Nuclear Fusion and Radiation
Lecture 5 (Meetings 9, 10, 11 & 12)

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Power Balance in Fusion Plasmas: P_{aux}, P_α, P_{loss}

- Now that we have established methods for calculating reaction rates in a plasma over the whole energy/speed M-B distribution, we are in a position to consider energy/power balances in a fusioning plasma.

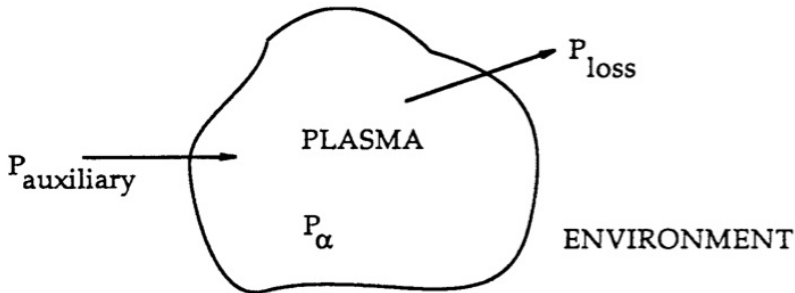- Consider the following schematic representation of a fusioning plasma:

Figure 1: Power balance.

Power Balance in Fusion Plasmas: P_{aux}, P_α, P_{loss}

- P_{aux} denotes the external power input required to heat the fuel to fusion conditions and to sustain temperature at fusion conditions.

- P_α denotes the power associated with DT produced α particles ($_2^4 He$) assumed to be deposited within the plasma.

$$_1^2 D +_1^3 T \rightarrow {}_2^4 He +_0^1 n \tag{1}$$

Note: $Q_{DT} = 17.6 MeV \rightarrow Q_{DT}^\alpha = 3.5 MeV, Q_{DT}^n = 14.1 MeV$.

- P_{loss} denotes the power loss from the plasma.

What mechanisms might result in energy loss from the plasma?

Power Balance in Fusion Plasmas: Energy Loss Mechanisms

- The temperature of the environment is much less than that of the plasma. Therefore, very steep gradients in temperature will exist between the plasma and its environment, driving heat out of the plasma (heat conduction).

- The density of ions and electrons in the environment is much less than in the plasma. Therefore, very steep gradients in density will exist between the plasma and its environment, driving particles out of the plasma (particle transport/diffusion). Note also that escaping particles carry energy out with them.

- Processes in the plasma can generate electromagnetic (EM) radiation and much of this radiation will not be absorbed in the plasma, but will escape from the plasma leading to radiative power loss from the plasma.

Power Balance in Fusion Plasmas: Key Equations

- The <u>plasma energy</u> is given by

$$E_{content} = \int_{volume} \left(\frac{3}{2} n_i k T_i + \frac{3}{2} n_e k T_e \right) dV \qquad (2)$$

  $n_e/T_e, n_i/T_i$: electron, ion density/temperature.

- The <u>quasineutrality</u> condition is written as

$$n_p = n_e \qquad (3)$$

  n_p, n_e : number (i.e., density) of protons, electrons.

- Consider energy content of **purely hydrogenic** 50/50 DT plasma:
  - No ions other than fuel ions (no impurities or α particles)), i.e. $n_i = n_D + n_T$. This is an approximation. There is always some α content in fusion plasmas. There is always some impurity content. Both are considered negligible in purely hydrogenic plasmas.
  - All species have same temperature, i.e. $T_i = T_e \triangleq T$. This is also an approximation. T_i and T_e may be different in general.
  - D and T have optimal 50/50 mixture, i.e. $n_D = n_T$.

Power Balance in Fusion Plasmas: Key Equations

- For the assumed **purely hydrogenic** DT plasma, we have

$$n_p = n_D Z_D + n_T Z_T = n_D + n_T = n_i \tag{4}$$

- In this case, the quasineutrality conditions reduces to

$$n_i = n_e \tag{5}$$

- The overall plasma density is always defined as

$$n = n_i + n_e \tag{6}$$

- Therefore, for the assumed **purely hydrogenic** DT plasma,

$$n = n_i + n_e = 2n_e = 2n_i \tag{7}$$

- Recalling that $T_i = T_e \triangleq T$, we can then rewrite

$$E_{content} = \int_{volume} 3n_e kT dV = \int_{volume} \frac{3}{2} nkT dV \tag{8}$$

Power Balance in Fusion Plasmas: Key Equations

- In general, densities and temperatures will be functions of position and time. After integrating over the volume, and assuming $T_i = T_e \triangleq T$, we can write

$$E_{content} = \frac{3}{2}\bar{n}k\bar{T}V \qquad (9)$$

  where $\bar{n}$ and $\bar{T}$ represent the appropriate volume average of n and T.

