IE 121 – Spring 2002

Midterm exam 1

1. (30 points) In this problem, you need to answer the following questions TRUE or FALSE inside the brackets:

1. (   ) Any unbiased estimator always gives a smaller mean square error than an unbiased one.
2. (   ) The standard error of an estimator $\hat{\Theta}$ is its standard deviation.
3. (   ) If $X_1, X_2, \ldots, X_n$ is a random sample of size n ($n > 50$) taken from a population with exponential distribution, then the sample mean $\bar{X}$ also follows the exponential distribution.
4. (   ) Suppose that a random sample of size n is taken from a normal population with mean μ. Then the mean of the sample mean is μ.
5. (   ) Suppose that a random sample of size n is taken from a normal population with variance σ^2. Then the variance of the sample mean is σ^2.
6. (   ) Failing to reject the null hypothesis when it is false is defined as a type I error.
7. (   ) There is no way to reduce the probability of both type I and type II error simultaneously if the sample size is fixed.
8. (   ) If the sample size is fixed, enlarging the acceptance region for a test will result in an increase of the probability of type I error.
9. (   ) If the sample size is increased, the probability of rejecting the null hypothesis when it is true is reduced.
10. (   ) The sample mean is the true mean of the population as long as the sample size is larger than 10.

Answers: 1.F, 2.T, 3.F, 4.T, 5.F, 6.F, 7.T, 8.F, 9.T, 10.F

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2. (30 points) A chocolate factory wants to make sure that the mean of sugar content μ of one of their products does not exceed 30%. To test this hypothesis H_0: $\mu = 30$ against the alternate one H_1: $\mu > 30$ they choose a sample of size 20. Assume the standard deviation of the sugar content is equal to 3.5%.

a) What is the probability of type I error if the critical region is defined to be $\mu > 31.5$?

b) Find the probability of type II error assuming the true mean sugar content is 31.8%. Would this probability increase or decrease if the sample size were 40?

Solution:

a) $\alpha = P(\bar{X} > 31.5 | \mu = 30) = P\left(\frac{\bar{X}-30}{3.5/\sqrt{20}} > \frac{31.5-30}{3.5/\sqrt{20}}\right) = P(Z > 1.917) = 1 - \Phi(1.917) = 0.027$

b) $\beta = P(\bar{X} < 31.5 | \mu = 31.8) = P\left(\frac{\bar{X}-31.8}{3.5/\sqrt{20}} < \frac{31.5-31.8}{3.5/\sqrt{20}}\right) = P(Z < -0.383) = \Phi(-0.383) = 0.352$

If the sample size were 40 we would have:

$\beta = P(\bar{X} < 31.5 | \mu = 31.8) = P\left(\frac{\bar{X}-31.8}{3.5/\sqrt{40}} < \frac{31.5-31.8}{3.5/\sqrt{40}}\right) = P(Z < -0.542) = \Phi(-0.542) = 0.295$, i.e. β would decrease.

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3. (40 points) Let X be uniformly distributed on the interval a to b. Suppose we take a random sample of X and obtain the data $x_1 = 2.6$, $x_2 = 3.8$, $x_3 = 1.1$, $x_4 = 4.3$, $x_5 = 4.9$.

a) Write down the likelihood function $L(a, b)$ for *all* values of a and b.

b) What is the maximum likelihood estimate of a based on our data?

c) What is the maximum likelihood estimate of b based on the data?

d) Is the corresponding estimator for a unbiased? What about the estimator for b? Explain your answer.

Solution:

a)

$$L(a,b) = \begin{cases} (\frac{1}{b-a})^5 & \text{for } a < 1.1 \text{ and } b > 4.9 \\ 0 & \text{otherwise} \end{cases}$$

b) The maximum likelihood estimate of a is $\min_{i=1,\dots,5}(x_i) = 1.1$

c) The maximum likelihood estimate of b is $\max_{i=1,\dots,5}(x_i) = 4.9$

d) Both estimators are biased. For example the estimator for a is $\hat{a} = \min_i(X_i)$, so in any case $\hat{a} > a$. Therefore $E(\hat{a}) = a$ only if $\hat{a} \equiv a$, which is false. The argument for $\hat{b} = \max_i(X_i)$ is the same.