Abstract—The central task of current profile control during the ramp-up phase of a tokamak discharge is to find the actuator trajectories that are necessary to achieve certain desired current profile at some time between the end of the ramp-up phase and early stage of the flattop phase. The magnetic diffusion partial differential equation (PDE) models the dynamics of the poloidal magnetic flux profile, which is closely related to the toroidal current density profile, and plays a key role in the model-based control synthesis. Given the initial and desired target profiles, splines are used in this work to generate evolutionary curves connecting their boundaries at both endpoints of the spatial domain. Then, a closed four-edge frame (initial profile, target profile, two boundary curves) in the three dimensional space (time, space, poloidal magnetic flux) is obtained without knowing the transient dynamics inside. The minimal surface theory is used in this work to define a surface spanned by the closed four-edge frame, which represents the desired transient dynamics for the poloidal magnetic flux. Then, the control task becomes a trajectory tracking problem. Once the desired transient dynamics is defined, the temporal and spatial derivatives of the poloidal magnetic flux in the magnetic diffusion equation can be computed, and the control-oriented PDE model can be reformulated into an algebraic equation where the control values at each time instant represent the to-be-determined unknown variables. Numerical simulation results show the effectiveness of this approach. This method is characterized by high speed computation and shows potential for real-time implementation in a closed-loop receding-horizon scheme, particularly for long-discharge tokamaks such as ITER.

I. INTRODUCTION

Nuclear fusion is the process by which two nuclei fuse together to form a heavier nucleus. This process is accompanied by a release of energy, which is the result of the mass “lost” in the reaction. The amount of released energy is given by Einstein’s famous equation (derived in 1905 as a part of his special theory of relativity), $E = (M_r - M_p)c^2$, where $E$ is the energy, $M_r$ the mass of the reactant nuclei, $M_p$ the mass of the product nuclei, and $c$ the speed of light. To make a fusion reaction possible, a certain amount of energy is required to bring two repellant nuclei carrying positive charges sufficiently close. To overcome the Coulomb barrier, the kinetic energy of the nuclei is increased by heating. The temperature required for a thermonuclear fusion reaction to take place is around 100 million degrees. At much lower temperatures (about 10 thousand degrees), the electrons and nuclei separate and create an ionized gas called plasma, also known as the fourth state of matter. The plasma conducts electricity and responds to magnetic fields, motivating a magnetic confinement approach to nuclear fusion. One type of magnetic confinement devices is the Tokamak, where a torus-shaped intangible bottle is created by magnetic fields to confine the high-temperature plasma.

During the ramp-up phase of a tokamak discharge (Fig. 1), multiple external sources (e.g., ohmic heating, neutral beam injection, radio frequency) can be used to control the spatial profile of many different plasma variables such as density, temperature, current, and rotation. Transport models usually governed by 1-D nonlinear coupled partial differential equations (PDEs) can be used to predict the plasma dynamics...
be computed. Thus, at each time instance the transport PDEs
degenerate to algebraic equations at every spatial point where
the control values are the only unknowns. Optimization
problems can be formulated to solve the algebraic equations
by taking into account the control constraints (see, e.g., [7],
[8]). By using this proposed technique, the ramp-up current
profile optimal control problem can be formulated into a
least-square problem with algebraic constraints, which is
much less computationally demanding.

The paper is organized as follows. The optimal control
problem for the current profile system is introduced in
Section II. The transient dynamics defined by the minimal
surface theory is presented in Section III. In Section IV,
algebraic equations for the unknown control values are
formulated at each time instant and are later solved by the
least square method. Simulation studies are presented in
Section V. The paper is closed in Section VI by stating
conclusions and future research remarks.

II. STATEMENT OF THE CONTROL PROBLEM

A. Control-oriented model

To enable model-based control of the current profile at
DIII-D, a control-oriented model for the dynamic evolu-
tion of the poloidal flux profile during and just following
the ramp-up of the plasma current has been recently pro-
posed [9]. During “Phase I” (see Fig. 1), mainly governed
by the ramp-up phase, the plasma current is mostly driven
by induction. In this case, it is possible to decouple the
equation for the evolution of the poloidal flux \( \psi(\hat{\rho}, t) \)
from the equation for the evolution of the temperature \( T_e(\hat{\rho}, t) \).
The magnetic diffusion equation is combined with empirical
 correlations obtained at DIII-D for the temperature and non-
inductive current to introduce a simplified dynamic model
describing the evolution of the poloidal flux during the
inductive phase of the discharge.

