ON THE COHOMOLOGY RING OF SPACES OF ORIENTED ISOMETRY CLASSES OF PLANAR POLYGONS

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Abstract. We determine much information about the mod-2 cohomology ring of the space $M(\ell_1,\ldots,\ell_n)$ of oriented isometry classes of planar n-gons with the specified side lengths. From this, we obtain bounds for the zero-divisor-cup-length (zcl) of these spaces, which provide lower bounds for their topological complexity (TC). In many cases our result about the cohomology ring is complete and we determine the precise zcl. We find that there will usually be a significant gap between the bounds for TC implied by zcl and dimensional considerations.

1. Introduction

The topological complexity, $\mathrm{TC}(X)$, of a topological space X is, roughly, the number of rules required to specify how to move between any two points of X. A "rule" must be such that the choice of path varies continuously with the choice of endpoints. (See [4, §4].) Information about the cohomology ring of X can be used to give a lower bound for $\mathrm{TC}(X)$.

Let $\bar\ell=(\ell_1,\ldots,\ell_n)$ be an n-tuple of positive real numbers. Let $M(\bar\ell)$ denote the space of oriented n-gons in the plane with successive side lengths $\ell_1,\ldots,\ell_n$, where polygons are identified under translation and rotation. Thus

$$M(\bar\ell)=\{(z_1,\ldots,z_n)\in(S^1)^n:\sum\ell_iz_i=0\}/SO(2).$$

If we think of the sides of the polygon as linked arms of a robot, then $\mathrm{TC}(M(\bar\ell))$ is the number of rules required to program the robot to move between any two configurations.

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Let $[n] = \{1, \ldots, n\}$ throughout. We say that $\bar{\ell}$ is *generic* if there is no subset $S \subset [n]$ for which $\sum_{i \in S} \ell_i = \frac{1}{2} \sum_{i=1}^{n} \ell_i$. For such $\bar{\ell}$, $M(\bar{\ell})$ is an $(n-3)$-manifold ([4, Thm 1.3]) and hence, by [4, Cor 4.15], satisfies

$$\mathrm{TC}(M(\bar{\ell})) \le 2n - 5. \tag{1.1}$$

In this paper, we obtain significant information about the mod-2 cohomology ring $H^*(M(\bar{\ell}))$ when $\bar{\ell}$ is generic, yielding lower bounds for $\mathrm{TC}(M(\bar{\ell}))$. Frequently, our description of the cohomology ring is complete, and we can give the best lower bound implied by cohomological methods. However, unlike the situation for isometry classes of polygons, i.e., when polygons are also identified under reflection, this lower bound is usually significantly less than $2n - 5$.

Indeed, for the space of isometry classes of planar polygons,

$$\overline{M}(\bar{\ell}) = \{(z_1, \ldots, z_n) \in (S^1)^n : \sum \ell_i z_i = 0\}/O(2),$$

the mod-2 cohomology ring was completely determined in [8], and in [1] and [2] we showed that for several large families of $\bar{\ell}$,

$$2n - 6 \le \mathrm{TC}(\overline{M}(\bar{\ell})) \le 2n - 5,$$

the latter because $\overline{M}(\bar{\ell}))$ is also an $(n-3)$-manifold when $\bar{\ell}$ is generic. Note that for motions in the plane, $M(\bar{\ell})$ would seem to be a more relevant space than $\overline{M}(\bar{\ell})$.

Although the abelian group structure of $H^*(M(\bar{\ell}))$ is known ([4, Thm 1.7]), the ring structure was apparently not. Nitu Kitchloo pointed out to the author that applying the Eilenberg-Moore spectral sequence (EMSS) to the fibration

$$M(\bar{\ell}) \to \overline{M}(\bar{\ell}) \to B\mathbf{Z}_2 \tag{1.2}$$

should yield much information about this ring structure. The author is grateful to Dr. Kitchloo for this observation and instruction regarding this spectral sequence.

