dot{x} = Ax
\]

Lyapunov

\[
V(x) = x^T P x > 0
\]

\[
\dot{V}(x) = x^T A^T P x + x^T P A x < 0
\]

LMI

\[
P > 0
\]

\[
A^T P + P A < 0
\]
What are Bilinear Matrix Inequalities?

\[ F(x,y) = F_0 + x_1 y_1 F_1 + \ldots + x_p y_p F_p > 0 \]

- \( x \) and \( y \) are the variables
- \( F_i = F_i^T \) are given matrices
- \( F > 0 \) means all eigenvalues > 0

Standard Problem

\[
\min c^T x \text{ subject to } F(x) > 0
\]
Significance of BMIs

- BMIs are nonconvex constraints
- Solutions are computable
  - Locally via gradient search methods
  - Globally via branch and bound
- In general, BMI significantly more difficult to solve
- No existing commercial software

Many problems in systems and control can be cast as BMIs
Outline

- Constraints in practice
- Approaches to handling constraints
- Anti-windup analysis and synthesis
- A Framework for Design of Scheduled Output Feedback MPC
- Local Convex Robust Solution with Scheduling
- Example
- Conclusions
Motivation

- Linear Control Theory: PID, IMC, $H_\infty$, $H_2$
- Nonlinear Control Theory: gain scheduling, nonlinear $H_\infty$
  feedback linearization, sliding mode

Constraints on $u$ and $y$ ignored
Detrimental Effect of Constraints

- Chernobyl 1986 nuclear explosion
  - cooling water constraints
- YF22 aircraft crash (April 1992)
  - constraints on control surface moves
- Gripen JAS 39 aircraft (August 1993)

Constraints → Limits on controller performance
Constraint Resolution through Overdesign

- Make valves larger
- Install higher capacity compressor

Impractical due to high cost

Economic feasibility  Operation at constraints
Approaches to Handling Constraints

Linear Plant Model

- Model Predictive Control
- Anti-windup Control
- Nonlinear Stabilization

Nonlinear Plant Model

- Local linearization + linear methods with gain scheduling
- Feedback linearization + linear methods
- Geometric nonlinear and adaptive methods
What is Anti-Windup Control?

• **Step 1**: Design linear controller ignoring constraints

\[ r \rightarrow K(s) \rightarrow u \rightarrow \hat{u} \rightarrow P(s) \rightarrow y \]

• **Step 2**: Add anti-windup compensation

\[ r \rightarrow K(s) \rightarrow u \rightarrow \hat{u} \rightarrow P(s) \rightarrow y \]

\[ \rightarrow R(s) \rightarrow \hat{u} \rightarrow P(s) \rightarrow y \]
Anti-Reset Windup
Anti-windup Internal Model Control

Without constraints

With constraints
Conditioning, anti-windup and bumpless transfer

Figure 1. Discrete-time reset integrator anti-windup compensator corresponding to (3.1). This configuration can be reduced to Fig. 4.

Figure 2. Hammerstein Structure for Saturating System

Figure 4. Incremental controller form I followed by an anti-windup compensated integrator. Only the integrator state is anti-windup compensated.

Figure 5. Incremental controller form II. The anti-windup is imposed on both the integrator state and the differentiator filter state.
• Which technique for which scenario?
• Extension to multivariable processes?
• Stability and robustness guarantees?

⇒ Need fresh and general approach
Two new parameters $\Lambda_1, \Lambda_2$

General Framework

Kothare et al. (1994), Automatica

\[ \dot{x} =Ax + B e + \Lambda_1(\hat{u} - u) \]
\[ u = C x + D e + \Lambda_2(\hat{u} - u) \]
Unification of Existing Anti-Windup Schemes

Kothare et al. (1994), Automatica

- Anti-Windup in a general setting
- Parameterization in terms of $\Lambda_1, \Lambda_2$