- For convenience, we drop the upper-bar notation from now on.

- All variables are time-dependent and volume-averaged. With the appropriate volume averages, we can write the following power balance for our plasma:

$$\frac{d(\frac{3}{2}nkTV)}{dt} = P_{aux}V + P_\alpha V - P_L V \qquad (10)$$

- At <u>steady state</u>, $d()/dt = 0$, and

$$P_{aux}V + P_\alpha V = P_L V \iff P_{aux} + P_\alpha = P_L \qquad (11)$$

  Left: Power balance $[W]$. Right: Power density balance $[W/m^3]$.

Power Balance in Fusion Plasmas: Power Losses

- For macroscopic calculations, the term P_L is generally defined in terms of the energy content of the plasma and a global/ overall energy confinement time τ_E. Thus,

$$P_L \triangleq \frac{\frac{3}{2}nkT}{\tau_E} \tag{12}$$

- The above equation says nothing about the physics or mechanisms for energy loss, it simply defines a global energy confinement time

$$\tau_E \triangleq \frac{\frac{3}{2}nkT}{P_L} \tag{13}$$

- For the **purely hydrogenic** DT plasma ($n = 2n_e$),

$$P_L = \frac{3n_ekT}{\tau_E}, \quad \tau_E = \frac{3n_ekT}{P_L} \tag{14}$$

- The α-power density, P_α, can be written as

$$P_\alpha = \underbrace{n_D n_T < \sigma V >_{DT}}_{R_{DT}} E_\alpha \tag{15}$$

where it is understood that the densities and the reaction rate parameter are volume averages. $E_\alpha \triangleq Q_{DT}^\alpha = 3.5$ MeV denotes the fraction of energy released by the DT fusion reactor that is carried by the α particle.

- For the **purely hydrogenic** 50/50 DT plasma ($n_D = n_T = n_e/2$),

$$P_\alpha = \frac{n_e^2}{4} < \sigma V >_{DT} E_\alpha \tag{16}$$

- We can also define a plasma condition in which the plasma is energy/ power self sufficient, a situation in which the losses, P_L, are exactly balanced by the α-power, P_α. Thus,

$$P_L = P_\alpha \iff P_{aux} = 0 \tag{17}$$

This condition is called "ignition".

- In terms of plasma parameters the ignition condition can be written as

$$n_D n_T < \sigma V >_{DT} E_\alpha = \frac{\frac{3}{2} n k T}{\tau_E} \tag{18}$$

Power Balance in Fusion Plasmas: Ignition Condition

- For the **purely hydrogenic** 50/50 DT plasma ($n_e = n_i = n_D + n_T$, $n = n_i + ne = 2n_e = 2n_i$, $n_D = n_T = \frac{n_i}{2} = \frac{n_e}{2}$),

$$\frac{n_e^2}{4} < \sigma V >_{DT} E_\alpha = \frac{3 n_e kT}{\tau_E} \qquad (19)$$

where the volume average symbol has been dropped for convenience.

- Rearranging the above yields

$$n_e \tau_E |_{IGN}^{DT} = \frac{12kT}{< \sigma V >_{DT} E_\alpha} \qquad (20)$$

$$n \tau_E |_{IGN}^{DT} = \frac{24kT}{< \sigma V >_{DT} E_\alpha} \qquad (21)$$

- Since

$$< \sigma V >_{DT} = f(kT) \qquad (22)$$

the ignition parameter is a function of kT ($n_e \tau_E |_{IGN}^{DT} = g(kT)$).