The way in which knowledge of the cohomology ring yields a lower bound for topological complexity is through the mod-2 zero-divisor-cup-length of X, $\mathrm{zcl}(X)$, which is the maximum number of elements $\alpha_i \in H^*(X \times X; \mathbf{Z}_2)$ satisfying $m(\alpha_i) = 0$ and $\prod_i \alpha_i \ne 0$. Here $m : H^*(X) \otimes H^*(X) \to H^*(X)$ denotes the cup product pairing, and α_i is called a zero divisor. Throughout the paper, all cohomology groups have coefficients in $\mathbf{Z}_2$. In [5, Thm 7], it was shown that

$$\mathrm{TC}(X) \geq \mathrm{zcl}(X) + 1.$$

In Section 2, we describe the information about $H^*(M(\bar{\ell}))$ readily obtainable from the EMSS. See especially Corollary 2.11, which gives a complete description of the algebra $H^*(M(\bar{\ell}))$ in some cases (64 out of 134 when $n = 7$). In Section 3, we apply these results to obtain information about $\mathrm{zcl}(M(\bar{\ell}))$ and hence $\mathrm{TC}(M(\bar{\ell}))$. Theorems 3.1 and 3.2 give upper and lower bounds for $\mathrm{zcl}(M(\bar{\ell}))$. See Table 3.4 for a tabulation when $n = 8$. In Section 4, we give an example showing how extension questions in the EMSS prevent us from making stronger zcl estimates.

2. The cohomology ring $H^*(M(\bar{\ell}))$

The version of the Eilenberg-Moore spectral sequence described in [12, Thm 6.2] says that if $F \to E \to B$ is a fibration with B simply-connected, there is a natural second quadrant spectral sequence of bigraded commutative algebras converging strongly to $H^*(F)$ with $E_2^{p,q} = \mathrm{Tor}^{p,q}_{H^*B}(\mathbf{Z}_2, H^*E)$. In [3], it is shown that this is also true when $\pi_1(B)$ acts nilpotently on H^*F. We apply this to the fibration (1.2), and temporarily remove $\bar{\ell}$ from the notation. Since $\pi_1(B\mathbf{Z}_2)$ has order 2, it acts nilpotently on any finite dimensional vector space over $\mathbf{Z}_2$.

Since $H^*(B\mathbf{Z}_2)$ is a polynomial algebra on a 1-dimensional class x, the $H^*(B)$-resolution of $\mathbf{Z}_2$ can be taken to be just

$$0 \leftarrow \mathbf{Z}_2 \leftarrow H^*B\mathbf{Z}_2 \xleftarrow{x} \Sigma H^*B\mathbf{Z}_2 \leftarrow 0.$$

Since x maps to an element in $H^1(\overline{M})$ usually called R, $\mathrm{Tor}^{*,q}_{H^*B\mathbf{Z}_2}(\mathbf{Z}_2, H^*(\overline{M}))$ is the homology in grading q of

$$H^*(\overline{M}) \xleftarrow{R} \Sigma H^*(\overline{M}),$$

where Σ increases grading by 1. The usual indexing of the EMSS, as discussed in [11, p.240], uses negative p-gradings. Thus

$$E_2^{p,q} = \mathrm{Tor}^{p,q}_{H^*B\mathbf{Z}_2}(\mathbf{Z}_2, H^*\overline{M}) = \begin{cases} H^q(\overline{M})/R & p = 0 \\ \ker(R|H^{q-1}(\overline{M})) & p = -1 \\ 0 & \text{otherwise.} \end{cases}$$

There are no possible differentials, and so the short exact sequence

$$0 \to E_\infty^{0,q} \to H^q(M) \to E_\infty^{-1,q+1} \to 0$$

becomes a degree-preserving SES

$$0 \to H^*(\overline{M})/R \to H^*(M) \to \ker(R|H^*(\overline{M})) \to 0, \tag{2.1}$$

and, by [12, Thm 6.2], these are morphisms of algebras over $H^*(\overline{M})$. This SES was obtained by a different method in [7, (10.3.17)].

We assume throughout that $\ell_1 \le \cdots \le \ell_n$. It is well-understood ([8, Prop 2.2]) that the homeomorphism type of $M(\bar\ell)$ and $\overline{M}(\bar l)$ is determined by which subsets S of $[n]$ are *short*, which means that $\sum_{i\in S} \ell_i < \frac{1}{2}\sum_{i=1}^{n} \ell_i$. For generic $\bar l$, a subset which is not short is called *long*.