The current density \( j \), that flows toroidally around the
tokamak and whose profile must be controlled, is related to
spatial derivatives of the poloidal magnetic flux \( \psi \). We let
\( \rho \) be an arbitrary coordinate indexing the magnetic surface.
Any quantity constant on each magnetic surface could be
chosen as the variable \( \rho \). We choose the mean geometric
radius of the magnetic surface as the variable \( \rho \), i.e.,
\( \pi B_{\phi,o} \rho^2 = \Phi \), where \( B_{\phi,o} \) is the reference toroidal magnetic
field at the geometric plasma center \( R_o \). The variable \( \hat{\rho} \)
denotes the normalized radius \( \frac{\rho}{R_o} \), and \( \rho_b \) is the radius of
last closed flux surface. The evolution of the poloidal flux in
normalized cylindrical coordinates is given by the magnetic
diffusion equation,

\[
\frac{\partial \psi}{\partial t} = \frac{\eta(T_e)}{\mu_o \rho^2} \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} \left( \hat{\rho} F G H \frac{\partial \psi}{\partial \hat{\rho}} \right) - R_o \hat{H} \eta(T_e) \frac{\hat{\rho}}{B_{\phi,o}} > \frac{\hat{\rho}}{B_{\phi,o}} .
\]

(1)

where \( t \) is the time, \( \psi \) is the poloidal magnetic flux, \( \eta \) is the
plasma resistivity, \( T_e \) is the plasma electron temperature, \( \mu_o \)
is the vacuum permeability, \( j_{NI} \) is the non-inductive source
of current density (neutral beam, electron cyclotron, etc.), \( B \)
is the toroidal magnetic field, and \( <> \) denotes flux-surface

![Fig. 3. Surface integral. The surface \( S \) over domain \( M \) is expressed
by \( z = \psi(\hat{\rho}, t), \forall (\hat{\rho}, t) \in M \). The coordinate of \( A \) is...
](image-url)
average. \( \hat{F}, \hat{G}, \hat{H} \) are geometric factors, which are functions of \( \hat{\rho} \). The boundary conditions of (1) are given by

\[
\frac{\partial \psi}{\partial \rho} \bigg|_{\rho=0} = 0, \quad \frac{\partial \psi}{\partial \rho} \bigg|_{\rho=1} = \frac{\mu_0}{2\pi} \frac{R_0}{G} \hat{H} \bigg|_{\rho=1},
\]

where \( I(t) \) denotes the total plasma current.

Highly simplified models for the temperature and non-inductive toroidal current density are chosen for the inductive phase of the discharge. Based on experimental observations at DIII-D, the shapes of the profiles are assumed to remain fixed and equal to the so-called reference profiles, which are identified from DIII-D discharges associated with the experiment of interest. The responses to the actuators are simply scalar multiples of the reference profiles.

The temperature \( T_e \) is assumed to follow

\[
T_e(\hat{\rho}, t) = k_{Te} T_{e,\text{profile}}(\hat{\rho}) \frac{I(t) \sqrt{T_e}}{\bar{n}(t)},
\]

where the reference profile \( T_{e,\text{profile}} \) is identified from DIII-D through Thomson scattering, and \( k_{Te} = 1.7295 \cdot 10^{10} \text{m}^{-3}\text{A}^{-1/2}\text{m}^{-1/2} \). The average density \( \bar{n} \) is defined as \( \bar{n}(t) = \int_0^1 n(\hat{\rho}, t) d\hat{\rho} \), where \( n \) denotes the plasma density.