All existing anti-windup schemes can be recovered as special cases for particular choices of $\Lambda_1, \Lambda_2$
Anti-Windup Stability Analysis
Kothare, Morari (1999), Automatica

- Analysis of specific anti-windup schemes
  - describing functions
  - Popov, Circle, Off-Axis Circle criterion
  - small gain theorem

Need to address anti-windup stability in a general multivariable setting
Passivity Theorem: Assume all $\gamma \in \mathbb{N}$ are passive, i.e.,

$$\langle x_T | \gamma x_T \rangle \geq d \| x_T \|^2 + e, \text{ for all } x \in L_{2e}$$

Then, feedback interconnection is stable for all $\gamma \in \mathbb{N}$ if $M(s)$ is strictly passive and stable.

- \textbf{Conservative!! Properties of }\gamma \text{ not incorporated}
- \textbf{Can use multipliers to reduce conservatism}
Multiplier Theory

- \( W_+(s), W_-(s) \) exist, are stable, proper with stable proper inverses
- \( W_-(s)^{-T} M(s) W_+(s) \) is stable, strictly passive
- \( W_+(s)^{-1} \) ? \( W_-(s)^T \) is passive for all \( ? \in \mathbb{N} \)

\[ u_1 \xrightarrow{W_+(s)^{-1}} W_+(s) \xrightarrow{\ ? \ } W_-(s)^T \xrightarrow{?^{-1}} W_-(s)^{-T} \xrightarrow{u_2} \]

System is stable for all \( ? \in \mathbb{N} \)

Properties of \( ? \) influence choices of \( W_+, W_- \)
Passivity using Linear Matrix Inequalities

Brocket, Willems (1965), Willems (1971), Anderson (1973)

System\[\frac{dx}{dt} = Ax(t) + Bu(t)\]
\[y(t) = Cx(t) + Du(t)\]

is strictly passive if and only if
\[H(j\omega) + H(-j\omega)^* \geq dI, \text{ for } \omega \text{ all } \in \mathbb{R}\]

Equivalent to existence of \(P=P^T, d > 0\) such that
\[
\begin{bmatrix}
A^TP + PA & PB - C^T \\
B^TP - C & dI - (D + D^T)
\end{bmatrix} \leq 0
\]

Convex Linear Matrix Inequality in \(P, d\)
Anti-Windup Stability Analysis

Kothare, Morari (1999), Automatica

- Saturation \( \hat{\text{u}} \) is static odd monotonic nonlinearity
- Corresponding multiplier characterization (Zames, Falb (1968))
  \[
  (I - W(j \omega)) M(j \omega) + M(j \omega)^*(I - W(j \omega)^*) \geq d I
  \]
- \( W = \text{diagonal}(W_1, W_2, \ldots, W_{nu}) \)
  \[
  W_i(j \omega) = \int_{-\infty}^{\infty} w_i(t) \exp(-j \omega t) \, dt, \quad \int_{-\infty}^{\infty} |w_i(t)| < 1
  \]
  \[
  w_i(t) \geq 0 \quad \text{for all} \ t.
  \]
- Stability condition and condition on \( w_i(t) \) translate to LMIs.
Anti-Windup Stability Analysis
Kothare, Morari (1999), Automatica

• The analysis tools:
  • Passivity Theorem
  • Multiplier Theory
  • Linear Matrix Inequalities (LMIs)

• Resulting stability conditions:
  • are least conservative
  • are convex and computationally tractable
  • generalize all previous stability conditions
Anti-Windup Synthesis as an LMI
Mulder et al. (2001), Automatica

The Synthesis Problem

“Unfortunately, it has been established that the optimal reduced order control design problem [static anti windup] CANNOT be formulated as such a convex LMI problem.”