Define a partial order on the power set of $[n]$ by $S \le T$ if $S = \{s_1, \ldots, s_\ell\}$ and $T \supset \{t_1, \ldots, t_\ell\}$ with $s_i \le t_i$ for all i. This order will be used throughout the paper, applied also to multisets. As introduced in [9], the *genetic code* of $\bar l$ is the set of maximal elements (called *genes*) in the set of short subsets of $[n]$ which contain n. The homeomorphism type of $M(\bar\ell)$ and $\overline{M}(\bar l)$ is determined by the genetic code of $\bar l$. Note that if $\bar l = (\ell_1, \ldots, \ell_n)$, then all genes have largest element n. We introduce the new terminology that if $\{n, i_r, \ldots, i_1\}$ is a gene, then $\{i_r, \ldots, i_1\}$ is called a *gee*. (Gene without the n.)

We recall the following result from [8].

Theorem 2.2. *If $\bar l$ has length n, then the ring $H^*(\overline{M}(\bar l))$ is generated by classes $R, V_1, \ldots, V_{n-1}$ in $H^1(\overline{M}(\bar l))$ subject to only the following relations:*

1. *For each i, $V_i^2 = RV_i$. Thus all monomials can be expressed as $R^e V_{i_1} \cdots V_{i_r}$ for some $e \ge 0$ and $i_1 < \cdots < i_r$ with $0 \le r \le n-1$.*
2. *If $S \subset [n-1]$ has $S \cup \{n\}$ long, then $\prod_{i\in S} V_i = 0$.*
3. *If $L \subset [n-1]$ is long and $|L| \le d+1$, then*

$$\sum_{\substack{S\subset L\\ S\cup\{n\}\ short}} R^{d-|S|} \prod_{i\in S} V_i = 0. \tag{2.3}$$

We immediately obtain

Corollary 2.4. *A basis for $H^*(\overline{M}(\bar{l}))/R$ consists of all elements $V_S := \prod_{i\in S} V_i$ such that $S \cup \{n\}$ is short. Equivalently, these are exactly those V_S for which $S \le G$ for some gee G of $\bar{\ell}$.*

It is well-known (e.g. [6, Expl 2.3]) that if the genetic code of $\bar{\ell}$ is $\langle\{n, n-3, n-4, \ldots, 1\}\rangle$, then $M(\bar{\ell})$ is homeomorphic to $(S^1)^{n-3} \sqcup (S^1)^{n-3}$. We will exclude this case from our analysis and use the following known result, in which, as always, $\bar{\ell} = (\ell_1, \ldots, \ell_n)$.

Proposition 2.5. ([6, Rmk 2.8]) *If the genetic code of $\bar{\ell}$ does not equal $\langle\{n, n-3, \ldots, 1\}\rangle$, then all genes have cardinality less than $n-2$, and $M(\bar{\ell})$ is a connected $(n-3)$-manifold.*

From now on, let $m = n-3$ denote the dimension of $\overline{M}(\bar{\ell})$ and $M(\bar{\ell})$, and let $W_\emptyset$ denote the nonzero element of $H^m(M(\bar{\ell}))$. We obtain

Theorem 2.6. *A basis for $H^*(M(\bar{l}))$ consists of the classes V_S described in Corollary 2.4 together with classes $W_S \in H^{m-|S|}(M(\bar{\ell}))$, for exactly the same S's, satisfying that*

$$V_S W_{S'} = \delta_{S,S'} W_\emptyset \quad \text{if } |S'| = |S|.$$

Also $V_S V_{S'} = V_{S\cup S'}$ if S and S' are disjoint and $S \cup S' \le$ some gee of $\bar{\ell}$, while $V_S V_{S'} = 0$ otherwise. Finally, $W_S W_{S'} = 0$ whenever $|W_S| + |W_{S'}| = m$.

Proof. By [4, Thm 1.7], for all i, the ith Betti number of $M(\bar{\ell})$ equals the number of V_S's described in Corollary 2.4 of degree i plus the number of such V_S's of degree $m-i$. By (2.1), our classes V_S are linearly independent in $H^*(M(\bar{\ell}))$ and all products $V_S V_{S'}$ are zero except those listed in our set. The nonsingularity of the Poincaré duality pairing implies that there are classes W_S which pair with the classes V_S and with each other in the claimed manner, and the Betti number result implies that there are no additional classes. ■

The following elementary lemma was used in the preceding proof.

