Example: Baseball

A member of the Little League baseball team can be a substitute, a starter or a star. Find the steady state distribution for the transition matrix given below.

	Subst	Starter	Star
Subst	0.2	0.5	0.3
Starter	0.1	0.5	0.4
Star	0.1	0.3	0.6

1

Baseball continued …

- What is the average number of seasons it would take to transition from a starter to a star?

2

Mean First Passage Times

- m_{ij} = E(# steps from i until j is reached)

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}$$

- Note that: $$m_{ii} = \frac{1}{\pi_i}$$

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Revisiting the Cola Example

- What is the average number of purchases before a Cola1 customer switches to Cola2?

Photo from http://www.angelfire.com/oh/cocacolaantiques/

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The Geometric Distribution

- A discrete random variable X follows a geometric distribution if:

$$P(X = x \mid p) = p(1 - p)^{x-1},$$

where $0 \le p \le 1, \quad x = 1,2,$☺

$$E(X) = \frac{1}{p}; \quad Var(X) = \frac{1-p}{p^2}$$

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Back to Baseball Example

Suppose there is one more state, that of quitting the team:

	Subst	Starter	Star	Quit
Subst	0.2	0.4	0.2	0.2
Starter	0.1	0.4	0.4	0.1
Star	0.1	0.2	0.6	0.1
Quit	0	0	0	1

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Absorbing Chain

- s = number of states
- m = number of absorbing states
- s – m = number of transient states

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

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The Fundamental Matrix

- The Fundamental Matrix = $(I-Q)^{-1}$
- Note that: $(I-Q)^{-1} = I + Q + Q^2 + \ldots$
- E(periods in transient state t_j before absorption | currently in t_i) = $[(I-Q)^{-1}]_{ij}$
- P(absorbed into a_j | currently in t_i) = $[(I-Q)^{-1}R]_{ij}$

These two results will be shown in class.

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Some Baseball Question …

- What is the probability that a new Little Leaguer will eventually quit?
- On average, how many times will a star player be a substitute before quitting?

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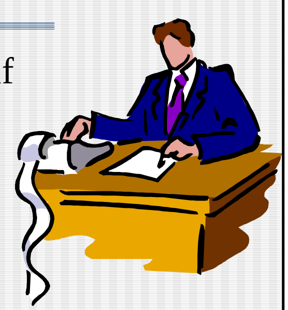
Example: Accounts Receivable

- Assume account uncollectable if more than 3 months overdue.
- After 3 months, debt is either collected or written off as a bad debt. Account closed.



1. P(new account collected)?
2. P(1mth acct becomes bad debt)?
3. Sales = $100K/mth. How much will go uncollected over a year?

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Accounts Receivable: Matrix

	new	*1mth*	*2mth*	*3mth*	*paid*	*baddebt*
new		0.60			0.40	
1mth			0.50		0.50	
2mth				0.40	0.60	
3mth					0.70	0.30
paid					1	
baddebt						1

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Application: Genetics

- a=dominant gene; b=recessive gene
- Two animals are mated. Two offspring are selected to mate.
- One offspring selected at random; offspring selects a mate. It is *k* times as likely to pick an animal unlike itself than one like itself. (aa and ab animals look alike.)
- States: (aa,aa), (bb,bb), (aa,ab), (bb,ab), (aa,bb), (ab,ab).

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Genetics continued …

- *k* measures how strongly "opposites attract each other"
- Why are (aa,aa) and (bb,bb) absorbing states?
- Write the resulting transition matrix.
- How does *k* affect the number of generations needed to reach a pure strain and the probability of getting pure dominants or pure recessives?

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Fundamental Matrix (I-Q)$^{-1}$

$$\frac{1}{(2k+1)(k+3)}\times\begin{pmatrix} 4(k^2+5k+2) & 2k(k+1)^2 & k(k+1) & (3k+1)(k+3) \\ 2(3k+1) & (4k^2+9k+3)(k+1) & k(k+1) & (3k+1)(k+3) \\ 4(3k+1) & 4k(k+1)^2 & (4k^2+9k+3) & 2(3k+1)(k+3) \\ 4(3k+1) & 4k(k+1)^2 & 2k(k+1) & 2(3k+1)(k+3) \end{pmatrix}\begin{matrix} (aa,ab) \\ (bb,ab) \\ (aa,bb) \\ (ab,ab) \end{matrix}$$

$$(I-Q)^{-1}R=\frac{1}{4(2k+1)(k+3)}\times\begin{pmatrix} 4k^2+23k+9 & 4k^2+5k+3 \\ 9k+3 & 8k^2+19k+9 \\ 18k+6 & 8k^2+10k+6 \\ 18k+6 & 8k^2+10k+6 \end{pmatrix}\begin{matrix} (aa,ab) \\ (bb,ab) \\ (aa,bb) \\ (ab,ab) \end{matrix}$$

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Example: Workforce Planning

- Mason & Burger employs 3 kinds of lawyers: Jr., Sr., Partner.

1. E(time Jr. lawyer in firm)?
2. P(Jr. lawyer makes Partner)?
3. E(time Partner in firm)?

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Workforce Planning: Matrix

	Jr.	*Sr.*	*P*	$L(NP)$	$L(P)$
Jr.	0.80	0.15		0.05	
Sr.		0.70	0.20	0.10	
Partner			0.95		0.05
Leave(nonpartner)				1	
Leave(partner)					1

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Work-force Planning Models

- H_i = number of hires of type i
- $N_i(t)$ = number of type i at period t
- N = $(N_1,N_2,\ldots,N_s)$ steady-state census

$$N_i = \lim_{t\to\infty} N_i(t)$$

What is the steady-state equation?

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Example: Workforce Planning

- Mason & Burger
- Classification of lawyers: Junior, Senior, Partner.
- Long-term goal: 50 junior lawyers, 30 senior lawyers, and 10 partners.
- How many junior lawyers should M&B hire each year?

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Workforce Planning: Matrix

	Jr.	*Sr.*	*P*	*L*(*NP*)	*L*(*P*)
Jr.	0.80	0.15		0.05	
Sr.		0.70	0.20	0.10	
P			0.95		0.05
L(*NP*)				1	
L(*P*)					1

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Example: Population Census

- Classification: children, working adults, or retired people.
- 0.959 of children remain children, 0.04 become working adults, 0.001 die. 0.96 working adults remain working, 0.03 retire, and 0.01 die. 0.95 of retired people remain retired, 0.05 die. (1000 children born each year.)

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Example: Population Census

- Determine the steady-state census.
- Pension for each retired person is $5000 per year. This is funded by payments from working adults. How much money must each working adult contribute annually?

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