Anti-Windup Synthesis as an LMI

Mulder et al. (2001) Automatica

Stability Criterion
• suitable multiplier criterion

Performance Objective
• suitable gain function

LMI SYNTHESIS for \( \Lambda_1, \Lambda_2 \)
Anti-Windup LMI Synthesis
Mulder et al. (2001), Automatica

Minimize $\Gamma$

$$\begin{bmatrix}
QA^T + AQ & B_w & B_v M - B_\xi X + QC_u^T & QC_z^T & 0 \\
B_w^T & -\Gamma & D_{uw}^T & D_{zw}^T & 0 \\
MB_v^T - X^TB_\xi^T + C_u Q & D_{uw}^T & -2M + D_{uw} M + M D_{uv}^T - D_{u_\xi} X - X^T D_{u_\xi}^T & M D_{zv}^T - X^T D_{z_\xi}^T & M \\
C_z Q & D_{zw} & D_{zv} M - D_{z_\xi} X & -\gamma^1 & 0 \\
0 & 0 & 0 & M & -\delta I \\
\end{bmatrix} < 0$$

$Q > 0 \hspace{1cm} M > 0 \hspace{1cm} \Gamma > 0 \hspace{1cm} \delta > 0 \hspace{1cm} X > 0$

Standard LMI Eigenvalue Problem

Solution: $\Lambda = XM^{-1}$
Anti-Windup LMI Synthesis
Mulder et al. (2001), Automatica

First successful attempt to develop a convex LMI-based solution to the Static Anti-windup Controller Synthesis Problem
Model Predictive Control (MPC)

- Reference input
- State estimate
- Manipulated variable
- Control horizon
- Prediction horizon
- Projected outputs
- Targets
On-line Optimization in MPC
(QP/LP – recently, MILP)

\[
\begin{align*}
\text{min} & \quad J_p(k) \\
\text{subject to} & \quad u_{\text{imin}} \leq u_i(k+j|k) \leq u_{\text{imax}} \\
& \quad x_{\text{imin}} \leq x_i(k+j|k) \leq x_{\text{imax}}
\end{align*}
\]

\[
J_p = \sum_{i=0}^{p} [x(k+i|k)^T Q_1 x(k+i|k) + u(k+i|k)^T R u(k+i|k)]
\]

- Physically meaningful objective \( J_p(k) \)
- Physically meaningful constraints
- Flexibility in model used to compute \( J_p(k) \)
Applications of Model Predictive Control

- Successfully applied in
  - chemical and petrochemical industries
  - paper and pulp industries
  - adhesive coating industries

- Reasons:
  - multivariable in nature
  - can use a variety of model forms directly
  - can handle constraints explicitly

Significant on-line computation unsuitable for fast processes
Nominal Stability of MPC

- Considered well-understood
- Well-known result (Rawlings, Muske (1993))

Take prediction horizon $p=\infty$
Then, feasibility of on-line MPC
$\Rightarrow$ nominal stability
Robustness of MPC
(Kothare et al. (1996), Automatica)

Example: \[ P(s) = \frac{k}{s(s + \alpha)} \]

-2 \leq u \leq 2

Model uncertainty: \[ 0.1 \leq \alpha \leq 10 \]

Nominal model: \( \alpha = 1 \)
Actual plant: \( \alpha = 9 \)

With \( p=\infty \), on-line feasibility \( \Rightarrow \) robust stability
**Problem Statement**

System: Discrete linear time-varying

\[ x(k+1) = A(k)x(k) + B(k)u(k) \]

\[ y(k) = Cx(k) \]

\([A(k) B(k)] \in \Omega = \text{uncertainty set}\)
Robust MPC Objective

**Goal:** At time $k$, synthesize $u(k+i|k) = Fx(k+i|k)$, $i \geq 0$ to minimize

$$\max_{[A(k+i) \ B(k+i)] \in \Omega} J_\infty$$

subject to

$$|u_r(k+i|k)| \leq u_{r,\text{max}}, \quad i \geq 0, \ r=1, \ldots, n_u$$
$$|y_q(k+i|k)| \leq y_{q,\text{max}}, \quad i \geq 1, \ q=1, \ldots, n_y$$