The non-inductive toroidal current density \( \bar{j}_{\text{NI}} = \bar{j}_{\text{NI,par}} \frac{T_{e,\text{profile}}}{\bar{n}(t)} \) is assumed to follow

\[
\bar{j}_{\text{NI,par}} \frac{T_{e,\text{profile}}}{\bar{n}(t)} = k_{NI,par}^\text{profile}(\hat{\rho}) \frac{I(t)^{1/2}}{\bar{n}(t)^{3/4}},
\]

where the reference profile \( k_{NI,par}^\text{profile} \) is identified from DIII-D through a combination of MSE diagnostics and the EFIT equilibrium reconstruction code, and \( k_{NI,par}^\text{profile} = 1.2139 \cdot 10^{18} \text{m}^{-9/2}\text{A}^{-1/2}\text{W}^{-5/4} \). The model for \( T_e \) and \( \bar{j}_{\text{NI,par}} \) presented above considers neutral beams as the only source of current and heating. In the case where more heating and current sources are considered, equations (3) and (4) should include the weighted contributions of the different sources, and reference profiles need to be identified for each heating and current source. The resistivity \( \eta \) scales with the temperature \( T_e \) as \( \eta(\hat{\rho}, t) = \frac{k_{eff} T_e}{\bar{T}_{e,\text{profile}}(\hat{\rho})^{3/2}} \), where \( k_{eff} = 1.5, \) and \( k_{eff} = 4.2702 \cdot 10^{-8} \Omega\text{m}(\text{keV})^{3/2} \). By introducing

\[
\hat{D}(\hat{\rho}) = \hat{F} \hat{G} \hat{H},
\]

\[
\hat{D}(\hat{\rho}) = R_0 \hat{H} \mu_0 B_{\rho,\phi} J_{NI,par}^\text{profile}(\hat{\rho}),
\]

the normalized poloidal magnetic flux can be rewritten as

\[
\frac{1}{\hat{D}} \hat{\rho} \frac{\partial \psi}{\partial \hat{\rho}} = u_1(t) \frac{1}{\hat{D}} \frac{\partial}{\partial \hat{\rho}} \left[ \hat{\rho} \hat{D}(\hat{\rho}) \frac{\partial \psi}{\partial \hat{\rho}} \right] - u_2(\hat{\rho}) u_2(t).
\]

The control inputs \( u_1 \) and \( u_2 \) are functions of physical actuators such as the total power \( P \) of the non-inductive current drive, the total plasma current \( I \), and the average density \( \bar{n} \), i.e.,

\[
u_1(t) = \bar{n}^{1.5} I^{-1.5} P^{-0.75}, \quad u_2(t) = P^{0.5} I^{-1}.
\]

The poloidal magnetic flux at the spatial boundaries is determined by the Neumann conditions

\[
\frac{\partial \psi}{\partial \rho} (0, t) = 0, \quad \frac{\partial \psi}{\partial \rho} (1, t) = k u_3(t), \quad u_3(t) = I.
\]

where \( k \) is a constant. The initial condition for the magnetic flux profile is given by \( \psi(\hat{\rho}, t_0) = \psi(0) \).

B. Cost functional and constraints

In practice, the toroidal current density is usually specified indirectly by the rotational transform \( \iota \) (or the safety factor \( q = \iota^{-1} \)), which is defined as \( \iota(\rho, t) = \frac{\partial \psi(\rho, t)}{\partial \rho} \). The constant relationship between \( \Phi \) and \( \rho, \rho = \sqrt{\frac{\Phi}{\pi B_{\rho,\phi}}} \), and the definition of the normalized radius \( \bar{\rho} \) allow us to rewrite the rational transform as \( \iota(\bar{\rho}, t) = \frac{\bar{\psi}}{\bar{\rho}} = \frac{\partial \psi}{\partial \rho} \). Since \( \iota \) is uniquely defined by the spatial derivative of the magnetic flux \( \psi \), in this work we define the system output as \( \iota(\bar{\rho}, t) = \frac{\partial \psi}{\partial \rho} \).

The control objective is to find control inputs \( P(t) \) and \( I(t) \) that minimize the cost functional

\[
J = \frac{1}{2} \int_0^1 \left[ \iota(\bar{\rho}, t_f)^2 - \iota_0^2(\bar{\rho}) \right] d\bar{\rho} + \frac{1}{2} \int_{t_0}^{t_f} \left( \gamma_1 I^2 + \gamma_2 P^2 + \gamma_3 \bar{n}^2 \right) dt,
\]

where \( \iota_0^2(\bar{\rho}) \) is the desired target profile at time \( t_f \), and the positive constants \( \gamma_1, \gamma_2, \gamma_3 \) are control weighting factors. The control actuators may need to satisfy constraints such as:

- Magnitude saturation: \( \left( I_0^{(0)} \leq I \leq I_u^{(0)} \right) \)
- Rate saturation: \( \left( \frac{dI(t)}{dt} \leq I_u^{(1)} \right) \)
- Initial and final values: \( I(t_0) = I_0 \), \( I(t_f) = I_f \).