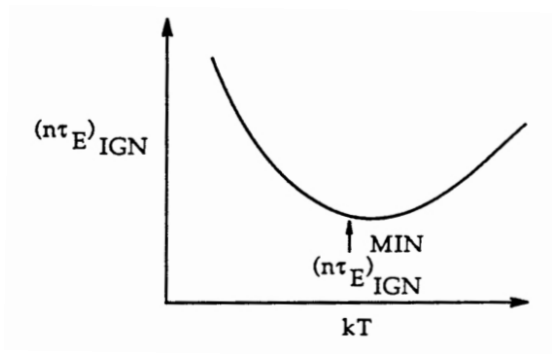# Power Balance in Fusion Plasmas: Ignition Condition

Figure 2: Ignition parameter.

- Note that $n_e\tau_E|_{IGN}$ as defined here is a requirement on the product $n_e\tau_E$. As we shall see later the value of $n_e\tau_E$ actually achieved for a given plasma must be compared with the required $n_e\tau_E$ to judge if the plasma can ignite.

Energy Gain in a Fusion Reaction: Power Gain Q

- We now turn to fusion power (or energy) gain in a fusioning plasma. In discussing energy gain in our beam-target reactor system, we defined the gain in terms of the ratio of fusion energy out to injected energy in the beam. For fusioning plasmas we have a similar definition:

$$\text{Power Gain} \equiv \frac{P_f}{P_{aux}} \equiv Q \tag{23}$$

- Historically the symbol Q has been used for the power gain (Note that this symbol is also used to denote energy release in fusion reactions!!!). From now on, we will use Q for power gain and E for energy release. For our steady-state plasma,

$$P_{aux} = P_L - P_\alpha \iff Q = \frac{P_f}{P_L - P_\alpha} \tag{24}$$

- Note that at ignition, $Q \to \infty$ ($P_L = P_\alpha$).

Energy Gain in a Fusion Reaction: Power Gain Q

- We can divide numerator and denominator of the above equation by P_α, yielding

$$Q = \frac{P_f/P_\alpha}{P_L/P_\alpha - 1} \qquad (25)$$

- In a DT reaction,

$$P_f = \underbrace{n_D n_T < \sigma V >_{DT}}_{R_{DT}} E_f \qquad (26)$$

where $E_f \triangleq Q_{DT} = 17.6$ MeV.

- What is the value of P_f/P_α for the DT reaction?

$$\frac{P_f}{P_\alpha} = \frac{n_D n_T < \sigma V >_{DT} E_f}{n_D n_T < \sigma V >_{DT} E_\alpha} = \frac{E_f}{E_\alpha} = \frac{17.6 MeV}{3.5 MeV} \sim 5 \qquad (27)$$

- What is the value of P_L/P_α for the 50/50 DT reaction in the case of a **purely hydrogenic** plasma?

$$\frac{P_L}{P_\alpha} = \frac{\frac{3 n_e kT}{\tau_E}}{\frac{n_e^2}{4} < \sigma V >_{DT} E_\alpha} = \frac{12 kT}{< \sigma V >_{DT} E_\alpha} \frac{1}{n_e \tau_E} \qquad (28)$$

Energy Gain in a Fusion Reaction: Power Gain Q

- But

$$\frac{12kT}{<\sigma V>_{DT} E_\alpha} \equiv n_e \tau_E|_{IGN}^{DT} = g(kT) \qquad (29)$$

Thus, at a given kT value

$$\frac{P_L}{P_\alpha} = \frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACTUAL}^{DT}} \qquad (30)$$

Then,

$$Q^{DT} = \frac{5}{\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT}^{DT}} - 1} \qquad (31)$$

- Note that at ignition, $Q \to \infty$ ($n_e \tau_E|_{IGN}^{DT} = n_e \tau_E|_{ACT}^{DT}$).

- Various goals of the fusion program are discussed in terms of values of Q.

Energy Gain in a Fusion Reaction: Breakeven Condition

- The goal of "breakeven" is defined as $Q = 1 \Leftrightarrow P_{aux} = P_f$. In this case,

$$Q^{DT} = \frac{5}{\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT}^{DT}} - 1} \equiv 1, \qquad (32)$$

yields

$$\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT(Q=1)}^{DT}} = 6 \Rightarrow n_e \tau_E|_{ACT(Q=1)}^{DT} = \frac{1}{6} n_e \tau_E|_{IGN}^{DT} \qquad (33)$$

- For $Q^{DT} = 1$, the achieved $n_e \tau$ must be $\sim 1/6$ of that required for ignition. At $kT = 10 keV$,

$$n_e \tau_E|_{IGN}^{DT} \sim 3 \times 10^{20} \frac{s}{m^3} \Rightarrow n_e \tau_E|_{ACT(Q=1)}^{DT} \sim 5 \times 10^{19} \frac{s}{m^3} \qquad (34)$$

- The goal of breakeven, achieving as much fusion power release as auxiliary power into the plasma, has been the major goal of the fusion program for over forty years. However, it should be emphasized that $Q = 1$ will not yield net electricity, $Q > 1$ is required for a useful reactor.

