ME 433 – STATE SPACE CONTROL

Lecture 1

State Space Control

• Time/Place: Room 290, STEPS Building
  M/W 12:45-2:00 PM

• Instructor: Eugenio Schuster,
  Office: Room 454, Packard Lab,
  Phone: 610-758-5253
  Email: schuster@lehigh.edu,
  Office hours: By appointment

• Webpage:
  http://www.lehigh.edu/~eus204/Teaching/ME433/ME433.html

• E-mail list: Make sure to be in the mailing list!!!
State Space Control

State-space methods of feedback control system design and design optimization for invariant and time-varying deterministic, continuous systems; pole positioning, observability, controllability, modal control, observer design, the theory of optimal processes and Pontryagin's Maximum principle, the linear quadratic optimal regulator problem, Lyapunov functions and stability theorems, linear optimal open loop control; introduction to the calculus of variations. Intended for engineers with a variety of backgrounds. Examples will be drawn from mechanical, electrical and chemical engineering applications. MATLAB is used extensively during the course for the analysis, design and simulation.

State Space Control – Part I

- Topics:
  - Course description, objectives, examples
  - Review of Classical Control
  - Transfer functions ↔ state-space representations
  - Solution of linear differential equations, linearization
  - Canonical systems, modes, modal signal-flow diagrams
  - Observability & Controllability
  - Observability & Controllability grammians; Rank tests
  - Stability
  - State feedback control; Accommodating reference inputs
  - Linear observer design
  - Separation principle
State Space Control – Part II

• **Topics:**
  - Static Optimization
    - Optimization without/with constraints
    - Numerical solution methods
  - Dynamic Optimization
    - Discrete-time and continuous-time systems
    - Open loop and closed loop control
    - Linear Quadratic Regulator (LQR)
    - Pontryagin’s Minimum Principle
  - Dynamic Programming
    - Bellman’s Principle of Optimality
    - Discrete-time and continuous-time systems
    - Hamilton-Jacobi-Bellman Equation
  - Optimal Estimation/Kalman Filtering
    - Discrete-time and continuous-time systems
    - Linear Quadratic Gaussian Control (LQG)

Modern Control

• **Books:**
  - B. Friedland, “Control System Design: An Introduction to State-Space Methods,”
  - Kailath, “Linear Systems”
  - Brogan, “Modern Control Theory”
  - Rugh, “Linear System Theory”
  - Dorf and Bishop, “Modern Control Systems”
  - Antsaklis and Michel, “Linear Systems”
  - Chen, “Linear system Theory and Design”
Optimal Control and Estimation

- Books:
  - Bryson and Ho, “Applied Optimal Control”
  - Lewis and Syrmos, “Optimal Control”
  - Anderson and Moore, “Optimal Filtering”
  - Gelb, “Applied Optimal Estimation”
  - Stengel, “Optimal Control and Estimation”

Model Classification

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<tr>
<th>Spatial Dependence</th>
<th>Control Technique</th>
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Spatial Dependence

Distributed Parameter Systems
\[ \frac{\partial n}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} \left( D_n \frac{\partial n}{\partial r} - n V_n \right) + S_n(t, r) \]

Lumped Parameter Systems
\[ \frac{dn}{dt} = -\frac{1}{\tau_e} n + S_e(t) \]

\[ \left. \frac{\partial n}{\partial r} \right|_{r=0} = 0 \quad \left. n \right|_{r=a} = n_a^i \]

Control: Interior Boundary

1. Reduction

2. Keep the PDE representation (problem specific)

Linearity: Nonlinear/Linear

Linear/Nonlinear Distributed Parameter Control
Linear/Nonlinear Lumped Parameter Control

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Linearity

Nonlinear (ODE) Systems
\[ y^{(s)} = g\left(y^{(s-1)}, \ldots, y, u^{(m-1)}, \ldots, u\right) \]
Linear (ODE) Systems
\[ y^{(s)} + a_{n-1} y^{(s-1)} + \cdots + a_0 y = b_{n-1} u^{(m-1)} + \cdots + b_0 u \]