where

$$J_\infty = \sum_{i=0}^{\infty} [x(k+i|k)^\text{T}Q_1 \ x(k+i|k) + u(k+i|k)^\text{T}R \ u(k+i|k)],$$
Robust MPC using LMIs
Kothare et al. (1996), Automatica

\[ \begin{align*}
\min & \quad \gamma \\
\begin{bmatrix}
1 & x(k|k)^T \\
x(k|k) & Q
\end{bmatrix} & \succeq 0 \\
\begin{bmatrix}
Q & QA_j^T + Y^T B_j^T & QQ_1^{1/2} & Y^T R_1^{1/2} \\
A_j Q + B_j Y & Q & 0 & 0 \\
Q_1^{1/2} Q & 0 & \gamma I & 0 \\
R_1^{1/2} Y & 0 & 0 & \gamma I
\end{bmatrix} & \succeq 0 \\
\end{align*} \]

\[ j=1,\ldots,L \]

Controller gain \( F = Y Q^{-1} \)
**Actuator Constraints**

**Invariant ellipsoid:**
\[ E = \{ x | x^T Q^{-1} x \leq 1 \} \]

- \( x(k|k) \in E \)
- \[ x(k+i|k) \in E, \ i \geq 0 \]

**Input constraints:**
\[ |u_r(k+i|k)| \leq u_{r, \text{max}}, \quad i \geq 0, \ r = 1, \ldots, n_u \]

\[ \begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \succeq 0 \quad \text{with } X_{rr} \leq u_{r, \text{max}}^2 \]
Robustness of MPC

(Kothare et al. (1996), Automatica)

Example: \[ P(s) = \frac{k}{s(s + \alpha)} \]

\[-2 \leq u \leq 2, \quad 0.1 \leq \alpha \leq 10 \]

on-line LMI feasibility \( \Rightarrow \) robust stability
Motivation: Efficient Nonlinear MPC

- Strongly nonlinear behavior with constraints
  - large operating region
  - transition between different local linear regimes

- Generic nonlinear MPC computationally demanding
  - Allgower et al., Mayne, Michalska, Mosca, Zheng,
    de Oliveira, Biegler, Kouvaritakis, Rawlings, Christofides, Magni,
    Mullary, Jadbabaie, etc.

- Typical approaches
  - Multiple shooting, reduced control horizon, reduced dimensionality, LTV approximation, hybrid switching, a priori CLF
Alternative: Gain Scheduling Predictive Controllers

Gain Scheduling (Leonessa et al, McConley et al.)
Problem Statement

Consider a nonlinear system:

\[
\dot{x} = f(x(t), u(t))
\]

or

\[
x(k+1) = f(x(k), u(k))
\]

**Goal:** Regulate the system to the desired equilibrium \((x_{ss}, u_{ss})\) transitioning through multiple operating regimes
Maximum Region of Stability

Invariant ellipsoid: \[ E = \{ x | x^T Q^{-1} x \leq 1 \} \]

\[ x(k|k) \in E \Rightarrow x(k+i|k) \in E, \; i \geq 0 \]

Problem 1

\[ \max_{u(k)} \max_{x(k+i|k)} \quad \log \det (Q) \]

subject to

\[ x(k+i|k) \in \Pi_x, \; u(k+i|k) \in \Pi_u, \; i \geq 0 \]

\[ V(x(k+i|k)) - V(x(k+i+1|k)) > 0, \text{ with } V(x) = x^T Q^{-1} x, \; Q > 0 \]
∀ x(0) ∈ E, Problem 1 always feasible and $x(k) ∈ E$ for all $k ≥ 0$, $x(k) → 0$ as $k → ∞$. 
Local LTV Representation of Nonlinear System