- Another goal of the fusion program is to achieve plasma conditions such that the α-power deposited in the plasma (P_α) is equal to the auxiliary power (P_{aux}) to the plasma–this is an interesting physics regime since now the α-power plays a major role in heating the plasma. For this condition,

$$P_{aux} = P_\alpha \Leftrightarrow Q = \frac{P_f}{P_{aux}} = \frac{P_f}{P_\alpha} = 5 \tag{35}$$

- Therefore,

$$Q^{DT} = \frac{5}{\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT}^{DT}} - 1} \equiv 5, \tag{36}$$

  which implies

$$\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT(Q=5)}^{DT}} = 2. \tag{37}$$

- For $Q^{DT} = 5$, the achieved $n_e \tau$ must be $\sim 1/2$ of that required for ignition. At $kT = 10 keV$, this results in the requirement

$$n_e \tau_E|_{ACT(Q=5)}^{DT} = \frac{1}{2} n_e \tau_E|_{IGN}^{DT} \sim 1.5 \times 10^{20} \frac{s}{m^3} \tag{38}$$

Particle Balances: F, B, L_p

- Up to now we have been considering power balances. It is also important to consider particle balances in a fusioning plasma. Schematically,
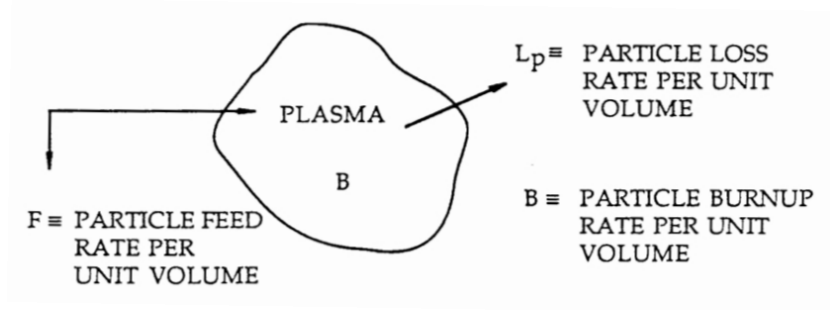

Figure 3: Particle balance.

Particle Balances: Key Equations

- Thus, for the assumed **purely hydrogenic** DT plasma,

$$\frac{d(n_D + n_T)}{dt} = F - L_p - B \tag{39}$$

where $n_D + n_T = n_i$ represents the appropriate volume average of the fuel density.

- As with the P_L term in the power balance we define a global/macroscopic particle confinement time, τ_p, such that

$$L_p \equiv \frac{(n_D + n_T)}{\tau_p} \tag{40}$$

- The burnup rate is given by (**purely hydrogenic** DT plasma: $n_e = n_i = n_D + n_T$)

$$B \equiv \underbrace{\frac{n_e^2}{4} <\sigma V>_{DT}}_{R_{DT}} \times 2 = \frac{n_e^2}{2} <\sigma V>_{DT} = \frac{(n_D + n_T)^2}{2} <\sigma V>_{DT} \tag{41}$$

Again, average symbol has been dropped for convenience.

Particle Balances: Key Equations

- At steady state, $d(n_D + n_T)/dt = 0$ and

$$F = L_p + B = \frac{(n_D + n_T)}{\tau_p} + \frac{(n_D + n_T)^2}{2} < \sigma V > \qquad (42)$$

- Defining $n_{DT} \triangleq n_D = n_T$, we can rewrite

$$F = \frac{(2n_{DT})}{\tau_p} + \frac{(2n_{DT})^2}{2} < \sigma V > \qquad (43)$$

- It is also useful to define a quantity called the fractional burnup, f_b, where

$$f_b \equiv \frac{B}{F} = \frac{\frac{(2n_{DT})^2}{2} < \sigma V >}{\frac{2n_{DT}}{\tau_p} + \frac{(2n_{DT})^2}{2} < \sigma V >} = \frac{n_{DT} < \sigma V >}{\frac{1}{\tau_p} + n_{DT} < \sigma V >} \qquad (44)$$

Particle Balances: Fractional Burnup

What is this quantity?

