Control of PDE Systems

Lecture 2: Backstepping Control

Eugenio Schuster

schuster@lehigh.edu

Mechanical Engineering and Mechanics
Lehigh University
Control Lyapunov Function (CLF)

We are interested in an extension of the Lyapunov function concept, called a *control Lyapunov function* (CLF). Let us consider the following system:

\[ \dot{\eta} = f(\eta, u), \quad \eta \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad f(0, 0) = 0, \]

Task: Find a feedback control law \( u = \phi(\eta) \) such that the equilibrium \( \eta = 0 \) of the closed-loop system

\[ \dot{\eta} = f(\eta, \phi(\eta)) \]

is globally asymptotically stable.
Control Lyapunov Function (CLF)

Task: Find a feedback control law \( u = \phi(\eta) \) and a Lyapunov function candidate \( V(\eta) \) such that

\[
\dot{V} = \frac{\partial V}{\partial \eta}(\eta) f(\eta, \phi(\eta)) \leq -W(\eta), \quad W(\eta) \text{ positive definite}
\]

A system for which a good choice of \( V(\eta) \) and \( W(\eta) \) exists is said to possess a CLF.
**Backstepping**

**Assumption:** There exist a stabilizing state feedback control law $\phi(\eta)$, with $\phi(0) = 0$, and a Lyapunov function $V(\eta)$ s.t.

$$\frac{\partial V}{\partial \eta} [f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \quad W(\eta) \text{ positive definite}$$

**Lemma [K] 14.2:** Integrator Backstepping

$$\begin{align*}
\dot{\eta} &= f(\eta) + g(\eta)\xi \\
\dot{\xi} &= u
\end{align*}$$

There is a whole integrator between $u$ and $\xi$. Under the previous assumption, the system has a CLF

$$V_a(\eta, \xi) = V(\eta) + \frac{1}{2}(\xi - \phi(\eta))^2, \quad (a: \text{augmented})$$
and the corresponding feedback that gives global asymptotical stability is

\[ u = -c(\xi - \phi(\eta)) + \frac{\partial \phi}{\partial \eta}(\eta)[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta}g(\eta), \quad c > 0 \]

Moreover, if all the assumptions hold globally and \( V(\eta) \) is radially unbounded, the origin will be globally asymptotically stable.

Backstepping: We have a “virtual” control \( \xi \) and we have to go back through an integrator.

Proof:
Backstepping

Figure 1: Original block diagram.
Figure 2: Block diagram after introducing $\phi(\eta)$.
Backstepping

Figure 3: Block diagram after “backstepping" $-\phi(\eta)$ through integrator.
Backstepping

Example:

\[ \dot{x}_1 = x_1 x_2 \]
\[ \dot{x}_2 = u \]
Backstepping

Example [K] 14.8:

\[
\begin{align*}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\
\dot{x}_2 &= u
\end{align*}
\]
Backstepping

In the case of more than one integrator

\[
\begin{align*}
\dot{\eta} &= f(\eta) + g(\eta)\xi_1 \\
\dot{\xi}_1 &= \xi_2 \\
&\quad\vdots \\
\dot{\xi}_{n-1} &= \xi_n \\
\dot{\xi}_n &= u
\end{align*}
\]

we only have to apply the backstepping lemma \( n \) times.
Example [K] 14.9:

\[ \begin{align*}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u
\end{align*} \]
Backstepping

In the more general case

\[
\dot{x} = f(x) + g(x)\xi \\
\dot{\xi} = f_a(x, \xi) + g_a(x, \xi)u
\]

If \(g_a(x, \xi) \neq 0\) over the domain of interest, the input transformation

\[
u = \frac{1}{g_a(x, \xi)}[v - f_a(x, \xi)]
\]

will reduce the system to

\[
\dot{x} = f(x) + g(x)\xi \\
\dot{\xi} = v
\]

and the backstepping lemma can be applied.
Backstepping

Strict Feedback Systems:

\[
\begin{align*}
\dot{x}_i &= x_{i+1} + \phi_i(\bar{x}_i) \quad i = 1, \ldots, n - 1 \\
\dot{x}_n &= u + \phi_n(x)
\end{align*}
\]

where \(\bar{x}_i = [x_1, \ldots, x_i]^T\), \(\phi_i(\bar{x}_i)\) are smooth and \(\phi_i(0) = 0\). We have a local triangular structure:

\[
\begin{align*}
\dot{x}_1 &= x_2 + \phi_1(x_1) \\
\dot{x}_2 &= x_3 + \phi_2(x_1, x_2) \\
\vdots \\
\dot{x}_n &= u + \phi_n(x_1, x_2, \ldots, x_n)
\end{align*}
\]

Linear part: Brunovsky canonical form \(\Rightarrow\) feedback linearizable
Backstepping

The control law

\[ z_i = x_i - \alpha_{i-1}(\bar{x}_{i-1}) \quad \alpha_0 = 0 \]

\[ \alpha_i(\bar{x}_i) = -z_{i-1} - c_iz_i - \phi_i + \sum_{j=1}^{i} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \phi_j), \quad c_i > 0 \]

\[ u = \alpha_n \]

guarantees global asymptotic stability of \( x = 0 \).

Proof:
Backstepping

The technique can be extended to more general Strict Feedback Systems:

\[
\begin{align*}
\dot{x}_i &= \psi_i(\bar{x}_i)x_{i+1} + \phi_i(\bar{x}_i) \quad i = 1, \ldots, n - 1 \\
\dot{x}_n &= \psi_n(x)u + \phi_n(x)
\end{align*}
\]

where \(\bar{x}_i = [x_1, \ldots, x_i]^T\) (\(\bar{x}_n = x\)), \(\phi_i(\bar{x}_i)\) are smooth and \(\phi_i(0) = 0\), and \(\psi_i(\bar{x}_i) \neq 0\) for \(i = 1, \ldots, n\) over the domain of interest.