Around equilibrium surface \( S: \ x_{eq} = f(x_{eq}, u_{eq}) \),

\[
\begin{align*}
\Delta \\
\Delta
\end{align*}
\]

at \( (x^{(1)}, u^{(1)}) \)

at \( (x^{(i)}, u^{(i)}) \)

at \( (x^{(0)}, u^{(0)}) \)

\[
\begin{align*}
Z^{-1} \ \\
Z^{-1}
\end{align*}
\]
Local Predictive Controller

Let $\Omega = \text{Co}\{[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial u}]\}$

Off-line, for $\bar{x} \in \Pi_x$, $\bar{u} \in \Pi_u$, solve Problem 1, define

$$E = \{ \bar{x} \in \mathbb{R}^n \mid \bar{x}^T R^{-1} \bar{x} \leq 1 \}$$

On-line, for $\bar{x}(k) \in E$, solve Problem 2, apply

$$u(k) = F(k) (x(k) - x^{eq}) + u^{eq}$$
Design of Scheduled MPC

Wan, Kothare (2003, IJRNC)

1. Design MPC \#i at \((x^{eq(i)}, u^{eq(i)})\) with

\[
E^{(i)} = \{ x \in \mathbb{R}^n \mid (x - x^{eq(i)})^T (R^{(i)})^{-1} (x - x^{eq(i)}) \leq 1 \}
\]

2. Select \((x^{eq(i+1)}, u^{eq(i+1)}) \in E^{(i)}\). Let \(i:=i+1\), go to step 1.
Implementation of Scheduled MPC

Wan, Kothare (2003, IJRNC)

Given \( x(0) \in E^{(1)} \cup \ldots \cup E^{(M)} \).

At time \( k \), apply MPC \# i = \text{argmin } i, \text{ s.t. } x(k) \in E^{(i)}.

Or, once \( x(k) \in E^{(i+1)} \cap E^{(i)} \), switch from Controller \#(i+1) to \# i.
Example: Two-Tank system

Wan, Kothare (2003, IJRNC)

Example: A two-tank system

\[
\begin{align*}
\rho S_1 \dot{h}_1 &= -\rho A_1 \sqrt{2gh_1} + u \\
\rho S_2 \dot{h}_2 &= \rho A_1 \sqrt{2gh_1} - \rho A_2 \sqrt{2gh_2}
\end{align*}
\]

Let \( \bar{h}_1 = h_1 - h_1^{\text{eq}}, \bar{h}_2 = h_2 - h_2^{\text{eq}}, \bar{u} = u - u^{\text{eq}}, \)

\[
\begin{bmatrix}
\bar{h}_1(k + 1) \\
\bar{h}_2(k + 1)
\end{bmatrix} \in \left(0.5 \sum_{i=1}^{4} \alpha_i \Phi_i + I\right) \begin{bmatrix}
\bar{h}_1(k) \\
\bar{h}_2(k)
\end{bmatrix} + \begin{bmatrix}
0.5 \\
\frac{\rho A_1}{\rho A_1}
\end{bmatrix} \bar{u}
\]
Region of Stability of Scheduled MPC
State Responses

State Responses

$t$, (sec)

$h_1$, (cm)

$h_2$, (cm)

$0 \quad 50 \quad 100 \quad 150 \quad 200 \quad 250 \quad 300 \quad 350 \quad 400$

$0 \quad 20 \quad 40 \quad 60 \quad 80 \quad 100 \quad 120$

LEHIGH University
Local Observer Design

\[ e(k+1) = f(x(k), u(k)) - f(x(k) - e(k), u(k)) - L_p (h(x(k)) - h(x(k) - e(k))) \]
Local Output Feedback MPC

![Diagram of Local Output Feedback MPC]

- **Reference Input**: \( r \)
- **State Feedback RMPC**:
  - \( x(k) \) to \( x(k+1) \)
  - \( u(k) \) to \( u(k) \)
  - \( z^{-1} \)
- **Observer**:
  - \( e(k) \) to \( e(k) \)
  - \( z^{-1} \)
- **Plant**:
  - \( u \) to \( y \)

Mathematical Formulas:

\[
\begin{align*}
\Delta & \quad \text{State feedback RMPC} \\
z^{-1} & \quad \text{Observer} \\
\end{align*}
\]
Stability Analysis