Non-Autonomous
\[ \dot{x} = f(t, x, u) \quad y = h(t, x, u) \]

Autonomous
\[ \dot{x} = f(x, u) \quad y = h(x, u) \]

1. Linearization
2. Keep the nonlinearities

Nonlinear Control
Linear Control

Output/State Feedback
Estimation: How to estimate states from input/output?
Linearity

Particular type of nonlinearities: Constraints

LTI
\[ \dot{x} = Ax + Bu \quad u, y \rightarrow \text{sat}(u), \text{sat}(y) \] input/output constraints
\[ y = Cx + Du \quad x_i < x_j < \bar{x}_j \] state constraints

1. A priori → Constraint is considered for control design
2. A posteriori → Constraint is NOT considered for control design

Anti-windup Techniques

Temporal Representation

Continuous-time Systems
\[ y(t) = a_n y(t-n) + \cdots + a_0 y(t) + b_m u(t-n) + \cdots + b_0 u \]

Discrete-time Systems
\[ y[kT] + a_n y[(k-1)T] + \cdots + a_0 y[(k-n)T] = b_m u[kT] + \cdots + b_0 u[(k-m)T] \]

Sampled-Data Systems

LTI
\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \]

Discrete-time Systems
\[ x[k+1] = Ax[k] + Bu[k] \]
\[ y[k] = Cx[k] + Du[k] \]

System Identification:
How to create models from data?
Fault Detection and Isolation:
How to detect faults from data?
Domain Representation

Continuous-time Systems

\[
y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y + a_0 y = b_n x^{(n)} + \cdots + b_0 x
\]

Discrete-time Systems

\[
y[k] + a_{n-1}y[k-1] + \cdots + a_1 y[k-n] + a_0 y[k-m] = b_n x[k] + \cdots + b_0 x[k-m]
\]

\[T(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 s + a_0}
\]

\[s = j\omega
\]

Frequency Response

\[T(j\omega) = \left| T(j\omega) \right| e^{j\omega T}\]

Optimality

Continuous-time Systems

\[
\dot{x} = A(t)x + B(t)u
\]
\[
y = C(t)x + D(t)u
\]

Discrete-time Systems

\[
x_{k+1} = A_k x_k + B_k u_k
\]
\[
y_k = C_k x_k + D_k u_k
\]

\[
\min_{u_k} \frac{1}{2} x_k^T S_k x_k \text{ with } x_N = 0, x_0
\]

\[
\min_{u(t)} \frac{1}{2} \int_0^T x^T(T) S_x x(T) + \frac{1}{2} \int_0^T \left[ u^T(t) Q(t) x(t) + u^T(t) R(t) u(t) \right] dt
\]

Optimal Control
Robustness

How to deal with uncertainties in the model?

A. Non-model-based control
   - PID
   - Extremum Seeking

B. Model-based control
   - 1. Robust Control → Design for a family of plants
   - 2. Adaptive Control → Update model (controller) in real time

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Dynamic Model

MECHANICAL SYSTEM:

\[ F = I\alpha \quad \text{Newton’s law} \]

\[ I\alpha = -lmg \sin \theta - b\dot{\omega} + T_c \]

\[ \omega = \dot{\theta} \quad \text{angular velocity} \]

\[ \alpha = \ddot{\omega} = \ddot{\theta} \quad \text{angular acceleration} \]

\[ I = ml^2 \quad \text{moment of inertia} \]

\[ \ddot{\theta} = -\frac{b}{ml^2} \dot{\theta} - \frac{g}{l} \sin \theta + \frac{T_c}{ml^2} \]

Which are the equilibrium points when \( T_c = 0 \)?