Lemma 2.7. *Suppose U and U' are t-dimensional vector spaces over $\mathbf{Z}_2$ and $\phi : U \times U' \to \mathbf{Z}_2$ is a nonsingular bilinear pairing. Suppose $\{u_1, \ldots, u_k\} \subset U$ is linearly* *independent, as is* $\{u'_{k+1},\dots,u'_t\}\subset U'$*, and* $\phi(u_i,u'_j)=0$ *for* $1\le i\le k<j\le t$*. Then there exist bases* $\{u_1,\dots,u_t\}$ *and* $\{u'_1,\dots,u'_t\}$ *of* U *and* U' *extending the given linearly-independent sets and satisfying* $\phi(u_i,u'_j)=\delta_{i,j}$*.*

Proof. For $1\le i\le k$, let $\psi_i:U\to\mathbf{Z}_2$ be any homomorphism for which $\psi_i(u_j)=\delta_{i,j}$ for $1\le j\le k$. By nonsingularity, there is $u'_i\in U'$ such that $\phi(u,u'_i)=\psi_i(u)$ for all $u\in U$. To see that $\{u'_1,\dots,u'_t\}$ is linearly independent, assume $\sum c_\ell u'_\ell=0$. Applying $\phi(u_i,-)$ implies that $c_i=0$, $1\le i\le k$, while linear independence of $\{u'_{k+1},\dots,u'_t\}$ then implies that $c_{k+1}=\cdots=c_t=0$. Nonsingularity now implies that there are classes u_i for $i>k$ such that $\phi(u_i,u'_j)=\delta_{i,j}$ for all j, and linear independence of the u_i's is immediate. ■

This lemma is applied to $U=H^i(M(\bar\ell))$, $U'=H^{m-i}(M(\bar\ell))$, $\{u_1,\dots,u_k\}$ the set of V_S's in $H^i(M(\bar\ell))$, and $\{u'_{k+1},\dots,u'_t\}$ the set of V_S's in $H^{m-i}(M(\bar\ell))$.

Let s denote the size of the largest gee of $\bar\ell$. The only V_S's occur in gradings $\le s$, and so the only W_S's occur in grading $\ge m-s$. If $2(m-s)\le m-1$ (i.e., $m\le 2s-1$), then there can be nontrivial products of W_S's, about which we apparently have little control, unless we go to the huge effort of obtaining explicit formulas for elements in the $\ker(R)$-part of (2.1). See Section 4 for an example of such an effort.

The following simple result gives excellent information about products of V classes times W classes. In particular, if $m\ge 2s$, the entire ring structure is determined! See Corollary 2.11. When $m=4$, this is the case for 64 of the 134 equivalence classes of $\bar\ell$'s. (See [10].) It seems quite remarkable that much (often all) of the algebra structure of $H^*(M(\bar\ell))$ can be obtained without using the complicated relation in Theorem 2.2(3).

Proposition 2.8. *Mod polynomials in* $V_1,\dots,V_{n-1}$*,*

$$V_iW_S\equiv\begin{cases}W_T & \text{if } S=T\cup\{i\}\\ 0 & \text{if } i\notin S.\end{cases}\tag{2.9}$$

In particular, if s *is the maximal size of gees and* $m-|S|\ge s$*, then*

$$V_iW_S=\begin{cases}W_T & \text{if } S=T\cup\{i\}\\ 0 & \text{if } i\notin S.\end{cases}\tag{2.10}$$

Proof. Write $V_iW_S = \sum \alpha_P V_P + \sum \alpha'_Q W_Q$ with $\alpha_P, \alpha'_Q \in \mathbf{Z}_2$. If V_T is any monomial in grading $|S|-1$, then

$$\delta_{S,T\cup\{i\}}W_\emptyset = V_T V_i W_S = \sum_Q \alpha'_Q \delta_{T,Q} W_\emptyset = \alpha'_T W_\emptyset,$$

as all monomials in the V's are 0 in grading m. The first result follows immediately.

The second part follows since $|V_iW_S| = m - |S| + 1$ and all polynomials in the V's are 0 in grading $> s$. ■

Corollary 2.11. *Let $M = M(\bar{\ell})$. If $m \geq 2s$, where s is the maximal gee size, then the complete product structure of $H^*(M)$ is given as follows. There are classes $V_i \in H^1(M)$ such that $V_i^2 = 0$ and $\prod_{i\in S} V_i$ is nonzero iff $S \leq$ some gee of $\bar{\ell}$, and these monomials are linearly independent. For all such S, there are also additional independent classes $W_S \in H^{m-|S|}(M)$. All products $W_SW_{S'}$ are 0, and V_iW_S is given by (2.10).*

We offer the following illustrative example, in which we have complete information about the product structure. Here we begin using the notation introduced in [9] of writing genes (and gees) which are sets of 1-digit numbers by just concatenating those digits.