Closed-loop system

\[ x(k+1) = f(x(k),u(k)) + (f(x(k),u(k)) - f(x(k),u(k)) \]
\[ \quad \wedge \quad \wedge \]
\[ x(k+1) = f(x(k),u(k)) + L_p(h(x(k)) - h(x(k))) \]

\[ \forall k \geq 0, \ x(k), \hat{x}(k) \in \Pi_x \]
\[ ||d_1(k)||_R \leq \beta_1 ||e(k)||_p \]
\[ ||d_2(k)||_R \leq \beta_2 ||e(k)||_p \]

Output feedback MPC

asymptotically stable within \( \mathbb{E} \), if

\[ x(0), \hat{x}(0) \in \mathbb{E}, \ ||e(0)||_p \leq \eta := \frac{1-\alpha}{\max\{\beta_1, \beta_2\}} \]
**Observability Analysis**

**Auxiliary system**

\[ \tilde{x}(k+1) = f(\tilde{x}(k), u(k)), \tilde{x}(k-T) = \hat{x}(k-T) \]

\[ \tilde{y}(k) = h(\tilde{x}(k)) \]

\[ \|x(k) - \hat{x}(k)\|^2 \leq \frac{\rho^T V_T}{\mu} \]

\[ V_T = \sum_{j=k-T}^{k} \|y(j) - \tilde{y}(j)\|^2 \]
Scheduled Output Feedback MPC


1. Design MPC \(i\) at \((x^{(i)}, u^{(i)})\) with \(E^{(i)}\)
2. Select \(x^{(i+1)} \in \theta E^{(i)}\). Let \(i:=i+1\), go to step 1.

3. At time \(k\), switch from MPC \(\#(i+1)\) to \(\# i\), if \(\hat{x}(k) \in \theta E^{(i)}\) and \(\|x(k)-\hat{x}(k)\|^2 \leq \frac{\rho^T V_T}{\mu} \leq \delta\) such that \(x(k) \in E^{(i)}\) and \(\|x(k)-\hat{x}(k)\|^2 \leq \eta^{(i)}\)
Example: CSTR

Example: CSTR with exothermic reaction

\[
\begin{align*}
\dot{C}_A &= \frac{q}{V} \left( C_{Af} - C_A \right) - k_0 \exp \left( - \frac{E}{RT} \right) C_A \\
\dot{T} &= \frac{q}{V} \left( T_f - T \right) + \frac{(- \Delta H)}{\rho C_p} k_0 \exp \left( - \frac{E}{RT} \right) C_A + \frac{UA}{V \rho C_p} \left( T_C - T \right)
\end{align*}
\]

Let \( \bar{C}_A = C_A - C_A^{eq}, \bar{T} = T - T^{eq}, \bar{u} = u - u^{eq}, \)

\[
\begin{bmatrix}
\bar{C}_A(k+1) \\
\bar{T}(k+1)
\end{bmatrix} \in \left( 0.03 \sum_{i=1}^{4} \alpha_i \Phi_i + I \right) \begin{bmatrix}
\bar{C}_A(k) \\
\bar{T}(k)
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{UA}{V \rho C_p}
\end{bmatrix} \bar{T}_C
\]
Steady State Multiplicity
Example: CSTR (Scheduled MPC vs. NMPC)

<table>
<thead>
<tr>
<th>Scheduled output feedback MPC</th>
<th>Output feedback NMPC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05 sec</td>
<td>5 sec</td>
</tr>
</tbody>
</table>
Summary of Contributions

Scheduled State Feedback MPC
- characterized explicit region of local MPC stability
- expanded region of stability by multiple predictive controllers
- developed computationally efficient formulation

Scheduled Output Feedback MPC
- characterized local upper bound for state estimation error
- established switching criteria for scheduled output feedback MPC
- established a framework for scheduled output feedback MPC
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