1. Write as a simple fraction $1/(\frac{1}{2}+\frac{1}{3})$.

2. Heavenly body B lies directly between bodies A and C. The distance from A to B is $6.2 \cdot 10^8$, and the distance from A to C is $6.11 \cdot 10^9$. What is the distance from B to C, expressed in a similar form?

3. For how many integer values of x does there exist a triangle whose sides have length $2\frac{1}{2}$, 5, and x?

4. Let $a = 11/15$, $b = 13/19$, and $c = 16/23$. Write the letters a, b, and c in increasing order (smallest to largest).

5. At how many points does the graph of

$$y = (x^2 - 5x + 9)(x^2 - 6x + 9)(x^2 - 7x + 9)$$

intersect the x-axis?

6. How many pairs (x, y) satisfy both $x + y = 2$ and $x + y^2 = 4$?

7. How many integers between 100 and 1000 are multiples of 7?

8. With the sun directly overhead, a woman shades herself with a parasol of radius 2 feet as she walks on a grassy lawn around a circle of radius 6 feet. This grass must be in constant sunlight, and so any section which ever lies beneath the parasol dies instantly. How many square feet of dead grass will result?

9. What is the perimeter of a right triangle whose base has length 5 and whose hypotenuse is 2 greater than the third side?

10. Each of w, x, y, and z equals 0 or 1. It is known that (a) if $x = 0$, then $y = 1$; (b) if $y = 0$, then $w = z$; and (c) if $z = 0$, then $w = 1$. Suppose $y = 0$. What is the value of $w + x + z$?

11. A used car salesman sold two cars and received \$840 for each car. One of these transactions yielded a 40% profit for the dealer (compared to his purchase price), whereas the other amounted to a 25% loss. What is the dealer's net profit (or loss) on the two transactions?

12. A bowl contains 50 colored balls: 13 green, 10 red, 9 blue, 8 yellow, 6 black, and 4 white. If you are blindfolded as you pick balls out of the bowl, what is the smallest number of balls you must pick in order to guarantee that you have at least 7 balls of the same color?

13. If Bill and Mary leave their houses at the same time, walking directly toward each other, each at their own constant rate, they will meet after 5 minutes. If Bill leaves 3 minutes later than Mary, they meet after he has walked for 3 minutes. How many minutes would it take him to walk all the way from his house to Mary's?

14. Two fair coins are flipped simultaneously. This is done repeatedly until at least one of the coins comes up heads, at which point the process stops. What is the probability that both coins came up heads on this last flip?

15. Let $p(x) = ax^2 + bx + c$ for certain coefficients a, b, and c. If $p(1) = 1$, $p(2) = 2$, and $p(3) = 4$, what is $p(5)$?

16. What is the largest value of $|x^2 - 4|$ for all x satisfying $|x + 2| < 0.1$?

17. George's car gets 3 more miles per gallon during highway driving than it does during city driving. On a recent trip, he drove 112 miles on the highway and 150 miles in the city and used exactly 10 gallons of gasoline. How many miles per gallon does his car get during city driving?

18. Five lines parallel to the base of a triangle divide each of the other sides into six equal segments and the area into six distinct parts. If the area of the largest of these parts is 33, then what is the area of the original triangle?

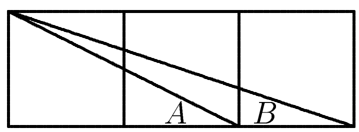
19. An equal number of juniors and seniors responded to the question "Do you like math?," each responding with Yes or No. If 70% of those who said Yes were seniors and 80% of those who said No were juniors, then what percentage of the seniors polled said Yes?

20. Recall that the iterated power a^{b^c} means $a^{(b^c)}$. If x is a real number which satisfies $2^{2^x} + 4^{2^x} = 56$, then what is the value of $2^{2^{2^x}}$?