III. TRANSIENT DYNAMICS DESIGN

A. Edge design

By noting the definition \( \iota(\bar{\rho}, t) = \frac{\partial \psi}{\partial \rho} \) for the system output, a desired target magnetic flux profile at the final time \( t_f \), i.e. \( \psi(\bar{\rho}, t_f) = \psi_d(\bar{\rho}) \), can be obtained by integrating the desired output \( \psi_d(\bar{\rho}) \) over \( [0, \bar{\rho}] \), \( 0 \leq \bar{\rho} \leq 1 \):

\[
\psi_d(\bar{\rho}) = \psi_d(0) + \int_0^{\bar{\rho}} \psi_d(\bar{\rho}) d\bar{\rho} = \psi_d(1) - \int_0^{\bar{\rho}} \psi_d(\bar{\rho}) d\bar{\rho}.
\]

where either \( \psi_d(0) \) or \( \psi_d(1) \) need to be fixed to obtain the desired \( \psi_d \)-curve shown in Fig. 2. We can determine the left \( (\psi(0, t_0) \psi(0, t_f)) \) and right \( (\psi(1, t_0) \psi(1, t_f)) \) boundary values by using the compatibility conditions: \( \psi(0, t_0) = \psi_0(0), \psi(1, t_0) = \psi_0(1), \psi(0, t_f) = \psi_d(0) \) and \( \psi(1, t_f) = \psi_d(1) \). Thus, we add a sequence of points \( \psi(0, t_j) \) such that \( t_0 < t_1 < \cdots < t_n < t_f \), between \( \psi(0, t_0) \) and \( \psi(0, t_f) \), and a sequence of points \( \psi(1, t_j) \) such that \( t_0 < t_1 < \cdots < t_m < t_f \), between \( \psi(1, t_0) \) and \( \psi(1, t_f) \) to represent the left and right boundary evolution conditions (Dirichlet boundary conditions) via spline interpolations. Therefore, we obtain the four-edge frame shown in Fig. 2, where the surface within this frame representing the desired transient dynamics still needs to be defined.
B. Minimal surface

In Fig. 3, we define \( M = \{0 \leq \hat{\rho} \leq 1, t_0 \leq t \leq t_f\} \) in the \( \hat{\rho}t \)-plane with the boundary denoted by \( \partial M \). We define a three dimensional curved \( \partial S = E \sim F \sim G \sim H \sim E \) over \( \partial M \), which can span a surface \( S \) in infinite many ways. In this work, the minimal surface theory is used to define a unique surface within the frame and minimize transient fluctuations. We discuss the detailed theory and algorithms in the rest of this subsection.

We use \( z = \psi(\hat{\rho}, t) \) to express the surface \( S \). As shown in Fig. 3, the minimal surface problem can be stated as the following optimization problem:

\[
\min_{\psi(\hat{\rho}, t)} \int_M \sqrt{1 + \psi_{\hat{\rho}}^2 + \psi_t^2} \, d\hat{\rho} dt,
\]

subject to: \( \psi(\hat{\rho}, t)|_{\partial M} = g(\hat{\rho}, t) \triangleq \psi_0(\hat{\rho}), \hat{\rho} \in [0, 1], t = t_0, \)

\[
\begin{cases}
\text{Spline} \left( \{0, t_0\}, \{\psi(0, T_i)\}_{i=1}^J \right), T_i \in (t_0, t_f), \\
\text{Spline} \left( \{1, t_0\}, \{\psi(1, T_j)\}_{j=1}^K \right), T_j \in (t_0, t_f), \\
\psi(\hat{\rho}, t) \big|_{\partial \hat{\rho} \in (0, 1), t = t_f},
\end{cases}
\]

where \( \partial M \) is the boundary of the domain \( M \). There are very few examples of minimal surfaces that can be expressed analytically. Nonlinear programming (NLP) can be used in general to find a numerical solution minimizing the area functional, but it is often computationally costly. Alternatively, by using the Euler-Lagrange equation in the calculus of variations [6], the minimal surface problem (12) can be reformulated as a nonlinear elliptic PDE:

\[
\frac{\partial}{\partial \hat{\rho}} \left( \frac{\psi_{\hat{\rho}}}{\sqrt{1 + \psi_{\hat{\rho}}^2 + \psi_t^2}} \right) + \frac{\partial}{\partial t} \left( \frac{\psi_t}{\sqrt{1 + \psi_{\hat{\rho}}^2 + \psi_t^2}} \right) = 0,
\]

\[
\psi(\hat{\rho}, t)|_{\partial M} = g(\hat{\rho}, t).
\]

This is called the minimal-surface equation, which is impossible to solve analytically in general and numerical algorithms such as the finite element method (FEM) [10] or the finite difference method (FDM) [11] can be used to obtain numerical solutions.

