Example: Cola

- Entire cola industry produces two types of colas.
- Given: last purchased was cola1, there is 90% chance next purchase is cola1.
- Given: last purchase was cola2, there is 80% chance next purchase is cola2.
- Draw the transition diagram.

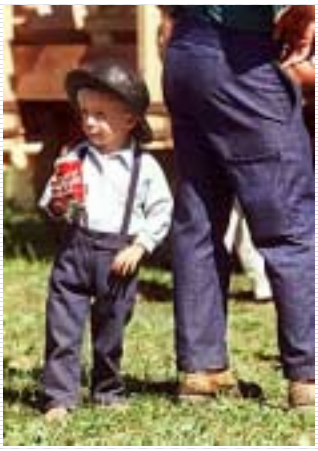

Photo from http://www.angelfire.com/oh/cocacolaantiques/

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Cola continued ...

- If Hermione is currently a cola2 purchaser, what is the probability that she will purchase cola1 two purchases from now?
- If Harry is currently a cola1 purchaser, what is the probability that he will purchase cola1 three purchases from now?

Photo from the Pepsi Cola Virtual Museum
http://www.pepsi100th.com/

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n-Step Transition Probabilities

- States = {1,2,...,s}
- 2-Step Transition Probabilities

$$p_{ij}(2) = \sum_{k=1}^{s} p_{ik} p_{kj}$$

- By extension,

$$p_{ij}(n) = [P^n]_{ij}$$

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Cola continued ...

Currently, 60% of Lehigh students drink cola1, and 40% cola2.

What fraction of purchasers will be drinking cola1 three purchases from now?

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n-Step Transition Probabilities

- Given initial probabilities

$$\mathrm{q} = [q_1, q_2, \ldots, q_s]$$

- Probability of being in state j at time n

$$\sum_{i=1}^{s} q_i p_{ij}(n) = \sum_{i=1}^{s} q_i [P^n]_{ij} = \mathrm{q}[P^n]_{\cdot j}$$

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Cola example: 1-Step Tran Prob

$$P = \begin{bmatrix} 0.90 & 0.10 \\ 0.20 & 0.80 \end{bmatrix}$$

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Cola example: 2-Step Tran Prob

$$P^2 = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

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Cola example: 3-Step Tran Prob

$$P^3 = \begin{bmatrix} 0.78 & 0.22 \\ 0.44 & 0.56 \end{bmatrix}$$

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Cola example: 4-Step Tran Prob

$$P^4 = \begin{bmatrix} 0.75 & 0.25 \\ 0.51 & 0.49 \end{bmatrix}$$

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Cola example: 5-Step Tran Prob

$$P^5 = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$

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Cola example: 10-step tran prob

$$P^{10} = \begin{bmatrix} 0.68 & 0.32 \\ 0.65 & 0.35 \end{bmatrix}$$

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Cola example: 20-step tran prob

$$P^{20} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

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Cola example: 30-step tran prob

$$P^{30} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

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Cola example: 40-step tran prob

$$P^{40} = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

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Cola example: steady state

Steady State Distribution
a.k.a. Equilibrium Distribution

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix}$$

$$\pi_1 = 0.67, \pi_2 = 0.33$$

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Steady State Probabilities

- Let P be the transition matrix of s-state ergodic Markov Chain
- There exists a vector $\boldsymbol{\pi} = [\pi_1, \pi_2, \ldots, \pi_s]$ such that

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi_1 & \pi_2 & \ldots & \pi_s \\ \pi_1 & \pi_2 & \ldots & \pi_s \\ \ldots & \ldots & \ldots & \ldots \\ \pi_1 & \pi_2 & \ldots & \pi_s \end{bmatrix}$$

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Cola continued ...

Photo from the Pepsi Cola Virtual Museum
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- Write the system of equations to solve for the steady-state probabilities
- Solve for $[\pi_1, \pi_2]$

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Solving for Steady State

- To find the steady state probabilities, solve the following system of equations:

$$\pi = \pi \cdot P$$

$$\sum_{i=1}^{s} \pi_i = 1$$

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More on the Cola Example

- 1 cola purchase/customer/week.
- 100 million cola customers.
- Production cost $1 per unit.
- Selling price $2 per unit.
- For $500M per year, an advertising firm guarantees to decrease from 10% to 5% the fraction of cola1 customers who switch to cola2 after one purchase.
- Should cola1 hire the firm?

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