21. The smallest positive integer n with the property that 3 divides n, 4 divides $n+1$, 5 divides $n+2$, 6 divides $n+3$, 7 divides $n+4$, 8 divides $n+5$, and 9 divides $n+6$ is 3. What is the next largest integer with this property?

22. The diagonals of a quadrilateral are perpendicular. Three of its sides, in some order, have length 2, 3, and 4. What are the possible values for the length of the fourth side?

23. The diagram below consists of three squares and lines connecting certain vertices as indicated. The angles that the slanting lines make with the horizontal are denoted by A and B. What is the radian measure of angle $A+B$?



24. The polar coordinates (r, θ) of a point are its distance r from the origin and the angle θ from the positive x-axis to its radius vector. Let P and Q have polar coordinates $(8, 5\pi/12)$ and $(8, -3\pi/12)$, and let M be the midpoint of the segment PQ. What are the polar coordinates of M?

25. For each point (m, n) in the plane, with m and n integers, draw a circle of radius $1/(2^{|m|+1}3^{|n|+1})$. What is the total area enclosed by all these circles?

26. Consider the following 52 curves:

- the 21 vertical lines $x = i$ for $i = -10, -9, \ldots, 10$;
- the 21 horizontal lines $y = j$ for $j = -10, -9, \ldots, 10$;
- the 10 circles centered at the origin with radius $k + \frac{1}{\pi}$ for $k = 0, 1, \ldots, 9$.

No three of the curves intersect in a common point. What is the total number of points of intersection of these curves?

27. Evaluate as a simple fraction $\displaystyle\sum_{i=1}^{100} \frac{1}{4i^2 - 1}$.

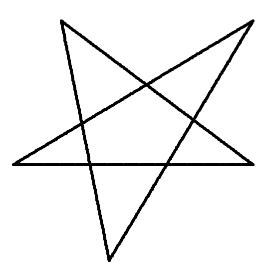
28. Let $a_1 = 20$ and $a_2 = 03$. For $n \geq 1$, define $a_{n+2} = a_n + a_{n+1}$. What is the remainder when $a_1^2 + a_2^2 + \cdots + a_{2003}^2$ is divided by 8?

29. For the table below, you can start at any letter, then move horizontally or vertically one letter at a time, up or down or left or right. How many different paths in the table will spell the word LEHIGH, with the letters occurring in the correct order? For example, going backwards along the bottom counts as 1 way.

L	E	H	I	G	H
E	L	E	H	I	G
H	E	L	E	H	I
I	H	E	L	E	H
G	I	H	E	L	E
H	G	I	H	E	L

30. Two matching decks have 52 cards each. One is shuffled and put on top of the second. For each card of the top deck, we count the number of cards between it and the corresponding card of the bottom deck (not including the cards themselves). What is the sum of these numbers?

31. Let P be a convex polygon such that, if sides are numbered consecutively $1, \ldots, n$, sides i and $i + 2$ eventually intersect when extended beyond side $i + 1$. (For $i = n - 1$ or n, $i + 2$ is interpreted as 1 or 2.) By prolonging each pair of sides i and $i + 2$ until they meet, an n-pointed star is obtained. What is the sum (in degrees) of the angles at the n points of the star? Your answer may be a formula involving n. We assume $n \geq 5$. (An example when $n = 5$ appears below.)



32. Let P denote the perimeter of a right triangle, and Q the sum of the squares of its three sides. What is the ordered triple of coefficients, (k_1, k_2, k_3), such that the area of the triangle equals $k_1 P^2 + k_2 P\sqrt{Q} + k_3 Q$?

33. Circles with centers at A, B, and C are exteriorly tangent to one another. The radius of the circle centered at A is 3, and that of the circle centered at B is 5. If in triangle ABC, $\cos(A) = 11/16$, what is $\cos(B)$?

34. How many integer solutions (x, y) are there for the following equation?

$$(x-4)(x-10) = 2^y$$

35. Chicken McNuggets come in packets of 6, 9, or 20. What is the largest number of Chicken McNuggets that you cannot buy by combining various packets?