- Remember that the mean life of a particle between fusion events is given by

$$\tau_{DT} = \frac{1}{n_{DT} < \sigma V >} \tag{45}$$

Thus,

$$f_b = \frac{\frac{1}{\tau_{DT}}}{\frac{1}{\tau_p} + \frac{1}{\tau_{DT}}} \frac{\tau_p}{\tau_p} = \frac{\frac{\tau_p}{\tau_{DT}}}{1 + \frac{\tau_p}{\tau_{DT}}} \tag{46}$$

- For ignition at $10 keV$ with standard plasma, $n_e = 10^{20}/m^3$:

$$n_e \tau_E |_{IGN}^{DT} \sim 3 \times 10^{20} \frac{s}{m^3}, \qquad \tau_E \sim 3s \tag{47}$$

- Assume $\tau_p = 2\tau_E \sim 6s$. From previous calculations $\tau_{DT} \sim 180s$. Thus,

$$f_b = \frac{\frac{6}{180}}{1 + \frac{6}{180}} \sim 0.032 \tag{48}$$

About 3.2% of the feed is burned up and the rest appears as particle loss!!

Radiative Processes in Plasmas

- We previously examined power/energy balances in plasmas on a macroscopic scale and we lumped all energy loss phenomena into a single term:

$$P_L \equiv \frac{\frac{3}{2}nkT}{\tau_E} \tag{49}$$

- For the **purely hydrogenic** DT plasma ($n = 2n_e$)

$$P_L = \frac{3n_e kT}{\tau_E} \tag{50}$$

- We noted P_L might consist of at least three contributions:
    - diffusion losses $\propto dn/dx$ (and whatever energy the "leaking" particle carries)
    - conduction losses $\propto dT/dx$
    - radiative losses

- We will now turn to a discussion of the radiative losses.

Radiative Processes in Plasmas: Bremsstrahlung Radiation

- There are a number of processes by which EM radiation can be produced in, and eventually lost from, a plasma.

- For fusioning plasmas the most important process is in general Bremsstrahlung (braking radiation). Whenever a charge experiences an acceleration/ deceleration (i.e., change in velocity), it will radiate electromagnetic energy.

- In a plasma, coulomb scattering collisions will cause acceleration/ deceleration of charged particles and will therefore result in the emission of Bremsstrahlung radiation.

Radiative Processes in Plasmas: Bremsstrahlung Radiation

- Consider a scattering collision between target ion Z_1 and projectile (ion or electron) Z_2. The acceleration, a_2, of Z_2 produced by the collision is:

$$a_2 = \frac{F_{12}}{M_2} \tag{51}$$

where

$$F_{12} = \text{ Force between } Z_1 \text{ and } Z_2 \propto Z_1 Z_2. \tag{52}$$

Therefore

$$a_2 \propto \frac{Z_1 Z_2}{M_2} \tag{53}$$

- It can be shown that the intensity of the emitted radiation is proportional to $\|a_2\|^2$ and thus the Bremsstrahlung radiation intensity is proportional to $(Z_1 Z_2 / M_2)^2$.

Radiative Processes in Plasmas: Bremsstrahlung Radiation

- In a plasma, the primary source of Bremsstrahlung radiation is that resulting from electron acceleration / deflections (note the mass dependence of the intensity) and essentially all from electron-ion collisions.

- It can be shown that the Bremsstrahlung radiation power density, P_b, is given by:

$$P_b^Z = C T_e^{1/2} n_e \sum_{\text{ions } i} n_i < \bar{Z}_i >^2 \qquad (54)$$

where C is a constant and $< \bar{Z}_i >$ represents the average charge state of species i, which need not be fully ionized.