Example 2.12. *Suppose the genetic code of $\bar{\ell}$ is $\langle 9421, 95\rangle$. Then a basis for $H^*(M(\bar{\ell}))$ is:*

0	1
1	V_1, V_2, V_3, V_4, V_5
2	$V_1V_2, V_1V_3, V_1V_4, V_2V_3, V_2V_4$
3	$V_1V_2V_3, V_1V_2V_4, W_{123}, W_{124}$
4	$W_{12}, W_{13}, W_{14}, W_{23}, W_{24}$
5	W_1, W_2, W_3, W_4, W_5
6	$W_\emptyset$.

The only products of V's are those indicated. All products of W's are 0. The multiplication of V_i by W_S is by removal of the subscript i, if $i \in S$, else 0.

In the above example, $m = 6$ and $s = 3$. It is quite possible that a similarly nice product structure might hold in some cases in which $m < 2s$. When it does not, we refer to nonzero products of W's or cases in which (2.10) does not hold as *exotic products*. In Section 4, we study the exotic products and their effect in a simple example.

3. Zero-divisor-cup-length

In this section we consider the zero-divisor-cup-length $\mathrm{zcl}(M(\bar\ell))$, where $\bar\ell = (\ell_1, \ldots, \ell_n)$, $\bar\ell$ is generic, and its genetic code does not equal $\langle\{n, n-3, \ldots, 1\}\rangle$. We also discuss the implications for topological complexity.

Our first result is an upper bound, which will sometimes be sharp. See Table 3.4 for a tabulation when $n = 8$. Recall that $m = n - 3$.

Theorem 3.1. *If s is the largest cardinality of the gees of $\bar\ell$, then* $\mathrm{zcl}(M(\bar\ell)) \le 2s+1$.

Proof. For $u \in H^*(M(\bar\ell))$, let $\bar u = u \otimes 1 + 1 \otimes u$. We first consider products of the form $\prod \overline{u_i}$. By symmetry, any such product in dimension $2m$ must be 0, as it will have an even number of terms $W_\emptyset \otimes W_\emptyset$.

A product of a $\overline{V_i}$'s and b $\overline{W_S}$'s has grading $\ge a+b(m-s)$. If $a > 2s$, then $\prod \overline{V_i} = 0$, so we may assume that $a \le 2s$. If $a + b \ge 2s + 2$, then $b \ge 2$ and

$$a+b(m-s) \ge 2s+2-b+b(m-s) = bm-(b-2)(s+1) \ge bm-(b-2)m = 2m,$$

and so the product must be 0. We have used that $s \le m - 1$ by Proposition 2.5.

Now we consider the possibility of more general zero divisors. Let α_j denote a zero divisor which contains a term $A \otimes B$ in which the total number of V-factors (resp. W-factors) in AB is p_j (resp. q_j) with $p_j + q_j \ge 2$. Its grading is $\ge p_j + q_j(m - s)$. A product of a $\overline{V_i}$'s, b $\overline{W_S}$'s, and c α_j's, with $a+b+c \ge 2s+2$ will be 0 if $a+\sum p_j > 2s$, so we may assume $a + \sum p_j \le 2s$. This product, with $c \ge 1$, has grading

$$\begin{aligned}
&\ge a + b(m-s) + \sum p_j + (m-s)\sum q_j \\
&\ge a + b(m-s) + \sum p_j + (m-s)(2c - \sum p_j) \\
&\ge a + (b+2c)(m-s) + (m-s-1)(a-2s) \\
&= (m-s-1)(a+b+2c-2s) + a + b + 2c \\
&\ge 2(m-s-1) + 2s + 2 + c \\
&= 2m+1,
\end{aligned}$$

and hence is 0. ■

Next we give our best result for lower bounds. Recall that the partial order described just before Theorem 2.2 is applied also to multisets.

Theorem 3.2.

a. *If G and G' are gees of $\bar{\ell}$, not necessarily distinct, and there is an inequality of multisets $G \cup G' \ge [k]$, then* $\mathrm{zcl}(M(\bar{\ell})) \ge k+1$.

b. *If there are no exotic products in $H^*(M(\bar{\ell}))$, then (a) is sharp in the sense that*

$$\mathrm{zcl}(M(\bar{\ell})) = 1 + \max\{k : \exists \textit{ gees of } \bar{\ell} \textit{ with } G \cup G' \ge [k]\}.$$

In particular, (b) holds if $m \ge 2s$, where s denotes the maximum size of the gees of $\bar{\ell}$.