36. For each side of an equilateral triangle of side length 2, a square is drawn containing the triangle and having that side as one of its edges. What is the side length of the smallest triangle containing these three squares?

37. For any nonempty subset of distinct positive integers, define its alternating sum to be the sum obtained by arranging its numbers in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. For example, the alternating sum of $\{1, 2, 5, 6\}$ is $6-5+2-1 = 2$, and for $\{5\}$ it is just 5. What is the sum of the alternating sums of all nonempty subsets of $\{1, 2, 3, 4, 5, 6\}$?

38. Let $f(x) = x^3 + 4x^2 + 8x + 16$. Express

$$f(x+7)-f(x+6)-f(x+5)+f(x+4)-f(x+3)+f(x+2)+f(x+1)-f(x)$$

as an explicit simple polynomial.

39. Let $f(x) = x^2 + kx$ with k a real number. What is the set of values of k for which the equations $f(x) = 0$ and $f(f(x)) = 0$ have exactly the same real solution sets?

40. Find all real numbers x such that $x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$.

SOLUTIONS TO 2003 MATH CONTEST

1. 6/5. It is $1/(\frac{3}{6}+\frac{2}{6})$.

2. $5.49\cdot 10^9$. It is $(6.11-0.62)10^9$.

3. 5. Since the sum of two sides must be greater than the third, x can be 3, 4, 5, 6, or 7.

4. b, c, a. $a = .73$, $b = .68$, and $c = .69$.

5. 3. Note that $y = 0$ if and only if one of the factors equals 0. For the three factors, $b^2 - 4ac$ equals -11, 0, and 13, so that they have respectively 0, 1, and 2 roots.

6. 2. Subtract the equations to get $y^2 - y = 2$. This implies $y = 2$ or -1, and each of these yields a unique value of x that works.

7. 128. 105 is the first. $105 + 128\cdot 7 = 1001$ is too large, so there are 127 more after 105.

8. 48π. It will be the area inside a circle of radius 8 feet which lies outside a concentric circle of radius 4 feet. This is $\pi(8^2 - 4^2)$.

9. $17\frac{1}{2}$ or $\frac{35}{2}$. If x denotes the other leg, then $(x+2)^2 = x^2 + 5^2$, so $4x + 4 = 25$. The perimeter is $2x + 7 = \frac{21}{2} + 7$.

10. 3. (a) implies $x = 1$. (b) implies $w = z$. If $z = 0$, then by (c), $w \neq z$. Hence $z \neq 0$. Thus $z = 1$ and then $w = 1$.

11. $40 loss. His purchase price x on the first car must satisfy $1.4x = 840$. Thus $x = 600$. Similarly, his purchase price on the second car must have been $840/.75 = 1120$. Thus he paid $1720 and only recouped $1680.

12. 35. You might choose 6 greens, 6 reds, 6 blues, 6 yellows, 6 blacks, and 4 whites. The next one will certainly give you 7 of some color.

13. 15. Let s_B and s_M denote the speeds of Bill and Mary, resp., and d the distance between their houses. Then $5s_B + 5s_M = d = 3s_B + 6s_M$. Hence $s_M = 2s_B$, and $15s_B = d$.

14. 1/3. The previous flips are irrelevant. Of the four equally likely outcomes of a double flip, HH, HT, TH, and TT, the first three have at least one head, and just one of them has the other coin also a head.

15. 11. For a quadratic, differences between values at equally spaced intervals increase linearly. Since $p(2) - p(1) = 1$ and $p(3) - p(2) = 2$, we must have $p(4) - p(3) = 3$ and $p(5) - p(4) = 4$, and hence $p(5) = p(3) + 3 + 4 = 11$. Alternatively, solve the equations $a + b + c = 1$, $4a + 2b + c = 2$, and $9a + 3b + c = 4$ to obtain that $p(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 1$ and then substitute $x = 5$.