One challenge arising during the implementation of the minimal surface theory for the definition of the transient dynamics of the magnetic flux is the satisfaction of the boundary conditions. In this problem, a Neumann boundary condition at \( \hat{\rho} = 0 \) must be satisfied. However, such spatial derivative requirement is not taken into account by the minimal surface equation (13). To overcome this challenge we decompose the domain into sub-domains and solve the minimal surface equation (nonlinear elliptic PDE) over each sub-domain with overlapping boundaries. In order to define a transient dynamics satisfying the zero Neumann boundary condition at \( \hat{\rho} = 0 \), we split the domain into two sub-domains \( M = M_1 \cup M_2 \), where \( M_1 \) is a narrow region of width \( \Delta \hat{\rho} \) along the \( \hat{\rho} = 0 \) boundary (Fig. 3). By properly defining Dirichlet boundary conditions for \( M_1 \), it is possible to approximately satisfy the zero Neumann boundary condition at \( \hat{\rho} = 0 \).

IV. CONTROL COMPUTATIONS

A. Scalar analysis

In plasma discharge experiments at the DIII-D tokamak, the total power \( P(t) \), the total plasma current \( I(t) \), and the average density \( \bar{n}(t) \) are of order \( 10^{19} \text{W}, 10^6 \text{A} \), and \( 10^{19} \text{m}^{-3} \) respectively. The coefficients \( \vartheta_1(\hat{\rho}) \) and \( \vartheta_2(\hat{\rho}) \), which vary with respect to \( \hat{\rho} \), are of order \( 10^{-15} \) and \( 10^{13} \), respectively. The poloidal magnetic flux \( \psi(\hat{\rho}, t) \), which varies with respect to both the normalized radius and time, is of order \( 10^{-1} \). The other variables in (5) are of order 1. Thus, we can estimate the orders of all the terms in (5) and (6):

\[
\frac{1}{\vartheta_1(\hat{\rho})} \frac{d\vartheta}{dt} \sim 10^{14}, \quad u_1(t) \sim 10^{15}, \quad u_1(t) \frac{\partial}{\partial \hat{\rho}} \left( \hat{\rho}D(\hat{\rho}) \frac{\partial \vartheta}{\partial \hat{\rho}} \right) \sim 10^{14}, \quad u_2(t) \sim 10^{-3}
\]

and \( \vartheta(\hat{\rho})u_2(t) \sim 10^{10} \). Therefore, the interior control term \( \vartheta(\hat{\rho})u_2(t) \) is small in comparison to other terms in (5).

B. Least square scheme

We consider a grid division \( (\hat{\rho}_i, t_j) \) in the temporal-spatial domain \( M = \{0 \leq \hat{\rho} \leq 1, 0 \leq t \leq t_f\} \):

\[
0 = \hat{\rho}_1 < \hat{\rho}_2 < \ldots < \hat{\rho}_i < \ldots < \hat{\rho}_M = 1 \quad \text{and} \quad t_0 = t_1 < t_2 < \ldots < t_j < \ldots < t_N = t_f.
\]

We assume that the desired transient dynamics is obtained by solving the minimal surface equation (13) and is denoted by \( \psi(\hat{\rho}, t) \) over \( M \). Then, we can compute the boundary control through
the Neumann boundary condition (7), \( u_3(t_n) = \frac{1}{\hat{\rho}} \frac{\partial \hat{\psi}}{\partial \hat{\rho}}(1, t_n) \), \( n = 1, 2, \ldots, N \).