At equilibrium:

\[ \ddot{\theta} = \dot{\theta} = 0 \Rightarrow 0 = -\frac{g}{l} \sin \theta \Rightarrow \theta = 0, \pi \]

Open loop simulations: pend_par.m, pendol01.mdl

Stable

Unstable

Linearization

What happens around \( \theta = 0 \)?

\[ \theta = y \Rightarrow \dot{y} = -\frac{b}{ml^2} \dot{y} - \frac{g}{l} \sin(y) + \frac{T_c}{ml^2} \]

By Taylor Expansion:

\[ \sin(y) = y + h.o.t. \Rightarrow \sin(y) \approx y \]

Linearized Equation:

\[ \ddot{y} = -\frac{b}{ml^2} \dot{y} - \frac{g}{l} y + \frac{T_c}{ml^2} \]
Laplace Transform

Time domain (t domain)

- Linear system
- Differential equation
- Classical techniques
- Response signal

Complex frequency domain (s domain)

- Laplace transform \( \mathcal{L} \)
- Algebraic equation
- Algebraic techniques
- Inverse Laplace transform \( \mathcal{L}^{-1} \)
- Response transform

Transfer Function

\[ u \equiv T_c \Rightarrow \ddot{y} = -\frac{b}{ml^2} \dot{y} - \frac{g}{l} y + \frac{T_c}{ml^2} \]

\[ \ddot{y} + \frac{b}{ml^2} \dot{y} + \frac{g}{l} y = \frac{u}{ml^2} \]

Laplace Transform

\[ \mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s), \quad U(s) = \mathcal{L}\{u\}, \quad Y(s) = \mathcal{L}\{y\} \]

Transfer Function

\[ G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + \frac{b}{ml^2} s + \frac{g}{l}} \]

Characteristic Equation
**Solution of the ODE**

\[ T_c = 0 \Rightarrow \ddot{y} + \frac{b}{ml^2} \dot{y} + \frac{g}{l} y = 0 \]

What is the solutions \( y(t) \)?

**Characteristic Equation**

\[ \lambda^2 + \frac{b}{ml^2} \lambda + \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{-b}{ml^2} \pm \sqrt{\left(\frac{b}{ml^2}\right)^2 - 4 \frac{g}{l}} \]

\[ G(s) = \frac{1}{s^2 + \frac{b}{ml^2} s + \frac{g}{l}} \]

\[ y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \]

The dynamics of the system is given by the roots of the denominator (poles) of the transfer function.

\[ \text{real}(\lambda_1, \lambda_2) < 0 \Rightarrow \text{STABLE SYSTEM} \]

We use feedback control for **PERFORMANCE**

---

**Closed-loop Control**

\[
\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}
\]

\[
E(s) = \frac{R(s) - Y(s)}{1 + C(s)G(s)}
\]

\[ C(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_I}{s} + K_D s \]

\[ u(t) = K_p e(t) + K_I \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt} \]

**PID**: Proportional – Integral – Derivative

Closed loop simulations: **pid.m**
Closed-loop Control

We can place the poles at the desired location to obtain the desired dynamics

CLASSICAL CONTROL (ME 343)

Linearization

What happens around \( \theta = \pi \)?

\[
\theta = \pi + x \Rightarrow \quad \ddot{x} = -\frac{b}{ml^2} \dot{x} - \frac{g}{l} \sin(\pi + x) + \frac{T_c}{ml^2} \\
\ddot{x} = -\frac{b}{ml^2} \dot{x} + \frac{g}{l} \sin(x) + \frac{T_c}{ml^2}
\]

By Taylor Expansion:

\[
\sin(x) = x + h.o.t. \Rightarrow \sin(x) \approx x
\]

Linearized Equation:

\[
\ddot{x} = -\frac{b}{ml^2} \dot{x} + \frac{g}{l} x + \frac{T_c}{ml^2}
\]
Reduce to first order equations:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{b}{ml^2} x_2 + \frac{g}{l} x_1 + \frac{T_c}{ml^2}
\end{align*}
\]

State Variable
Representation

\[
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u = T_c \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u = Ax + Bu
\]

\[
eig(A) = \{ \lambda : |\lambda I - A| = 0 \} = \{ \lambda : \lambda^2 + \frac{b}{ml^2} \lambda - \frac{g}{l} = 0 \}
\]