16. 0.41 or None. This problem was misstated. The intent was to have $|x+2| \le 0.1$. Then the answer would have been 0.41, with the following solution: For $-2.1 \le x \le -1.9$, $|x^2 - 4|$ has a minimum value of 0 at $x = -2$ and increases as you move in either direction. Its maximum will occur either at -2.1 or -1.9. Since it equals $|x+2||x-2|$, at these points it will equal 0.1 times 4.1 or 3.9. As stated, with $|x + 2| < 0.1$, no maximum is achieved. The value of the function gets arbitrarily close to 0.41. Both answers were accepted as correct, along with such variations as $0.409999\ldots$.

17. 25. If x denotes the requested number, then $\frac{112}{x+3} + \frac{150}{x} = 10$. Factoring out a 2, this reduces to $0 = 5x^2 - 116x - 225$. This can be solved by the quadratic formula, although the numbers are pretty large. It can be factored as $0 = (5x + 9)(x - 25)$. Or you might guess the solution from looking at the original equation, especially after it has been divided by 2.

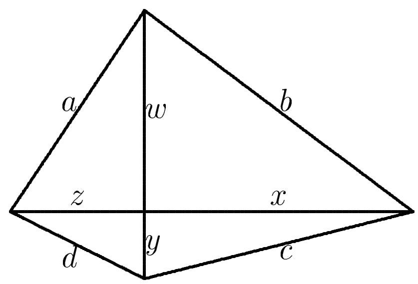
18. 108. The portion of the whole triangle obtained by excluding the bottom part is a triangle similar to the whole, with dimensions 5/6 as large. Thus its area is 25/36 times that of the whole. Thus the area of the bottom portion is 11/36 times that of the whole. So the area of the triangle is $33 \cdot 36/11$.

19. 84. Let j (resp. s) denote the fraction of juniors (resp. seniors) saying Yes. Then $j = \frac{3}{7}s$, while $1 - j = 4(1 - s)$. Thus $1 - \frac{3}{7}s = 4 - 4s$, and hence $\frac{25}{7}s = 3$, so that $s = \frac{21}{25}$.

20. 128. If $y = 2^{2^x}$, then $y + y^2 = 56$, from which we deduce $y = 7$. The desired answer is 2^7.

21. 2523. $n-3$ must be divisible by 3, 4, 5, 6, 7, 8, and 9. Thus it must be divisible by $8\cdot 9\cdot 5\cdot 7=2520$.

22. $\sqrt{11}$ or $\sqrt{21}$. (Both are required.) Refer to the diagram below. Because the triangles are "right," $a^2+c^2=w^2+x^2+y^2+z^2=b^2+d^2$. Thus the possible lengths d for the fourth side satisfy $d^2=2^2+3^2-4^2$ (impossible) or $2^2+4^2-3^2$ or $3^2+4^2-2^2$.



23. $\pi/4$. $\tan(A+B)=(\tan(A)+\tan(B))/(1-\tan(A)\tan(B))=(\frac{1}{2}+\frac{1}{3})/(1-\frac{1}{2}\cdot\frac{1}{3})=\frac{5}{6}/\frac{5}{6}=1$.

24. $(4,\pi/12)$. If O is the origin, then triangle OPQ is isosceles, with $\angle O=120^{\circ}$. Thus the angle OM is halfway between $-3\pi/12$ and $5\pi/12$, and the length OM is $8\cos(60^{\circ})$.