Proof. Under the hypothesis of (a), there is a partition $[k] = S \sqcup T$ with $G \ge S$, and $G' \ge T$. Then the following product of $k+1$ zero-divisors is nonzero:

$$\prod_{i \in S} \overline{V_i} \cdot \overline{W_S} \cdot \prod_{j \in T} \overline{V_j}.$$

Indeed, this product contains the nonzero term $W_\emptyset \otimes V_T$, and this term cannot be cancelled by any other term in the expansion, since the only way to obtain $W_\emptyset$ is as $V_U W_U$ for some set U.

Now we prove (b) by showing that for any nonzero product of $k+1$ $\overline{u_i}$'s, $[k]$ is $\le$ a union of two gees. If $\prod_{i \in U} \overline{V_i} \cdot \overline{W_R} \ne 0$, then we can partition $U = S \sqcup T$ with

$S\subset R\le G$ and $T\le G'$ for gees G and G'. The number of factors is $|U|+1$, and $[|U|]\le G\cup G'$. A similar argument works for a product without any $\overline{W}$'s.

Because, under the hypothesis of no exotic products, products of W's are 0, the most $\overline{W}$'s that could occur in a nonzero product is two. Denoting $\prod_{i\in S}\overline{V}_i$ as $\overline{V}_S$,[1] we consider products $\overline{V}_S\overline{W_R}\cdot\overline{V}_T\overline{W_{R'}}$ with S and T disjoint, $S\subset R$, $T\subset R'$, and R and $R'\le$ gees. First we consider the case in which S and R' are disjoint, as are T and R. Then $\overline{V}_S\overline{W_{R'}}=V_S\otimes W_{R'}+W_{R'}\otimes V_S$ and similarly for $\overline{W_R}\cdot\overline{V}_T$. We obtain

$$\overline{V}_S\overline{W_R}\cdot\overline{V}_T\overline{W_{R'}}=W_{R-S}\otimes W_{R'-T}+W_{R'-T}\otimes W_{R-S}.$$

If $R-S=R'-T$, this is 0. If $R-S\ne R'-T$, then $R\cup R'$ contains at least $|S|+|T|+1$ distinct elements, and so, with $k=|S|+|T|+1$, the multiset $R\cup R'$ is $\ge[k]$ and is $\le$ the union of two gees. Since there are $k+1$ factors in the product, the claim of (b) holds in this case.

Now suppose that we are in the situation of the preceding paragraph, except that T and R are not disjoint. For simplicity, we assume $T\cap R=\{j\}$ and $S\cap R'=\emptyset$. Let $R=S\sqcup\{j\}\sqcup D_1$ and $R'=T\sqcup D_2$, with D_1 and D_2 possibly empty. Working as in the preceding paragraph, we find that

$$\overline{V}_S\overline{W_R}\cdot\overline{V}_T\overline{W_{R'}}=W_{D_1}\otimes W_{D_2\cup\{j\}}+W_{D_1\cup\{j\}}\otimes W_{D_2}+W_{D_2}\otimes W_{D_1\cup\{j\}}+W_{D_2\cup\{j\}}\otimes W_{D_1}.$$

If $D_1=D_2=\emptyset$, this is 0. Otherwise, $R\cup R'$ contains at least $|S|+|T|+1$ elements, and so the claim follows as in the preceding paragraph. A similar argument works if $(T\cap R)\cup(S\cap R')$ contains more than one element. ■

Our zcl results depend only on the gees, and not on the value of n. (Recall $m=n-3$.) However the possible gees depend on n. Of course, the numbers which occur in the gees must be less than n, but also, if G and G' are gees (not necessarily distinct), then we cannot have $[n-1]-G'\le G\cup\{n\}$, for then $G\cup\{n\}$ would be both short and long. Thus, for example, 8531 is an allowable gene, but 7531 is not, since $642<7531$ but $7642\not<8531$.