Characteristic Equation

\[
T_c = 0 \Rightarrow \ddot{x} + \frac{b}{ml^2} \dot{x} - \frac{g}{l} x = 0
\]

What is the solution \( x(t) \)?

\[
\lambda^2 + \frac{b}{ml^2} \lambda - \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = -\frac{b}{ml^2} \pm \sqrt{\left(\frac{b}{ml^2}\right)^2 + \frac{4g}{l}}
\]

\[
eig(A) = \{ \lambda : |\lambda I - A| = 0 \} = \{ \lambda : \lambda^2 + \frac{b}{ml^2} \lambda - \frac{g}{l} = 0 \}
\]

The dynamics of the system is given by the eigenvalues of the system matrix

\[
x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}
\]

\[
\text{real(eig}(A)) > 0 \ (\text{real}(\lambda_1, \lambda_2) > 0) \Rightarrow \text{INSTABILITY}
\]

We use feedback control for STABILIZATION
Linear State Feedback

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u = Ax + Bu
\]

\[
u = -Kx = -\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} x
\]

\[
\dot{x} = (A - BK)x = \begin{bmatrix} 0 \\ \frac{g}{l} \end{bmatrix} - \frac{1}{ml^2} K_1 - \frac{b}{ml^2} \frac{1}{ml^2} K_2
\]

How do we choose \(K_1\) and \(K_2\) to make \(\text{real}\left(\text{eig}(A-BK)\right)<0\)? Always possible?

How do we choose \(K_1\) and \(K_2\) to satisfy optimality condition?

How do we proceed when states are not measurable?

MODERN CONTROL (ME 433)

Closed loop simulations: pend_par.m, statevar_control_lin.m, pendcllin01.mdl

Nonlinear State Feedback

\[
\dot{x} = \begin{bmatrix} 0 \\ \frac{g}{l} \sin(x_1) \end{bmatrix} - \frac{b}{ml^2} x_2 + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u = -mg\sin(x_1) + ml^2v
\]

Feedback Linearization

\[
\dot{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{1}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v = A^*x + B^*u
\]

\[
v = -Kx = -\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} x \Rightarrow \dot{x} = (A^* - B^*K)x = \begin{bmatrix} 0 \\ -K_1 \\ -\frac{b}{ml^2} \end{bmatrix} x
\]

We choose \(K_1\) and \(K_2\) to make \(\text{real}\left(\text{eig}(A^*-B^*K)\right)<0\)

\[
u = -mg\sin(\theta - \pi) - ml^2\left[K_1(\theta - \pi) + K_2 \dot{\theta} \right]
\]

NONLINEAR CONTROL (ME 350/450)

Closed loop simulations: pend_par.m, statevar_control_nolin.m, pendclnolin01.mdl
Nonlinear State Feedback

\[
\dot{x} = \begin{bmatrix}
0 & -\frac{x_2}{l^2} \\
\frac{g}{l} \sin(x_1) & -\frac{b}{ml^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{u}{ml^2} \end{bmatrix}
\]

Parameters \((m, l, b)\) are not well known:

- MULTIVARIABLE ROBUST CONTROL (ME 350/450)
- SYSTEM IDENTIFICATION AND ADAPTIVE CONTROL (ME 350/450)

Flexible pendulum \(\Rightarrow\) ODE \(\Rightarrow\) PDE:

- DISTRIBUTED PARAMETER SYSTEMS (ME 350/450)

Controls Education at Lehigh

- ME 343: CLASSICAL CONTROL FALL
- ME 389: CONTROLS LAB SPRING
- ME 433: MODERN & OPTIMAL CONTROL FALL
- ME 350: ADVANCED TOPICS IN CONTROL SPRING

- NONLINEAR CONTROL
- MULTIVARIABLE ROBUST CONTROL
- SYSTEM IDENTIFICATION AND ADAPTIVE CONTROL
- DISTRIBUTED PARAMETER SYSTEMS