25. $25\pi/432$. It is π times

$$\begin{aligned}
&(\tfrac{1}{4}+2(\tfrac{1}{4^2}+\tfrac{1}{4^3}+\cdots))(\tfrac{1}{9}+2(\tfrac{1}{9^2}+\tfrac{1}{9^3}+\cdots)) \\
=\ &(\tfrac{1}{4}+\tfrac{2}{16}(1/(1-\tfrac{1}{4}))(\tfrac{1}{9}+\tfrac{2}{81}(1/(1-\tfrac{1}{9})) \\
=\ &(\tfrac{1}{4}+\tfrac{1}{6})(\tfrac{1}{9}+\tfrac{2}{72}) \\
=\ &\tfrac{5}{12}\cdot\tfrac{5}{36}=25/432.
\end{aligned}$$

26. 841. Each of the 21 vertical lines meets each of the 21 horizontal lines, yielding $21\cdot 21=441$ points of intersection. Consider, for the circle with radius $k+\frac{1}{\pi}$, the quarter beginning on the positive x-axis and continuing up to, but not including, the point where it meets the positive y-axis. It meets $k+1$ horizontal lines and k vertical lines. Thus the whole circle has $4(2k+1)$ intersections. Summing this as k goes from 0 to 9 yields $8(1+\cdots+9)+4\cdot 10=8\cdot 45+40=400$. Thus the total is $441+400$.

27. 100/201. It equals

$$\sum_{i=1}^{100} \frac{1}{(2i-1)(2i+1)} = \sum_{i=1}^{100} \tfrac{1}{2}(\tfrac{1}{2i-1} - \tfrac{1}{2i+1}) = \tfrac{1}{2}(1 - \tfrac{1}{201}) = \tfrac{100}{201},$$

since all terms between 1 and $\frac{1}{201}$ occur with both positive and negative signs.

28. 7. The mod 8 value of a^2 depends only on the mod 4 value of a. The mod 4 values of the successive a_i are $0, 3, 3, 2, 1, 3$ and this sequence of six numbers then repeats forever. The mod 8 value of the sum of the squares in this repeating block is $0+1+1+4+1+1=0$. 2003 has 333 of these blocks and then the next 5 numbers, yielding $0+1+1+4+1$.

29. 64. The number of paths ending at the H in the bottom left corner equals the number ending in the top right corner, and so we count the latter and double it. We label a letter with the number of allowable ending paths beginning with that letter. The H in position (1,6) is labeled with 1, as are the two G's adjacent to it. The I's in position (1,4), (2,5), and (3,6) will be labeled with 1, 2, and 1, respectively. Note that the label equals the sum of the labels of the letters following it which are adjacent to it. Then the labels of the four H's are 1, 3, 3, and 1, those of the five E's 1, 4, 6, 4, 1, and finally those of the six L's 1, 5, 10, 10, 5, and 1. This sum $(1+5+10+10+5+1)$ multiplied by 2, to account for the other direction, is our desired answer.

30. 2652. Label the pairs of matching cards $1, 2, \ldots, 52$. Let a_i be the location from the bottom of the pile of the top card labeled i, and b_i the location of the bottom card labeled i. We wish to find $\sum_{i=1}^{52}(a_i - b_i - 1)$. This equals

$$\sum_{i=1}^{52} a_i - \sum_{i=1}^{52} b_i - 52 = (53+\cdots+104)-(1+\cdots+52)-52 = 52^2-52 = 2652.$$

31. $(n-4)180$. There are n triangles exterior to the polygon with a vertex at a vertex of the star. The set of the base angles of all these triangles is all exterior angles of the polygon. The sum of the exterior angles of a polygon is 360 degrees, but that only counts one of the two exterior angles at each vertex. Thus the sum of the base angles of our n triangles is 720, which when subtracted from $180n$ yields the desired angle sum.

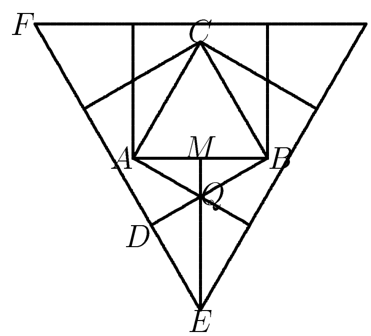
32. $(\frac{1}{4}, -\frac{1}{4}\sqrt{2}, 0)$. If a and b are the legs and c the hypotenuse, then $a+b+\sqrt{a^2+b^2} = P$ and, using the Pythagorean Theorem again, $2(a^2+b^2) = Q$. Thus $a+b = P - \sqrt{Q/2}$. Squaring this yields $a^2+b^2+2ab = P^2 - 2P\sqrt{Q/2} + Q/2$. The terms a^2+b^2 and $Q/2$ cancel. Noting that the area is $\frac{1}{2}ab$, we obtain $\frac{1}{4}(P^2 - \sqrt{2}P\sqrt{Q})$.