- In discussing plasma radiation it is useful to define the
  <u>effective atomic number</u> of the plasma as

$$Z_{eff} = \frac{\sum_i n_i < \bar{Z}_i >^2}{n_e} \qquad (55)$$

Radiative Processes in Plasmas: Bremsstrahlung Radiation

- The Bremsstrahlung radiation power density is rewritten as

$$P_b^Z = C n_e^2 T_e^{1/2} Z_{eff} \tag{56}$$

- Note that for a **purely hydrogenic** DT plasma

$$<\bar{Z}_i> = 1 \text{ and } \sum_i n_i <\bar{Z}_i>^2 = n_D + n_T \tag{57}$$

and by charge neutrality

$$n_e = n_D + n_T = n_i. \tag{58}$$

Then, $Z_{eff} \equiv 1$, and

$$P_b = C T_e^{1/2} n_e^2 \tag{59}$$

Radiative Processes in Plasmas: Bremsstrahlung Radiation

- Once again, the Bremsstrahlung radiation power density is rewritten as

$$P_b^Z = \underbrace{Cn_e^2 T_e^{1/2}}_{\text{Hydrogenic Term}} Z_{eff} \tag{60}$$

where $[T_e] = keV$. The Bremsstrahlung constant is given by

$$C = 5 \times 10^{-37} \frac{Wm^3}{\sqrt{keV}} = 5 \times 10^{-43} \frac{MWm^3}{\sqrt{keV}} \tag{61}$$

- Consider the standard purely hydrogenic plasma, $n_e = n_i = 10^{20}/m^3$, $T_e = T_i = 10keV$, where $Z_{eff} = 1$. Then,

$$P_b^{Z=1} = 5 \times 10^{-43}(10^{20})^2(10)^{1/2} \sim 0.016 MW/m^3 \tag{62}$$

- In addition, recall that for the standard plasma $P_\alpha \sim 0.15 MW/m^3$ (Lecture 4 - Slide 43). Thus, for this case $P_b \sim P_\alpha/10$.

- The power balance equation in Slide 7 needs to be modified to include the effect of the radiation loss.

- In this case, we can write the following power balance for our plasma:

$$\frac{d(\frac{3}{2}nkTV)}{dt} = P_{aux}V + P_\alpha V - P_L V - P_b V \tag{63}$$

- At <u>steady state</u>, $d()/dt = 0$, and

$$P_{aux}V + P_\alpha V = P_L V + P_b V \iff P_{aux} + P_\alpha = P_L + P_b \tag{64}$$

Power Balance with P_b: Purely Hydrogenic 50/50 DT plasma

- We will now examine the power balance equation for a purely hydrogenic 50/50 DT plasma ($Z_{eff} = 1$) with P_b included explicitly for the ignition case ($P_{aux} = 0$).

- For this case,

$$
\begin{align}
P_\alpha &= P_L + P_b \tag{65} \\
\frac{n_e^2}{4} < \sigma V >_{DT} E_\alpha &= \frac{3 n_e kT}{\tau_E} + C T_e^{1/2} n_e^2 Z_{eff} \tag{66}
\end{align}
$$

- This yields,

$$
n_e \tau_E |_{IGN,P_b}^{DT} = \frac{12 kT}{< \sigma V >_{DT} E_\alpha - 4 C T^{1/2}} \tag{67}
$$

- Comparing this with the result in the absence of radiation loss (see (20)),

$$
n_e \tau_E |_{IGN}^{DT} = \frac{12 kT}{< \sigma V >_{DT} E_\alpha} \tag{68}
$$

we note that $n_e \tau_E |_{IGN,P_b}^{DT} > n_e \tau_E |_{IGN}^{DT}$ (tougher requirement!).

Power Balance with P_b: Purely Hydrogenic 50/50 DT plasma

- Consider the ratio

$$\frac{n_e\tau_E|_{IGN,Pb}^{DT}}{n_e\tau_E|_{IGN}^{DT}} = \frac{<\sigma V>_{DT} E_\alpha}{<\sigma V>_{DT} E_\alpha - 4CT^{1/2}} \tag{69}$$

- Multiply numerator and denominator by $n_e^2/4$

$$\frac{n_e\tau_E|_{IGN,Pb}^{DT}}{n_e\tau_E|_{IGN}^{DT}} = \frac{\frac{n_e^2}{4} <\sigma V>_{DT} E_\alpha}{\frac{n_e^2}{4} <\sigma V>_{DT} E_\alpha - Cn_e^2T^{1/2}} \tag{70}$$

- For the standard case this yields

$$\frac{n_e\tau_E|_{IGN,Pb}^{DT}}{n_e\tau_E|_{IGN}^{DT}} = \frac{0.15}{0.15 - 0.016} = 1.12 \tag{71}$$

- The required $n_e\tau_E|_{IGN}$ increases by 12% compared to the case where Bremsstrahlung radiation loss is neglected!