Based on the results of our order analysis, we let the interior control \( u_2 = 10^{-3} \) and rewrite the PDE system (5) as the following linear system

\[
\frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} \left[ \hat{\rho} D(\hat{\rho}) \frac{\partial \hat{\psi}}{\partial \hat{\rho}} \right] u_1(t) = \frac{1}{\hat{\rho}_1(\hat{\rho})} \frac{\partial \hat{\psi}}{\partial \mu} + \hat{\vartheta}_2(\hat{\rho}) u_2(t),
\]

where \( u_1 \) must be obtained at \( t_n, n = 1, 2, \ldots, N \). For each \( t_n, n = 1, 2, \ldots, N \), the equation (14) can be satisfied at each spatial node \( \hat{\rho}_m, m = 2, 3, \ldots, M \), i.e.,

\[
A^n_{2,M} u_1(t_n) = b^n_{2,M}, \quad n = 1, 2, \ldots, N,
\]

where

\[
A^n_{2,M} = \begin{pmatrix}
\frac{1}{\hat{\rho}_2} \frac{\partial}{\partial \hat{\rho}_2} \left[ \hat{\rho}_2 D(\hat{\rho}_2) \frac{\partial \hat{\psi}_2(\hat{\rho}_2, t_n)}{\partial \hat{\rho}_2} \right] \\
\vdots \\
\frac{1}{\hat{\rho}_M} \frac{\partial}{\partial \hat{\rho}_M} \left[ \hat{\rho}_M D(\hat{\rho}_M) \frac{\partial \hat{\psi}_M(\hat{\rho}_M, t_n)}{\partial \hat{\rho}_M} \right] \\
\frac{1}{\hat{\mu}_1(\hat{\rho}_2)} \frac{\partial \hat{\psi}_2(\hat{\rho}_2, t_n)}{\partial \mu} + \hat{\vartheta}_2(\hat{\rho}_2) u_2(t_n) \\
\vdots \\
\frac{1}{\hat{\mu}_1(\hat{\rho}_M)} \frac{\partial \hat{\psi}_M(\hat{\rho}_M, t_n)}{\partial \mu} + \hat{\vartheta}_2(\hat{\rho}_M) u_2(t_n)
\end{pmatrix},
\]

\[
b^n_{2,M} = \begin{pmatrix}
\frac{1}{\hat{\rho}_2} \left[ \hat{\rho}_2 D(\hat{\rho}_2) \frac{\partial \hat{\psi}_2(\hat{\rho}_2, t_n)}{\partial \hat{\rho}_2} \right] \\
\vdots \\
\frac{1}{\hat{\rho}_M} \left[ \hat{\rho}_M D(\hat{\rho}_M) \frac{\partial \hat{\psi}_M(\hat{\rho}_M, t_n)}{\partial \hat{\rho}_M} \right] \\
\frac{1}{\hat{\mu}_1(\hat{\rho}_2)} \left[ \hat{\mu}_1(\hat{\rho}_2) \frac{\partial \hat{\psi}_2(\hat{\rho}_2, t_n)}{\partial \mu} \right] + \hat{\vartheta}_2(\hat{\rho}_2) u_2(t_n) \\
\vdots \\
\frac{1}{\hat{\mu}_1(\hat{\rho}_M)} \left[ \hat{\mu}_1(\hat{\rho}_M) \frac{\partial \hat{\psi}_M(\hat{\rho}_M, t_n)}{\partial \mu} \right] + \hat{\vartheta}_2(\hat{\rho}_M) u_2(t_n)
\end{pmatrix}.
\]

Without considering any actuation constraint, we can obtain the least square solution of the linear system (15) as

\[
u^n_{1} = u_1(t_n) = \left( A^n_{2,M} \right)^T A^n_{2,M} \left( A^n_{2,M} \right)^T b^n_{2,M}. \quad (18)
\]

In general, we can formulate the following optimization problem in the presence of actuation constraints:

\[
\min_{u_1 \in \mathcal{U}} \frac{1}{2} \beta_1 \| u_1 \|^2 + \frac{1}{2} \sum_{n=1}^{N} \beta_{2,n} \| A^n_{2,M} u_1(t_n) - b^n_{2,M} \|^2,
\]

where \( u_1 = (u_1(t_1), \ldots, u_1(t_n), \ldots, u_1(t_f))^T \), \( \beta_1 \) and \( \beta_{2,n} \) are positive weighting constants and \( \mathcal{U} \) is the admissible control set defined by (8)-(10) at \( t = t_n, n = 1, 2, \ldots, N \). This is a quadratic programming problem which can be solved relatively fast.