33. $7/8$. Let r denote the radius of the circle centered at C. Then $(r+5)^2 = 8^2 + (r+3)^2 - 2\cdot 8(r+3)11/16$. This implies $r = 1$. Now $4^2 = 8^2 + 6^2 - 2\cdot 6\cdot 8\cos(B)$, which implies $\cos(B) = 7/8$.

34. 2. $x-4$ and $x-10$ must be 2-powers that differ by 6, or their negatives. The only such are 2 and 8, or -2 and -8. This yields the solutions $y = 4$ and $x - 10 = -8$ or 2.

35. 43. Any multiple of 3 which is greater than 3 can be obtained from packets of 6 and 9. Since 43 and $43 - 20$ are not multiples of 3, and $43 - 2\cdot 20$ is too small to achieve, 43 cannot be achieved. To see that any number larger than 43 can be achieved, first note that since $36 = 9+9+9+9$, $38 = 20+9+9$, and $40 = 20+20$, any even number ≥ 36 can be achieved, by adding 6's to these. By adding another 9, any odd number ≥ 45 can be achieved.

36. $4\sqrt{3} - 2$. Refer to the diagram below, in which ABC is the given triangle. Triangle QMB implies that $BQ = 2/\sqrt{3}$, and hence $QD = 2 - 2/\sqrt{3}$. Now triangle QDE implies that $DE = \sqrt{3}\cdot QD = 2\sqrt{3} - 2$, and hence $EF = 2(2\sqrt{3} - 2) + 2$.



37. 192. There are $2^5 = 32$ subsets of $\{1, 2, 3, 4, 5\}$, including the empty set. (To see this, note that a subset is described by telling for each

of the 5 elements whether or not it is in the set.) For each of these 32 subsets S, S and $S \cup \{6\}$ give distinct subsets of $\{1,2,3,4,5,6\}$, and all subsets of $\{1,2,3,4,5,6\}$ are obtained this way. The sum of the alternating sums of S and $S \cup \{6\}$ is 6, since the other terms will appear once with each sign. Thus the answer is $6 \cdot 32$.

38. 48. Let $g(x) = f(x+1) - f(x)$ and $h(x) = g(x+2) - g(x)$. Then the given polynomial equals $g(x+6) - g(x+4) - g(x+2) + g(x) = h(x+4) - h(x)$. Note that if Ax^n is the leading term of a polynomial $p(x)$, then the leading term of $p(x+k) - p(x)$ is $Ankx^{n-1}$. Thus the leading term of $g(x)$ is $3x^2$, the leading term of $h(x)$ is $12x$, and the leading (and hence only) term of our given polynomial is 48.

39. $[0,4)$ or $0 \leq k < 4$. Since $f(f(x)) = f(x)(f(x)+k) = f(x)(x^2+kx+k)$, we seek the values of k for which $x^2+kx+k=0$ has no real solutions, except perhaps for $x=0$ or $x=-k$. These latter two are solutions of $x^2+kx+k=0$ iff $k=0$, which works as a possible value. $x^2+kx+k=0$ has no real solutions if and only if $k^2-4k<0$ if and only if $0<k<4$.

40. $(1+\sqrt{5})/2$. Note that the equation implies $x>0$. Isolating $\sqrt{1-\frac{1}{x}}$ and squaring simplifies to $(x^2-1) - 2\sqrt{x(x^2-1)} + x = 0$, which says $(\sqrt{x^2-1} - \sqrt{x})^2 = 0$, and hence $x = x^2-1$, from which the result follows.