Power Balance with P_b: Non-Purely Hydrogenic 50/50 DT plasma

- We will now examine the power balance equation for a non-purely hydrogenic 50/50 DT plasma (plasma with impurities ($Z_{eff} \neq 1$)) with P_b included explicitly for the ignition case ($P_{aux} = 0$).

- We are particularly interested in comparing this non-purely hydrogenic 50/50 DT plasma with the purely hydrogenic 50/50 DT plasma (plasma without impurities ($Z_{eff} = 1$)) studied before.

- We consider the case with iron impurity (which could be introduced into the plasma from surrounding walls). Assume $n_{Fe}/n_e = 0.01(1\%)$ and $< \bar{Z}_{Fe} > = 14$. Thus, iron ($Z = 26$) is not fully ionized. We assume that this value of the charge state is obtained from spectroscopic measurements.

- Thus,

$$
Z_{eff} = \frac{\sum_i n_i < \bar{Z}_i >^2}{n_e} \quad = \quad \frac{n_D + n_T + n_{Fe} < \bar{Z}_{Fe} >^2}{n_e} \tag{72}
$$

$$
= \quad \frac{n_D + n_T}{n_e} + \frac{n_{Fe} < \bar{Z}_{Fe} >^2}{n_e} \tag{73}
$$

Power Balance with P_b: Non-Purely Hydrogenic 50/50 DT plasma

- We are given the second term in Z_{eff}, how do we obtain $(n_D + n_T)/n_e$?.

  We apply charge neutrality!!

- Charge neutrality requires that the number of electrons per unit volume equals the number of positive charges per unit volume. Thus,

$$n_e = n_p \tag{74}$$
$$n_e = n_D Z_D + n_T Z_T + n_{Fe} < \bar{Z}_{Fe} > \qquad (Z_D = Z_T = 1) \tag{75}$$

- Dividing both sides of this equation by n_e yields

$$1 = \frac{n_D + n_T}{n_e} + \frac{n_{Fe} < \bar{Z}_{Fe} >}{n_e} \tag{76}$$

and

$$\frac{n_D + n_T}{n_e} = 1 - \frac{n_{Fe}}{n_e} < \bar{Z}_{Fe} > \tag{77}$$

Power Balance with P_b: Non-Purely Hydrogenic 50/50 DT plasma

- Thus,

$$Z_{eff} = 1 - \frac{n_{Fe}}{n_e} < \bar{Z}_{Fe} > + \frac{n_{Fe}}{n_e} < \bar{Z}_{Fe} >^2 \qquad (78)$$

For our case

$$Z_{eff} = 1 - (0.01)(14) + (0.01)(14)^2 = 2.82 \qquad (79)$$

- With the standard plasma (Slide 27: $P_b^{Z=1} = 0.016 MW/m^3$)

$$P_b^{Fe} = P_b^{Z=1} 2.82 = (0.016 \frac{MW}{m^3})(2.82) = 0.045 \frac{MW}{m^3} \qquad (80)$$

- Now that we have already analyzed the impact of the iron impurities on the radiation losses, it is also of interest to consider the impact of the iron impurities on both the fuel density and the plasma power density.

Power Balance with P_b: Non-Purely Hydrogenic 50/50 DT plasma

- We still assume $n_D = n_T \equiv n_{DT}$. The neutrality condition

$$\frac{n_D + n_T}{n_e} = 1 - \frac{n_{Fe}}{n_e} < \bar{Z}_{Fe} > \tag{81}$$

  yields (note that $n_D + n_T = 2n_{DT}$)

$$n_{DT}^{Fe} = \frac{1}{2}\left[1 - \frac{n_{Fe}}{n_e} < \bar{Z}_{Fe} >\right] n_e \iff n_{DT}^{Fe} = 0.43 n_e \tag{82}$$

- In the case with no impurities (**purely hydrogenic** plasma), the neutrality condition $n_e = n_D + n_T = 2n_{DT}$ yields $n_{DT}^{Z=1} = 0.5 n_e$.