There are 2469 equivalence classes of spaces $M(\overline{\ell})$ with $n=8$. Genes for these are listed in [10]. We perform an analysis of what we can say about the zcl and TC of these. Since $n=8$, each satisfies $\mathrm{TC}(M(\overline{\ell}))\le 11$ by (1.1). As we discuss

[1]Note the distinction: $\overline{W_S}=W_S\otimes1+1\otimes W_S$, whereas $\overline{V}_S=\prod_{i\in S}(V_i\otimes1+1\otimes V_i)$.

below in more detail, for most of them we can assert that $\mathrm{zcl}(M(\bar{\ell})) \geq 6$, and so $\mathrm{TC}(M(\bar{\ell})) \geq 7$. For most of them we can only assert lower bounds for zcl, due to the possibility of exotic products. We emphasize that the following analysis pertains to the case $n = 8$.

As discussed in Proposition 2.5 and the paragraph which preceded it, there is only one $\bar{\ell}$ with a gee of size 5. This $M(\bar{\ell})$ is homeomorphic to $T^5 \sqcup T^5$ with topological complexity 6. This is a truly special case, as it is the only disconnected $M(\bar{\ell})$.

Another very special case is the $\bar{\ell}$ whose only gee is empty; i.e., its genetic code is $\langle 8 \rangle$. This $M(\bar{\ell})$ is homeomorphic to S^5, with topological complexity 2.

Other slightly special cases which we wish to exclude from the analysis below are those whose only gee is 1, 21, 321, or 4321. For these, the zcl is ≥ 2, 3, 4, and 5, respectively. This is sharp in the first two cases, since $m \geq 2s$. All of the cases just mentioned are excluded from the following discussion.

There are 768 $\bar{\ell}$'s whose largest gee has size 4. For all of them, we can deduce only $\mathrm{zcl}(M(\bar{\ell})) \geq 6$, using Theorem 3.2(a) and the following result.

Proposition 3.3. *Suppose G and G' are subsets of $[7]$, not necessarily distinct, with neither strictly less than the other and with $\max(|G|, |G'|) = 4$. Assume also that it is not the case that $G = G' = 4321$, and it is not the case that $[7] - G' \leq G \cup \{8\}$. Then $G \cup G' \geq [5]$ but $G \cup G' \not\geq [6]$.*

Proof. The first conclusion follows easily from the observation that if $G = 4321$, then $5 \in G'$. For the second, if $G \cup G' \geq [6]$ then applying $\cup G'$ to the false statement $[7] - G' \leq G \cup \{8\}$ would yield a true statement, and the ordering that we are using for multisets has a cancellation property for unions. ∎

There are 1569 $\bar{\ell}$'s whose largest gee has size 3. For these, we again cannot rule out exotic products, so can only assert lower bounds for zcl. By Theorem 3.1, these all satisfy $\mathrm{zcl} \leq 7$. Of these, 929 have a gee $G \geq 531$, and this satisfies $G \cup G \geq [5]$, hence $\mathrm{zcl} \geq 6$. In addition to these, there are 524 with distinct gees satisfying $G \cup G' \geq [5]$. Combined with the 768 $\bar{\ell}$'s with some $|\mathrm{gee}| = 4$, 2221 of the 2469 $\bar{\ell}$'s with $n = 8$ satisfy $\mathrm{zcl}(M(\bar{\ell})) \geq 6$. There are another 116 $\bar{\ell}$'s with largest gee of size 3 for which we can only assert $\mathrm{zcl} \geq 5$. An example of a genetic code of this type is $\langle 8421, 843, 862, 871 \rangle$.

There are 120 $\bar{\ell}$'s whose largest gee has size 2. For these, exotic products are not possible and we can assert the precise value of zcl. Of these, 85 have a gee $G \geq 42$ and since $G \cup G \geq [4]$, they have $\text{zcl}(M(\bar{\ell})) = 5$. In addition to these, there are 10 having distinct gees satisfying $G \cup G' \geq [4]$ and so again zcl$= 5$. There are 25 others for which zcl$= 4$. Finally, there are 6 $\bar{\ell}$'s with largest gee of size 1. These satisfy $\text{zcl}(M(\bar{\ell})) = 3$.

In Table 3.4, we summarize what we can say about zcl when $n = 8$, omitting the six special cases described earlier. Keep in mind that

$$1 + \text{zcl} \leq \text{TC} \leq 11.$$

In the table, s denotes the size of the largest gee, and # denotes the number of distinct homeomorphism classes of 8-gons having the property.

