Terminology

- Input or Arrival Process
  - Single/bulk
  - Unaffected by number of customers
- Output or Service Process
  - Service time distribution
  - Parallel/series
- Queue Discipline
  - FCFS/LCFS/SIRO
- Method Used to Join Queue
  - Which line to join?
  - Is switching/jockeying allowed?

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Arrival Process: Exponential

- $E(A) = 1/\lambda$, $Var(A) = 1/\lambda^2$
- Memoryless Property
- Interarrival times are exponential with parameter λ $\Leftrightarrow$ number of arrivals occuring in an interval of length t follows a Poisson distribution with parameter λt.

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Arrival Process: Exponential

- If arrivals defined on nonoverlapping time intervals are independent, and
- P(arrival occuring between t and $t+\Delta t$) $= \lambda \Delta t + o(\Delta t)$, P(no arrival between t and $t+\Delta t$) $= 1 - \lambda \Delta t + o(\Delta t)$, P(more than one arrival between t and $t+\Delta t$) $= o(\Delta t)$,
- Then $\mathbf{N}_t$ follows a Poisson distribution with parameter λt, and interarrival times are exponential with parameter λ; that is, $a(t) = \lambda e^{-\lambda t}$

Example: Dick's Pub

- The number of glasses of beer ordered per hour at Dick's Pub follows a Poisson distribution, with an average of 30 beers per hour being ordered.
- Find the probability that exactly 60 beers are ordered between 10pm and 12midnight.
- Find the mean and standard deviation of the number of beers ordered between 9pm and 1am.
- Find the probability that the time between two consecutive orders is between 1 and 3 minutes.

Arrival Process: Erlang

- $E(A) = k/\lambda$, $Var(A) = k/\lambda^2$

$$f(t) = \frac{\lambda(\lambda t)^{k-1}e^{-\lambda t}}{(k-1)!}, t \geq 0$$

The Service Process

Customer Service

- Service time, S.
- Mean $1/\mu$, service rate μ.
- Modeled using Exponential
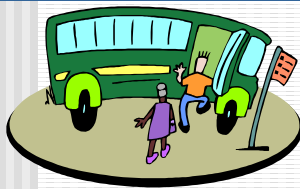- Modeled using Erlang

The Kendall-Lee Notation

- 1/2/3/4/5/6
- 1: M = iid exponential, D = iid deterministic, E_k = iid Erlang parameter k, G = iid general
- 2: M = iid exponential, D = iid deterministic, E_k = iid Erlang parameter k, G = iid general
- 3: number of parallel servers
- 4: FCFS/LCFS/SIRO/GD
- 5: max allowable # of customers in the system
- 6: population size

The Waiting Time Paradox

Suppose the time between arrival of buses at the student center is exponentially distributed with a mean of 60 minutes. If we arrive at the student center at a randomly chosen instant, what is the average amount of time that we will have to wait for a bus?

Example

Buses arrive at the downtown bus stop and leave for the mall stop. Past experience indicates that 20% of the time, the interval between buses is 20 minutes; 40% of the time, the interval is 40 minutes; and 40% of the time, the interval is 2 hours. If I have just arrived at the downtown bus stop, how long, on the average, should I expect to wait for a bus?

Birth-Death Process

- Continuous-time stochastic process
- State at any time is non-negative integer
    - State: # pple present in queuing system
- Transition probabilities
    - $P_{ij}(t)$: n-step transition probability
    - $\lim_{t\to\infty} P_{ij}(t) = \pi_j$: steady-state, equilibrium

B-D Process: Laws of Motion

- Birth rate, λ_j (arrival)
  - $P(j,j+1) = \lambda_j \Delta t + o(\Delta t)$
- Death rate, μ_j (service completion)
  - $P(j,j-1) = \mu_j \Delta t + o(\Delta t)$
- Births/deaths independent of each other