**C. Computational derivatives**

The matrices in (15) include both the temporal and spatial derivatives of the desired transient dynamics \( \hat{\psi}(\hat{\rho}, t) \) over \( M_{m,n} = (\hat{\rho}_m, t_n), m = 1, 2, \ldots, M \) and \( n = 1, 2, \ldots, N \). Using the discrete values \( \hat{\psi}(\hat{\rho}_{m-1}, t_n), \hat{\psi}(\hat{\rho}_m, t_n) \) and \( \hat{\psi}(\hat{\rho}_{m+1}, t_n) \) defined on a uniform grid, we can obtain the
Thus, we can obtain the physical actuators (denoted by $M$) minimal surface equation (13) over the discrete nodes in both division over the domain $M$ of the magnetic flux $\psi$ model (5) are shown in Fig. 4. The initial and desired profiles are compared. The error at the final time $t_f$ is the spatial step length. The term $\frac{1}{\rho} \frac{\partial}{\partial \rho} \hat{D}(\rho) \frac{\partial \psi(\rho, t_n)}{\partial \rho}$ is computed using similar second-order difference formulas in terms of the previously obtained $\frac{\partial \psi(\rho, t_n)}{\partial \rho}$ for $m = 1, 2, \ldots, M$. The temporal difference formulas are obtained following identical procedure.

V. Numerical Example

The geometrical parameters $D(\rho)$, $\psi_1(\rho)$ and $\psi_2(\rho)$ in model (5) are shown in Fig. 4. The initial and desired profiles of the magnetic flux $\psi(\rho, t)$ are given in Fig. 5. We use splines to define the boundary evolutions over $\partial M$, which are shown in Fig. 6. Then, we generate a triangular grid division over the domain $M = M_1 \cup M_2$. We solve the minimal surface equation (13) over the discrete nodes in both $M_1$ and $M_2$ using the finite element method.

In order to formulate the linear algebraic equation (15), we compute every term in (5) in terms of both the temporal and spatial derivatives, which are shown in Fig. 7. We obtain the control functions shown in Fig. 8 where we let $u_2(t) = 10^{-3}$. Thus, we can obtain the physical actuators (denoted by $P_a$, $I_a$ and $n_a$) taking into account the definitions (6) and (7). Using the obtained actuator functions, we simulate the PDE system (5) with boundary conditions (6). The obtained control functions can drive the system to the vicinity of the desired profile with a similar shape of the desired profile. This is illustrated in Fig. 9, where desired and actual flux profiles are compared. The error at the final time $t_f$ seems to be rather constant with respect to space, which implies that the desired shape for the $t$ variable in order to eliminate relatively spatially-constant matching errors that can appear in the $\psi$ variable, which are indeed not important. Alternatively, an iterative scheme can be designed where the matching error is used to redefine the desired magnetic flux target profile, and therefore the transient dynamics, for the following iteration.

VI. Conclusion

The open-loop finite-time optimal current profile control problem arising in tokamak plasmas during the ramp-up phase of the discharge is solved by using the minimal surface theory and the least square method (or the quadratic programming method when actuation constraints are taken into account). The minimal surface theory is used to generate the desired transient dynamics and then a tracking problem can be formulated for the current profile control. Knowing the transient dynamics, every term containing both temporal and spatial derivatives in the control-oriented PDE model can be computed using the finite difference method. Thus, the control-oriented PDE model becomes a set of algebraic equations where the only unknowns are the control functions at different instants of time.

Taking into account the control constraints, these algebraic equations can be reformulated as a quadratic programming problem. When no actuator constraint needs to be taken into account, the quadratic programming problem simplifies to a simple least square problem. Numerical studies demonstrate that this approach can significantly reduce computational effort, showing potential for real-time implementation in a closed-loop receding-horizon scheme, particularly for long-discharge tokamaks such as ITER.

Simulation results show that the numerical optimization procedure can generate control trajectories driving the final $\psi$-profile to the proximity of a predefined desired profile. Future work includes the implementation of this method directly in terms of the $t$ variable in order to eliminate relatively spatially-constant matching errors that can appear in the $\psi$ variable, which are indeed not important. Alternatively, an iterative scheme can be designed where the matching error is used to redefine the desired magnetic flux target profile, and therefore the transient dynamics, for the following iteration.

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