Table 3.4. *Number of types of 8-gon spaces*

s	zcl	#
1	3	6
2	4	25
2	5	95
3	$5, 6$ or 7	116
3	6 or 7	1453
4	$6, 7, 8$ or 9	768

For general $m(= n - 3)$, the largest gees (with one exception) have size $s = m - 1$, and so Theorem 3.1 allows the *possibility* of zcl as large as $2m-1$, which would yield a lower bound for TC within 1 of the upper bound $2m+1$ given by (1.1). However, this would require many nontrivial exotic products. By an argument similar to Proposition 3.3, all we can assert from Theorem 3.2(a) is zcl $\geq m + 1$ (when $s = m - 1$). If $s \leq [m/2]$, then we can determine the precise zcl, which can be as large as $2s + 1$, so we can obtain m or $m + 1$ as zcl, depending on parity, yielding a lower bound for TC only roughly half the upper bound given by (1.1).

4. Exotic products in an example

When $m < 2s$, the previous analysis of $H^*(M(\bar{\ell}))$ fails to resolve two things: (a) products $W_S W_{S'}$ when $|W_S| + |W_{S'}| < m$, and (b) products $V_i W_S$ when $|W_S| + 1 \leq s$. In this section, we present a simple example with $m = 3$ and $s = 2$, showing how the detailed relation (2.3) enables us to resolve (a), but the inability to resolve (b) prevents us from strengthening our lower bound for zcl.

Let $\bar{\ell}$ have genetic code $\langle 631\rangle$. Then a basis for $H^*(M(\bar{\ell}))$ is:

$$\begin{array}{ll} 0 & 1 \\ 1 & V_1, V_2, V_3, W_{1,2}, W_{1,3} \\ 2 & V_1V_2, V_1V_3, W_1, W_2, W_3 \\ 3 & W_\emptyset. \end{array}$$

The previous analysis says that

$$V_S W_{S'} = \delta_{S,S'} W_\emptyset \text{ if } |S| = |S'| \tag{4.1}$$

and, mod $\langle V_1V_2, V_1V_3\rangle$, $V_iW_{j,k} \equiv W_k$ if $i = j$, and is $\equiv 0$ if $i \notin \{j, k\}$. It also shows that $\overline{V_1}\ \overline{V_2}\ \overline{W_{1,2}}\ \overline{V_3}$ contains the nonzero term $W_\emptyset \otimes V_3$ and so $\mathrm{zcl}(M(\bar{\ell})) \geq 4$. It does not yield information about $W_{1,2}W_{1,3}$. We will show, using (2.3), that mod $\langle V_1V_2, V_1V_3\rangle$, $W_{1,2}W_{1,3} \equiv W_1$. However, the "mod $\langle V_1V_2, V_1V_3\rangle$" aspect of various products, essentially an extension question in (2.1), prevents us from deciding whether $\overline{V_1}\ \overline{V_2}\ \overline{V_3}\ \overline{W_{1,2}}\ \overline{W_{1,3}} \neq 0$, and hence whether $\mathrm{zcl}(M(\bar{\ell})) = 5$.

The relations (2.3) show that a basis for $\ker(R|H^*(\overline{M}(\bar{\ell})))$ is

$$\begin{array}{ll} 1 & R + V_3, R + V_2 \\ 2 & R^2 = RV_2 = RV_3, RV_1 + V_1V_3, RV_1 + V_1V_2 \\ 3 & RV_1V_2 = RV_1V_3 = R^2V_1. \end{array}$$

Since $V_iW_j = \delta_{i,j}W_\emptyset$, we deduce that the three elements listed in grading 2 are W_1, W_2, and W_3, respectively, and then similarly that $R + V_3 = W_{1,2}$ and $R + V_2 = W_{1,3}$. Since

$$(R + V_3)(R + V_2) = R^2 + RV_2 + RV_3,$$

we deduce that $W_{1,2}W_{1,3} = W_1$ in $\ker(R|H^*(\overline{M}(\bar{\ell})))$.

Using that products into the top grading are given by (4.1), we find that, mod $\langle V_1V_2, V_1V_3\rangle$,

$$\overline{V_1}\ \overline{V_2}\ \overline{V_3}\ \overline{W_{1,2}}\ \overline{W_{1,3}} \equiv W_\emptyset \otimes (V_3W_{1,3} + V_2W_{1,2}) + (V_3W_{1,3} + V_2W_{1,2}) \otimes W_\emptyset \equiv 0.$$

If $V_3W_{1,3}$ or $V_2W_{1,2}$ contain terms V_1V_2 or V_1V_3 in $H^*(M(\bar{\ell}))$ due to an extension in (2.1), this could be nonzero, but we have no way of obtaining this information.

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