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Examples

- M/M/1 Queue: between time t and t+ Δt
  - P(birth) = ?
  - P(death) = ?
  - Transition diagram?
- M/M/3 Queue, λ=4, μ=5
  - Transition diagram?
- When is the B-D model not appropriate?
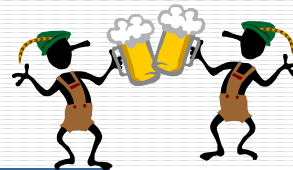
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Steady-state Probabilities

$$\pi_j = c_j \pi_0 \quad \text{where} \quad c_j = \frac{\lambda_0 \lambda_1 \otimes \lambda_{j-1}}{\mu_1 \mu_2 \otimes \mu_j}$$

$$\sum_{j=0}^{\infty} \pi_j = 1 \quad \Rightarrow \quad \pi_0 \left(1 + \sum_{j=1}^{\infty} c_j \right) = 1$$

$$\pi_0 = \frac{1}{\left(1 + \sum_{j=1}^{\infty} c_j \right)} \quad \text{if} \quad \sum_{j=1}^{\infty} c_j \text{ is finite.}$$

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Sample Problem

- There are 5 students and one keg of beer at a wild and crazy campus party. The time to draw a glass of beer follows an exponential distribution, with an average time of 2 minutes. The time to drink a beer also follows an exponential distribution, with a mean of 18 minutes. After finishing a beer, each student immediately goes back to get another beer.
- What fraction of time is the keg not in use?
- If the keg holds 500 glasses of beer, how long, on average, will it take to finish the keg?

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M/M/1/GD/∞/∞ System

- Traffic intensity of queuing system, ρ
- Steady-state probabilities:
  - $\pi_0 = 1 - \rho$
  - $\pi_j = \rho^j(1 - \rho)$
- If $\rho \geq 1$, no steady-state exists.

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Notation: L

- $L = E(\#\ \text{customers present in system})$
- $L_q = E(\#\ \text{customers waiting in line})$
- $L_s = E(\#\ \text{customers in service})$

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M/M/1/GD/∞/∞ System

- $L = \rho/(1-\rho) = \lambda/(\mu-\lambda)$
- $L_q = \rho^2/(1-\rho) = \lambda^2/[\mu(\mu-\lambda)]$
- $L_s = \rho = \lambda/\mu$

Notation: W

- $W = E(\text{time in system})$
- $W_q = E(\text{time waiting in line})$
- $W_s = E(\text{time in service})$

Little's Law

- If steady-state exists,
  - $L = \lambda W$
  - $L_q = \lambda W_q$
  - $L_s = \lambda W_s$
- Result is independent of # of servers, interarrival time distribution, service discipline, and service time distribution.

M/M/1/GD/∞/∞ System

- $W = 1/(\mu - \lambda)$
- $W_q = \lambda/[\mu(\mu - \lambda)]$
- As $\rho \rightarrow 1$,
  - $L \rightarrow$? and $W \rightarrow$?
  - $L_q \rightarrow$? and $W_q \rightarrow$?
- As $\rho \rightarrow 0$,
  - $W_q \rightarrow$? and $W \rightarrow$?

Example 1

- An average of 10 cars per hour arrive at a single-server drive-in. Assume the average service time for each customer is 4 minutes, and both interarrival times and service times are exponential.
- What is P(server is idle)?
- What is E(# cars waiting in line)?
- What is E(time spent in drive-in)?
- What is E(# customers served per hour)?

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Example 2

- Suppose that all car owners fill up when their tanks are exactly half full. At the present time, an average of 7.5 customers per hour arrive at a single-pump gas station. It takes an average of 4 minutes to service a car. Assume that interarrival times and service times are both exponential.
- Compute L and W
- Suppose a gas shortage occurs and panic buying takes place. Suppose all car owners now purchase gas when their tanks are exactly three quarters full, and service time is now 3.5 minutes. How has panic buying affected L and W?