- Then,

$$\frac{P_\alpha^{Fe}}{P_\alpha^{Z=1}} = \frac{(0.43 n_e)^2}{(0.5 n_e)^2} \sim 0.74 \tag{83}$$

- For the standard plasma (Lecture 4 - Slide 43: $P_\alpha^{Z=1} = 0.15 MW/m^3$),

$$P_\alpha^{Fe} = 0.74 \times P_\alpha^{Z=1} = 0.74 \times 0.15\frac{MW}{m^3} \sim 0.11\frac{MW}{m^3} \tag{84}$$

Power Balance with P_b: Non-Purely Hydrogenic 50/50 DT plasma

- In this case, the power balance equation is written as

$$
\begin{aligned}
P_\alpha &= P_L + P_b & (85) \\
n_D n_T <\sigma V>_{DT} E_\alpha &= \frac{\frac{3}{2} nkT}{\tau_E} + C T_e^{1/2} n_e^2 Z_{eff} & (86) \\
(0.43 n_e)^2 <\sigma V>_{DT} E_\alpha &= \frac{\frac{3}{2}(1.87 n_e)kT}{\tau_E} + 2.82 C T_e^{1/2} n_e^2 & (87)
\end{aligned}
$$

where we have used

$$
\begin{aligned}
n = n_e + n_i &= n_e + n_D + n_T + n_{Fe} & (88) \\
&= n_e + 2 n_{DT}^{Fe} + 0.01 n_e & (89) \\
&= n_e + 2(0.43) n e + 0.01 n_e & (90) \\
&= 1.87 n_e & (91)
\end{aligned}
$$

Power Balance with P_b: Non-Purely Hydrogenic 50/50 DT plasma

- This yields,

$$n_e\tau_E|_{IGN,P_b}^{Fe} = \frac{\frac{3}{2}\frac{1.87}{0.43^2}kT}{<\sigma V>_{DT} E_\alpha - \frac{2.82}{0.43^2}CT^{1/2}} \qquad (92)$$

- Comparing the above with the result in the absence of impurities (Slide 29),

$$n_e\tau_E|_{IGN,P_b}^{DT} = \frac{12kT}{<\sigma V>_{DT} E_\alpha - 4CT^{1/2}} \qquad (93)$$

we compute the ratio

$$\frac{n_e\tau_E|_{IGN,P_b}^{Fe}}{n_e\tau_E|_{IGN,Pb}^{DT}} = \frac{\frac{3}{2}\frac{1.87}{0.43^2}}{12}\frac{<\sigma V>_{DT} E_\alpha - 4CT^{1/2}}{<\sigma V>_{DT} E_\alpha - \frac{2.82}{0.43^2}CT^{1/2}} \qquad (94)$$

Power Balance with P_b: Non-Purely Hydrogenic 50/50 DT plasma

- Multiply numerator and denominator by $n_e^2/4$, we obtain

$$\frac{n_e \tau_E |_{IGN,P_b}^{Fe}}{n_e \tau_E |_{IGN,Pb}^{DT}} = \frac{3}{2} \frac{1.87}{0.43^2} \frac{\frac{n_e^2}{4} < \sigma V >_{DT} E_\alpha - n_e^2 C T^{1/2}}{12 \frac{n_e^2}{4} < \sigma V >_{DT} E_\alpha - \frac{2.82}{4 \times 0.43^2} n_e^2 C T^{1/2}} \quad (95)$$

- For the standard case this yields

$$\frac{n_e \tau_E |_{IGN,P_b}^{Fe}}{n_e \tau_E |_{IGN,Pb}^{DT}} = \frac{3}{2} \frac{1.87}{0.43^2} \frac{0.15 - 0.016}{12} \frac{0.15 - 0.016}{0.15 - \frac{2.82}{4 \times 0.43^2} 0.016} \approx 1.9 \quad (96)$$

The required $n_e \tau_E |_{IGN}$ increases by $\sim 90\%$ compared to the case where there are no impurities!

- Impurities (high Z materials) are deleterious to the plasma power balance in two ways:
    1. Enhancement of P_b by Z_{eff} factor
    2. Reduction of P_α by fuel depletion effect

- This turns into an increase in the $n_e \tau_E |_{IGN}$ requirement!