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M/M/1/GD/c/∞ System

- Steady-state probabilities:
  - $\pi_0 = (1 - \rho)/(1 - \rho^{c+1})$
  - $\pi_j = \rho^j \pi_0$
- If $\lambda \neq \mu$:
  - $L = \rho[1-(c+1)\rho^c + c\rho^{c+1}]/[(1-\rho^{c+1})(1-\rho)]$
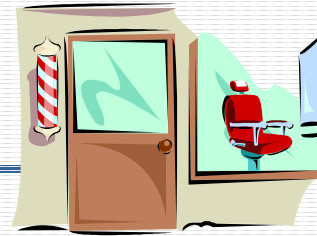- If $\lambda = \mu$:
  - $\pi_j = 1/(c+1)$
  - $L = c/2$

M/M/1/GD/c/∞ System

- $L_s = 1 - \pi_0$
- $L_q = L - L_s$
- Actual arrival rate of system?
- $W = ?$ and $W_q = ?$
- Steady-state always exists. Why?

Example 3

- A one-man barber shop has a total of 10 seats. Interarrival times are exponentially distributed, and an average of 20 prospective customers arrive each hour at the shop. Those customers who find the shop full do not enter. The barber takes an average of 12 minutes to cut each cusomer's hair. Haircut times are exponentially distributed.
- E(# haircuts per hour)?
- E(time spent in shop by customer who enters)?

M/M/s/GD/∞/∞ System

- Draw the transition diagram
- Steady-state probabilities:
    - $\rho = \lambda/(s\mu)$
    - $c_j = (s\rho)^j/j!$, j=0,1,2,...,s; $(s\rho)^j/s!s^{j-s}$, j=s+1,...
    - $\pi_j = c_j\pi_0$; $\pi_0 = 1/(1+c_1+c_2+...)$
- $P(j \geq s)$ and L_q

Example 2

- Consider a bank with two tellers. An average of 80 customers per hour arrive at the bank and wait in a single line for an idle teller. The average time it takes to serve a customer is 1.2 minutes. Assume that interarrival times and service times are exponential. Determine
- E(# customers present in the bank)
- E(length of time a customer spends in bank)
- Fraction of time a particular teller is idle

GI/M/∞/GD/∞/∞ System

- $W = 1/\mu$; $L = \lambda W = \lambda/\mu$
- If GI ~ M, L ~ Poisson(λ/μ)
 - $\pi_j = (\lambda/\mu)^j \exp(-\lambda/\mu) / j!$

Example 3

- During each year, an average of 3 ice-cream shops open up in Smalltown. The average time that an ice-cream shop stays in business is 10 years. On January 1, 2525, what is the average number of ice-cream shops that you would find in Smalltown? If the time between the opening of ice-cream shops is exponential, what is the probability that on January 1, 2525, there will be 25 ice-cream shops in Smalltown?

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Example with Cost 1
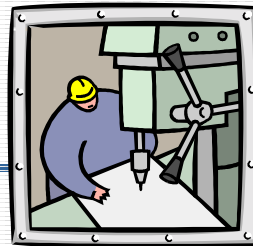
- Machinists who work at a tool-and-die plant must check out tools from a tool center. On average, ten machinists per hour arrive seeking parts. At present, the tool center is staffed by a clerk who is paid $6/hr and who takes an average of 5 min to handle each request for tools. Each hour that a machinist spends at the tool center costs the company $10. The company is deciding whether to hire (at $4/hr) a helper for the clerk. If the helper is hired, the clerk will take an average of only 4 min to process requests for tools. Assume that service, interarrival times are exponential. Should the helper be hired?

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Example with Cost 2

- The manager of a bank must determine how many tellers should work on Fridays. For every minute that a customer stands in line, the manager believes that a delay cost of 5-cents is incurred. An average of 2 customers/min arrive at the bank. On the average, it takes a teller 2 min to complete a customer's transaction. It costs the bank $9/hr to hire a teller. Interarrival times and service times are exponential. To minimize the sum of service costs and delay costs, how many tellers should the bank have working on Fridays?